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EXCISING STATES OF C^*-ALGEBRAS

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0. Introduction. A net $\{a_\alpha\}$ of positive, norm one elements of a C^*-algebra A *excises* a state f of A if

$$\lim \|a_\alpha a a_\alpha - f(a)a_\alpha^2\| = 0 \quad \text{for every } a \text{ in } A.$$

This notion has been used explicitly by the second author [**4**, **5**, **6**] for pure states, but the present paper will explore it more fully. The name is motivated by the following example. Let K be the unit disk in the complex plane, $A = C(K)$ and $f(a) = a(0)$. Define $a_n(re^{i\theta}) = \phi_n(r)$, where

$$\phi_n(r) = \begin{cases} 0 \text{ if } 0 \leqq r \leqq \dfrac{1}{n+2} \text{ or } r > \dfrac{1}{n} \\ 1 \text{ if } r = \dfrac{1}{n+1} \\ \text{linear elsewhere.} \end{cases}$$

Note that the sets $\{t \in K: a_n(t) > 0\}$ form rings about 0 with radii tending to 0. In this sense the sequence $\{a_n\}$ "cuts out" the state f and, in the limit, isolates it from all other states. It turns out (see Proposition 2.2) that a state f of A can be excised if and only if it is in the weak* closure $P(A)^-$ of $P(A)$. If A is separable, then excising can always be done with a sequence.

Suppose that $\{a_n\}$ is an orthogonal, positive sequence of norm one elements of a unital C^*-algebra A and $\{f_n\} \subset P(A)$ (the set of pure states of A) such that $f_n(a_n) = 1$ for all n. What can we say about the set L of weak* limit points of the set $\{f_n\}$ in $S(A)$ (the state space of A)? If A is abelian, then $L \subset P(A)$ since $P(A)$ is closed in $S(A)$. If A is not abelian, then $P(A)$ may even be dense in $S(A)$ (see [**8**, 11.2.4]), and easy examples show that the set L described above need not lie in $P(A)$. However, by [**6**, Theorem 1] we see that if $A = \mathbf{B}(H)$, the algebra of all bounded operators on a Hilbert space H, and the set $\{a_n\}$ consists of finite rank projections, then $L \subset P(A)$. In Theorem 4.2 we generalize this result to the context of the multiplier algebra $M(A)$ of a non-unital, σ-unital C^*-algebra A with the sequence $\{a_n\} \subset A$ "tending to infinity" rapidly enough. If we assume the Continuum Hypothesis, assume that A isn't too

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large and find some way to form infinite sums (for example, if we assume A is a separably represented von Neumann algebra), then we prove in Theorem 3.6 that $L \cap P(A)$ is non-void, but we can neither prove nor disprove the conjecture that $L \subset P(A)$ in this generality. Several applications of these results appear in [**2**, Section 2].

In order to deal with these questions we introduce in Section 3 the concept of an l^{∞}-embedding of a family $\{b_{\alpha}\}_{\alpha \in I}$ of mutually orthogonal, positive, norm one elements of A. Essentially this means that sums of the form

$$\sum_{\alpha \in I} (b_{\alpha}^{1/2} a_{\alpha} b_{\alpha}^{1/2})$$

make sense in A for any bounded family $\{a_{\alpha}\}_{\alpha \in I}$. This abstraction allows us to handle the case in which A is a von Neumann algebra at the same time as the case in which A is the multiplier algebra $M(B)$ of a non-unital C^*-algebra B. A more detailed discussion of the latter more complicated case appears in Section 4. Also included in Section 4 are a few results relating maximal abelian C^*-subalgebras (MASA's) of a C^*-algebra A to certain MASA's of $M(A)$.

1. Notation and preliminaries. Generally we follow the notation of [**10**]. The letters A and B will always denote C^*-algebras with elements $a, b, c, d, e, p, q, r, s, u, v, w, x, y$. The letters f, g, h will denote generic elements of A^*, the dual space of A. We shall frequently consider A as canonically embedded in its double dual A^{**}, identified with the weak closure of A in its universal representation (see [**10**, p. 60]). For any elements $a, b, c \in A$ and $f \in A^*$ define $(afb) \in A^*$ by

$$(afb)(c) = f(acb).$$

Let $S(A)$ denote the state space of A, $Q(A)$ the quasi-state space of A and $P(A)$ the pure state space of A. Convergence in A^* will default to weak* convergence, while the default convergence in A^{**} is strong*. The letter z will be reserved for the central projection in A^{**} covering the reduced atomic representation of A (see [**10**, p. 103]). Any $g \in Q(A)$ with $g(z) = g(1)$ is called *atomic* while any $f \in Q(A)$ with $f(z) = 0$ is called *diffuse*.

Let $Q_{at}(A)$ denote the set $\{g \in Q(A) \ : \ g(z) = g(1)\}$. Each f in $Q(A)$ is a normal state on the von Neumann algebra A^{**} and, as such, has a support projection $p = \text{supp}(f)$ in A^{**} such that $f(1 - p) = 0$ and $f|_{pA^{**}p}$ is faithful (see [**13**, p. 31]). If $f \in P(A)$, supp(f) is a rank one projection in A^{**} [**10**, p. 87]. By the Schwarz inequality, if $a \geqq 0$ with $f(a) = 1 = \|a\|$, then $(afa) = f$; so, in particular, if $p = \text{supp}(f)$, $pfp = f$. We let $\mathbf{B}(H)$ denote the algebra of all bounded operators on the Hilbert space H with inner product $\langle , \rangle$ and we let Tr denote the canonical trace on $\mathbf{B}(H)$ (see [**13**], p. 26]). For b in $\mathbf{B}(H)$, let

$$|b| = (b^*b)^{1/2} \quad \text{and} \quad \|b\|_1 = \mathrm{Tr}(\,|b|\,).$$

Then any normal, bounded linear functional f on $\mathbf{B}(H)$ has the form $f(x) = \mathrm{Tr}(bx)$ for some b in $\mathbf{B}(H)$ with $\|b\|_1 = \|f\|$. (see [**13**, p. 38]).

LEMMA 1.1. *Fix f in $S(A)$ and x in A with $f(x^*x) > 0$. Put*

$$\epsilon = |f(x)|f(x^*x)^{-1/2}$$

and consider the state $g = f(x^*x)^{-1}(x^*fx)$. Then

$$\|g - f\| \leqq 2(1 - \epsilon^2)^{1/2}.$$

If f is pure, then so is g.

Proof. Let (π, H, ξ) be the cyclic representation of A associated with f via the GNS construction [**10**, p. 46]. Then

$$g(a) = \langle \pi(a)\eta, \eta \rangle, \quad \text{where } \eta = \|\pi(x)\xi\|^{-1}\pi(x)\xi.$$

Changing η with a phase factor we may assume that $\langle \eta, \xi \rangle = \epsilon$. Clearly the distance $\|g - f\|$ is dominated by the distance between the vector functionals ω_ξ, ω_n on $\mathbf{B}(H)$ given by ξ and η (see [**13**, pp. 36-38]); but this distance only depends on operators in the 2-dimensional subspace H_2 spanned by ξ and η. Writing $\eta = \epsilon\xi + \delta\xi^\perp$ where $\delta = (1 - \epsilon^2)^{1/2}$ and $\xi^\perp$ is orthogonal to ξ, the density matrices in $\mathbf{B}(H_2)$ for ω_ξ and ω_η are given by

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} \epsilon^2 & \epsilon\delta \\ \epsilon\delta & \delta^2 \end{pmatrix}$$

respectively.
An easy computation shows that the eigenvalues for the matrix $p - q$ are $\pm(1 - \epsilon^2)^{1/2}$. Consequently

$$\|g - f\| \leqq \|p - q\|_1 = \mathrm{Tr}(\,|p - q|\,) = 2(1 - \epsilon^2)^{1/2}.$$

If f is pure, then G is also pure because by [**10**, 2.7.5] there is a unitary $u \in \tilde{A}$ with $ugu^* = f$.

LEMMA 1.2. *If $f \in P(A)$ with* $\mathrm{supp}(f) = p$ *and $g \in S(A)$ with $g(p) \geqq \epsilon$, then*

$$\|f - g\| \leqq 2(1 - \epsilon)^{1/2}.$$

Proof. Passing to the universal representation of A on the Hilbert space H_u, we find a trace class operator $h \geqq 0$ on H_u such that

$$g(x) = \mathrm{Tr}(xh) \quad \text{for all } x \text{ in } A.$$

Let $h = \sum \lambda_n q_n$ be a resolution of h in terms of orthogonal minimal projections q_n, so that $\sum \lambda_n = 1$, and write

$$pq_np = \epsilon_n p \quad \text{for some } \epsilon_n \geqq 0.$$

The assumption $g(p) \geqq \epsilon$ implies that $php \geqq \epsilon p$, whence $\sum \lambda_n \epsilon_n \geqq \epsilon$. Furthermore, with $\|a\|_1 = \mathrm{Tr}|a|$, we have

$$(*) \quad \|g - f\| \leqq \|h - p\|_1 \leqq \sum \lambda_n \|q_n - p\|_1.$$

As in Lemma 1.1 the matrix for $q_n - p$, where

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q_n = \begin{pmatrix} \epsilon_n & \bar{\delta}_n \\ \delta_n & 1 - \epsilon_n \end{pmatrix},$$

with

$$|\delta_n|^2 = \epsilon_n(1 - \epsilon_n),$$

has the eigenvalues $\pm(1 - \epsilon_n)^{1/2}$. Thus

$$(**) \quad \sum \lambda_n \|q_n - p\|_1 = \sum \lambda_n 2(1 - \epsilon_n)^{1/2} \leqq 2(\sum \lambda_n(1 - \epsilon_n))^{1/2}$$
$$= 2(1 - \sum \lambda_n \epsilon_n)^{1/2} \leqq 2(1 - \epsilon)^{1/2},$$

using the fact that $t \to t^2$ is a convex function [**12**, p. 63]. Combine (*) and (**), and we get the result.

LEMMA 1.3. *If $f, g \in S(A)$ with f diffuse and g atomic, then* $\|f - g\| = 2$.

Proof. By the hypothesis,

$$f(1 - z) = 1 = g(z), \quad \text{and} \quad 2 \geqq \|f - g\|.$$

Since $z = z^* = z^2$, $\|1 - 2z\| = 1$. Thus $(f - g)(1 - 2z) = 2$ implies $\|f - g\| = 2$.

2. Excising states.

Definition 2.1. *A* net $\{a_\alpha\}$ of positive, norm one elements of A *excises* $f \in S(A)$ if

$$\lim \|a_\alpha a a_\alpha - f(a)a_\alpha^2\| = 0 \quad \text{for every } a \text{ in } A.$$

First we develop some elementary properties of excising nets.

PROPOSITION 2.2. *Every pure state g of A is excised by a decreasing net $\{x_\lambda : \lambda \in \Lambda\}$ such that $g(x_\lambda) = 1$ for every λ in Λ. Moreover, for each element d in A_+ with $g(d) = \|d\| = 1$, the elements of the net can be chosen as*

$$x_\lambda = d^{1/2}(1 - u_\lambda)d^{1/2},$$

where $\{u_\lambda : \lambda \in \Lambda\}$ is an (increasing) approximate unit for the hereditary kernel N of g. Finally, if N is σ-unital, in particular if A is separable, the net can be chosen to be a sequence of mutually commuting elements.

Proof. Since g is pure, we can use Kadison's transitivity theorem to find d in A_+ with $g(d) = \|d\| = 1$, cf. [**10**, 2.7.5] (or if d is given, use it). The left kernel of g is

$$L = \{x \in A \quad : \quad g(|x|) = 0\}$$

and the hereditary kernel is $N = L \cap L^*$. Choose by [**10**, 1.4.2] an approximate unit $\{u_\lambda : \lambda \in \Lambda\}$ for N and put

$$x_\lambda = d^{1/2}(1 - u_\lambda)d^{1/2}.$$

Note that the net $\{x_\lambda\}$ is decreasing, majorized by d and satisfies $g(x_\lambda) = 1$ for all λ.

If $x \in A$ then

$$g(d^{1/2}(x - g(x))d^{1/2}) = 0.$$

Since $\ker g = L + L^*$ by [**10**, 3.13.6], we have

$$d^{1/2}(x - g(x))d^{1/2} = a + b^*$$

for some a, b in L. But then

$$\begin{aligned} \|x_\lambda(x - g(x))x_\lambda\| &\leqq \|(1 - u_\lambda)(a + b^*)(1 - u_\lambda)\| \\ &\leqq \|a(1 - u_\lambda)\| + \|(1 - u_\lambda)b^*\| \\ &= \|a(1 - u_\lambda)\| + \|b(1 - u_\lambda)\| \to 0. \end{aligned}$$

If N is σ-unital with a strictly positive element e of norm one, we put

$$y = d^{1/2}(1 - e)d^{1/2}$$

and define $x_n = y\phi_n(y)$ for some decreasing sequence $\{\phi_n\}$ of continuous functions on $[0, 1]$ for which $\phi_n(1) = 1$ but $\phi_n(t) \to 0$ for every $t < 1$. Set

$$w = (1 - e)^{1/2}d(1 - e)^{1/2}.$$

The formula

$$(1 - e)^{1/2}d^{1/2}y^n = w^n(1 - e)^{1/2}d^{1/2}$$

in conjunction with the Weierstrass approximation theorem shows that

$$(1 - e)^{1/2}d^{1/2}\phi(y) = \phi(w)(1 - e)^{1/2}d^{1/2}$$

for every continuous function ϕ on $\mathbf{R}_+$. With our choice of functions this implies that

$$x_n = d^{1/2}(1 - e)^{1/2}\phi_n(w)(1 - e)^{1/2}d^{1/2}.$$

If $x \in A$ then

$$d^{1/2}(x - g(x))d^{1/2} = a + b^*$$

as above, with a and b in L. Since Ae is dense in L, we may assume without loss of generality that $a = ce$ and $b = ve$ for some c, v in A. Thus

$$\begin{aligned}&\|x_n(x - g(x))x_n\| \\ &\leqq \|\phi_n(w)(1 - e)^{1/2}(ce + ev^*)(1 - e)^{1/2}\phi_n(w)\| \\ &\leqq \|\phi_n(w)e\|\,\|v\| + \|e\phi_n(w)\|\,\|c\|.\end{aligned}$$

Since $e + w \leqq 1$ it follows that

$$\begin{aligned}\|\phi_n(w)e\| &\leqq \|\phi_n(w)e^{1/2}\| \leqq \|\phi_n(w)(1 - w)^{1/2}\| \\ &\leqq \sup_{0 \leqq t \leqq 1} \phi_n(t)(1 - t)^{1/2} \to 0,\end{aligned}$$

and we conclude that $\{x_n\}$ excises g, as desired.

PROPOSITION 2.3. *A state g of A is excised by some net $\{x_\lambda:\lambda \in \Lambda\}$ if and only if $g \in P(A)^-$. If g is not pure, then $\{x_\lambda:\lambda \in \Lambda\}$ has no non-zero cluster points in A^{**} for the strong operator topology.*

Proof. If $g \in P(A)^-$, take $\lambda = (x_1, \dots, x_n)$ in A and $\epsilon = n^{-1}$. By assumption there is a pure state f of A such that

$$|g(x_m) - f(x_m)| < \frac{1}{2}\epsilon \quad \text{for all } x_m \text{ in } \lambda,$$

and by Proposition 2.2 we can find x_λ in A_+ with $\|x_\lambda\| = 1$ (and $f(x_\lambda) = 1$) such that

$$\|x_\lambda(x_m - f(x_m))x_\lambda\| < \frac{1}{2}\epsilon \quad \text{for all } x_m \text{ in } \lambda.$$

It follows that

$$\|x_\lambda(x_m - g(x_m))x_\lambda\| < \epsilon,$$

so that g is excised by $\{x_\lambda\}$.

Conversely, if the net $\{x_\lambda\}$ excises the state g, we choose for each λ a pure state g_λ of A such that $g_\lambda(x_\lambda) = 1$, cf. [**10**, 4.3.10]. Adjoining if necessary a unit to A and extending all states in the canonical manner, we see that the net $\{x_\lambda\}$ still excises g on the enlarged C^*-algebra. Assuming therefore that A is unital we have, for each x in A,

$$\begin{aligned}|g_\lambda(x) - g(x)| &= |g_\lambda(x - g(x)1)| \\ &= |g_\lambda(x_\lambda(x - g(x))x_\lambda)| \leqq \|x_\lambda(x - g(x))x_\lambda\|,\end{aligned}$$

which shows that the net $\{g_\lambda\}$ converges weak* to g.

If e is a non-zero strong limit point of a net $\{x_\lambda\}$ that excises the state g, then $0 \leqq e \leqq 1$. Since the norm is strongly lower semi-continuous, we have, for every x in A,

$$\|e(x - g(x))e\| \leqq \liminf \|x_\lambda(x - g(x))x_\lambda\| = 0.$$

As A is strongly dense in A^{**} and g is strongly continuous on A^{**}, it follows that $exe = g(x)e^2$ for every x in A^{**}. Consequently $g(e)^{-1}e$ is a minimal projection supporting g, so g is pure by [**10**, 3.13.6].

Remark. A net that excises a non-pure state need not converge weakly to zero. In fact, in the Fermion algebra [**10**, 6.4] we can find two inequivalent pure states g_1 and g_2 and a sequence $\{p_n\}$ of projections that excises the state $g = \frac{1}{2}(g_1 + g_2)$, but for which

$$g_1(p_n) = g_2(p_n) = \frac{1}{2} \quad \text{for all } n.$$

If g is diffuse, the situation is different; see [**2**, Corollary 2.15].

PROPOSITION 2.4. *If $f \in S(A)$, the net $\{x_\alpha\}$ excises f, and $\{f_\alpha\} \subset Q(A)$ is a similarly indexed net with*

$$\lim(x_\alpha f_\alpha x_\alpha) = g \quad \textit{in } Q(A) \quad \textit{and} \quad \lim \|x_\alpha f_\alpha x_\alpha\| = \lambda,$$

then $g = \lambda f$.

Proof. Since

$$\lambda = \lim \|x_\alpha f_\alpha x_\alpha\| = \lim f_\alpha(x_\alpha^2),$$

then for every a in A,

$$\begin{aligned} g(a) &= \lim f_\alpha(x_\alpha a x_\alpha) \\ &= \lim f_\alpha(x_\alpha(a - f(a))x_\alpha) + \lim f_\alpha(x_\alpha f(a) x_\alpha) \\ &= 0 + f(a)\lambda = \lambda f(a). \end{aligned}$$

3. l^∞-embeddings.

Notation. Throughout Section 3 we shall assume that A is unital.

Definition 3.1. We say that a family $\{b_\alpha\}_{\alpha \in I}$ of mutually orthogonal, positive, norm one elements in A is *l^∞-embedded* if there is a positive, linear map Ψ from the direct product C^*-algebra $\prod A_\alpha$ (each A_α being isomorphic to A) into A, such that

$$\Psi(\{x_\alpha\})\Psi(\{y_\alpha\}) = \Psi(\{x_\alpha b_\alpha y_\alpha\})$$

for all elements $\{x_\alpha\}$ and $\{y_\alpha\}$ in $\prod A_\alpha$, and such that

$$\Psi(\{x_\alpha\}) = \sum b_\alpha^{1/2} x_\alpha b_\alpha^{1/2} \quad \text{if } x_\alpha = 0$$

for all but finitely many α's. In the applications A will either be a von Neumann algebra with

$$\Psi(\{x_\alpha\}) = \sum b_\alpha^{1/2} x_\alpha b_\alpha^{1/2}$$

(strong* convergence), or A will be the multiplier algebra of a C^*-algebra with

$$\Psi(\{x_\alpha\}) = \sum b_\alpha^{1/2} x_\alpha b_\alpha^{1/2}$$

(strict convergence). For this reason we will often write

$$\Psi(\{x_\alpha\}) = \sum b_\alpha^{1/2} x_\alpha b_\alpha^{1/2}$$

to help the intuition.

If $\{b_\alpha\}$ is l^∞-embedded in A and σ is a subset of I with characteristic function χ_σ, we write

$$b_\sigma = \Psi(\{\chi_\sigma(\alpha)\}) = \sum_{\alpha\in\sigma} b_a.$$

Finally we say that $\{b_\alpha\}$ *supports* a family $\{f_\alpha\}_{\alpha\in I}$ of states of A if $f_\alpha(b_\alpha) = 1$ for each α in I (whence $f_\alpha(b_\beta) = 0$ for $\alpha \neq \beta$). Note that l^∞-embedding of $\{b_\alpha\}$ implies that

$$f_\beta(\Psi(\{x_\alpha\})) = f_\beta(x_\beta)$$

for every element $\{x_\alpha\}$ in $\prod A_\alpha$, because

$$f_\beta(\Psi(\{x_\alpha\})) = f_\beta(b_\beta \Psi(\{x_\alpha\})) = f_\beta(\Psi(\{\chi_{\{\beta\}}(\alpha) b_\alpha x_\alpha\}))$$
$$= f_\beta(b_\beta^{3/2} x_\beta b_\beta^{1/2}) = f_\beta(x_\beta).$$

We shall be interested in the limit points of the set $\{f_\alpha\}$ in $S(A)$. Are they pure? Here are two examples related to this question.

Examples 3.2. (A) (See [**4**, Section 1].) In the Fermion algebra we fix a diagonal algebra D, and take a sequence $\{p_n\}$ of orthogonal projections in D that excise a pure state f of D. If $\{f_n\}$ is any sequence of states supported by $\{p_n\}$, then every weak* limit point g of $\{f_n\}$ is pure. Indeed, $g|_D = f$, and since f has a unique state extension, which is pure, it follows that g is pure (and $f_n \to g$).

(B) Let $\{p_n\}$ be an orthogonal sequence of projections of finite rank in $\mathbf{B}(H)$. Clearly $\{p_n\}$ is l^∞-embedded in $\mathbf{B}(H)$. If $\{f_n\}$ is any sequence of pure states of $\mathbf{B}(H)$ supported by $\{p_n\}$, then every weak* limit point of $\{f_n\}$ is pure by [**6**, Theorem 1].

Write $\beta(I)$ for the set of ultrafilters of subsets of I. If $\{f_\alpha\}_{\alpha\in I}$ is a family of states on A and $\mathcal{U}$ is in $\beta(I)$, define a state

$$f_{\mathcal{U}} = \lim_{\mathcal{U}} f_\alpha$$

by the formula

$$f_{\mathcal{U}}(x) = \bigcap_{\sigma\in\mathcal{U}} \{f_\alpha(x):\alpha \in \sigma\}^-.$$

It is easy to see that $f_{\mathscr{U}}$ is in the weak*-closure of $\{f_\alpha\}_{\alpha \in I}$. In the l^∞-embedded case the converse is also true.

PROPOSITION 3.3. *If $\{b_\alpha\}_{\alpha \in I}$ is an l^∞-embedded family in A supporting the family $\{f_\alpha\}_{\alpha \in I}$ of states on A, and f is a weak* limit point of $\{f_\alpha\}_{\alpha \in I}$, then there is a unique ultrafilter $\mathscr{U}$ in $\beta(I)$ such that*

$$f = f_{\mathscr{U}}.$$

Proof. Using the notation of Definition 3.1 we set

$$\mathscr{U} = \{\sigma \subset I : f(b_\sigma) = 1\}.$$

We claim that $\mathscr{U}$ is an ultrafilter. Indeed, note first that since $b_\alpha b_\sigma = b_\alpha^2$ if $\alpha \in \sigma$ and $b_\alpha b_\sigma = 0$ if $\alpha \notin \sigma$, we must have $f(b_\sigma)$ equal to 0 or 1 for every subset σ. In particular we have $f(b_I) = 1$. Since

$$b_I = b_\sigma + b_{I \setminus \sigma},$$

it follows that, if σ is not in $\mathscr{U}$, then $I \setminus \sigma$ must be in $\mathscr{U}$. If $\sigma \in \mathscr{U}$ and $\sigma \subset \tau$, then

$$b_\sigma \leqq b_\tau \leqq b_I,$$

and therefore $f(b_\tau) = 1$ and $\tau \in \mathscr{U}$. Since $b_{\sigma \cap \tau} \geqq b_\tau b_\sigma b_\tau$, we see that $\mathscr{U}$ is closed under intersections, so $\mathscr{U}$ is an ultrafilter. To see that $f = f_{\mathscr{U}}$ fix x in A with $f(x) \neq 0$ and $\epsilon > 0$. We claim that

$$\sigma = \{\alpha \in I : |f_\alpha(x) - f(x)| < \epsilon\}$$

is in $\mathscr{U}$. Otherwise, we would have

$$f(x) = f(b_{I \setminus \sigma} x b_{I \setminus \sigma}),$$

and therefore we could approximate $f(x)$ arbitrarily well by $f_\alpha(x)$'s with α in $I \setminus \sigma$ and conclude that

$$|f(x) - f(x)| \geqq \epsilon.$$

If $\mathscr{V} \in \beta(I)$ with $\sigma \in \mathscr{U} \setminus \mathscr{V}$, then

$$f(b_\sigma) - f_{\mathscr{V}}(b_\sigma) = f_{\mathscr{U}}(b_\sigma) - f_{\mathscr{V}}(b_\sigma) = 1.$$

Thus $\mathscr{U}$ is unique.

In what follows we assume that $\{b_\alpha : \alpha \in I\}$ is an l^∞-embedded family supporting a family $\{f_\alpha : \alpha \in I\}$ of pure states of A. The optimal conclusion from our point of view is that every weak* limit point of $\{f_\alpha\}$ is a pure state of A. This may very well be the case, but we can only prove it under certain "normality" conditions on the f_α's (Theorem 4.2). We can, however, show that at least some of the weak* limit points are pure, by selecting the ultrafilter corresponding to the limit point very carefully (Theorem 3.7). To be sure that this selection is possible we need to assume the Continuum Hypothesis (actually less would do but our construction cannot be carried out without some extra set-theoretic axiom). We also need a restriction on the size of A.

Let f be a weak* limit point of the family $\{f_\alpha\}$. We say that the associated ultrafilter $\mathscr{U}$ (see Proposition 3.3) in $\beta(I)$ is *good* for $\{f_\alpha\}$ if, for each x in A and $\epsilon > 0$ there is a set $\sigma = \sigma(x, \epsilon)$ in $\mathscr{U}$ such that, for each β in σ, there is a finite subset $\theta = \theta(x, \epsilon, \beta)$ of σ satisfying

$$f_\beta(x^*b_\sigma x) - f_\beta(x^*b_\theta x) < \epsilon.$$

Note that if $\{f_\alpha\}$ consists of normal states on a von Neumann algebra M and $b_\sigma = \sum_{\alpha\in\sigma} b_\alpha$, then

$$f_\beta(x^*b_\sigma x) = \sup\{f_\beta(x^*b_\theta x):\theta \subset \sigma \text{ and } \theta \text{ is finite}\},$$

so every ultrafilter is good for $\{f_\alpha\}$. Also, if $\{f_\alpha\}$ (not necessarily normal) is supported by a family $\{p_\alpha\}$ of orthogonal central projections in M satisfying $p_\alpha b_\beta = 0$ if $\alpha \neq \beta$, then

$$f_\beta(x^*b_\sigma x) = f_\beta(p_\beta x^*b_\sigma x) = f_\beta(x^*b_\beta x),$$

so again all ultrafilters are good for $\{f_\alpha\}$.

In the proof of the next theorem we need a combinatorial result [**6**, Theorem 2] which is restated here for convenience as Lemma 3.4.

LEMMA 3.4. *If $\{t_{\alpha\beta}\}_{\alpha,\beta\in I}$ is a set of non-negative numbers such that $t_{\alpha\alpha} = 0$ for all α in I and $\sum_{\alpha\in I} t_{\alpha\beta} < \infty$ for each β in I, then there is a partition $\{\sigma_1, \sigma_2, \sigma_3\}$ of I such that for each $\beta \in \sigma_i$, $i = 1, 2, 3$, we have*

$$\sum_{\alpha\in\sigma_i} t_{\alpha\beta} \leqq \frac{2}{3} \sum_{\alpha\in I} t_{\alpha\beta}.$$

THEOREM 3.5. *If $\{b_\alpha\}_{\alpha\in I}$ is an l^∞-embedded family in A supporting the pure states $\{f_\alpha\}_{\alpha\in I}$ and f is a weak* limit point of $\{f_\alpha\}_{\alpha\in I}$ such that the associated ultrafilter $\mathscr{U}$ (see Proposition 3.3) is good for $\{f_\alpha\}_{\alpha\in I}$, then $f \in P(A)$.*

Proof. Let (π, H, ξ) be the GNS representation of A associated with f [**10**, Section 3.3]. We shall show that π is irreducible, whence $f \in P(A)$ by [**10**, 3.13.2].

Applying Proposition 2.2 we choose for each α in I a decreasing net $\{x_{\alpha\gamma}:\gamma \in \Gamma_\alpha\}$ of positive, norm one elements of A, such that the net

$$\{b_\alpha^{1/2} x_{\alpha\gamma} b_\alpha^{1/2}:\gamma \in \Gamma_\alpha\}$$

excises f_α. Write

$$\Lambda = \prod_{\alpha\in I} \Gamma_\alpha,$$

and give Λ the product (partial) ordering. Since $\{b_\alpha\}$ is l^∞-embedded in A, we have, for each subset σ of I and each λ in Λ, an element

$$x_{\sigma\lambda} = \Psi(\{\chi_\sigma(\alpha)x_{\alpha\lambda(\alpha)}\}_{\alpha\in I}) = \sum_{\alpha\in\sigma} b_\alpha^{1/2}x_{\alpha\lambda(\alpha)}b_\alpha^{1/2},$$

using the notation of Definition 3.1. If $\mathcal{U} \times \Lambda$ is given the product ordering, then

$$\{x_{\sigma\lambda}:(\sigma, \lambda) \in \mathcal{U} \times \Lambda\}$$

is a decreasing net, and therefore the image net $\{\pi(x_{\sigma\lambda})\}$ converges strongly to a positive operator p in $\pi(A)''$. Since $f(x_{\sigma\lambda}) = 1$ for every (σ, λ) in $\mathcal{U} \times \Lambda$ we know that

(1) $\quad p\xi = \xi.$

To establish the irreducibility of π it suffices to show

(2) $\quad p\pi(x)\xi = f(x)\xi$

for each x in A. Indeed (1) and (2) imply that p is the rank one projection onto the span of ξ. If $y \in \pi(A)'$, then because $p \in \pi(A)''$,

$$y\xi = yp\xi = py\xi,$$

and so $y\xi = t\xi$ for some scalar t. Since ξ is a separating vector for $\pi(A)'$ [**10**, p. 32], we get that $y = tI$, so π is irreducible [**10**, 3.13.2].

To prove (2) it is enough to show

(3) $\quad p\pi(y)\xi = 0 \quad$ for y in A_0,

where A_0 consists of those y in A such that, for some σ in $\mathcal{U}$,

$$f_\beta(y) = 0 \quad \text{for all } \beta \text{ in } \sigma.$$

To see this, fix x in A and $\epsilon > 0$. Write

$$\sigma = \{\alpha: |f_\alpha(x) - f(x)| < \epsilon\} \quad \text{and}$$

$$y = x - f(x)1 + \Psi(\{\chi_\sigma(\alpha)(f(x) - f_\alpha(x))\}_{\alpha\in I}).$$

As in the proof of Proposition 3.3 we see that $\sigma \in \mathcal{U}$. Also for β in σ

$$f_\beta(y) = f_\beta(x) - f(x) + (f(x) - f_\beta(x))f_\beta(b_\beta) = 0,$$

so that $y \in A_0$. By (1) and (3)

$$\begin{aligned}\|p\pi(x)\xi - f(x)\xi\| &= \|p(\pi(x) - \pi(y) - f(x)1)\xi\| \\ &\leqq \|x - y - f(x)1\| \\ &= \sup\{(f_\beta(x) - f(x)):\beta \in \sigma\} \\ &\leqq \epsilon.\end{aligned}$$

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As ϵ and x were arbitrary, (2) follows.

To prove (3) fix $\epsilon > 0$ and y in A_0 with $\|y\| = 1$. By assumption there is a set σ_0 in $\mathcal{U}$ with

(4) $\quad f_\beta(y) = 0 \quad$ for all β in σ_0.

Since $\mathcal{U}$ is good for $\{f_\alpha\}_{\alpha \in I}$, we may select σ_1 in $\mathcal{U}$ so that for each β in σ_1 there is a finite set $\theta(\beta)$ with

(5) $\quad (y^* f_\beta y)(b_{\sigma_1 \setminus \theta(\beta)}) < \epsilon.$

Next write $\phi(\beta) = \theta(\beta) \setminus \{\beta\}$ and, for α, β in I,

$$(6) \qquad t_{\alpha\beta} = \begin{cases} f_\beta(y^* b_\alpha y) \text{ if } \alpha \in \phi(\beta) \\ 0 \text{ otherwise.} \end{cases}$$

Choose an integer n so that $(2/3)^n < \epsilon$. Since $t_{\alpha\alpha} = 0$ for all α, we may apply Lemma 3.4 n times to obtain a partition $\{\tau_1, \ldots, \tau_p\}$, $p = 3^n$, of I such that, if $\beta \in \tau_m$ for some m in $\{1, 2, \ldots, q\}$, then

$$(7) \qquad \sum_{\alpha \in \tau_m} t_{\alpha\beta} \leqq (2/3)^n \sum_{\alpha \in I} t_{\alpha\beta} < \epsilon \sum_{\alpha \in I} t_{\alpha\beta}.$$

Since $\mathcal{U}$ is an ultrafilter, exactly one of the τ_m's is in $\mathcal{U}$; call it σ_2. If β is in σ_2, then by (6) and (7) we get

$$(8) \qquad \sum_{\alpha \in \sigma_2} t_{\alpha\beta} = \Sigma\{t_{\alpha\beta} : \alpha \in \sigma_2 \cap \phi(\beta)\}$$

$$< \epsilon \Sigma \{t_{\alpha\beta} : \alpha \in \phi(\beta)\} = \epsilon f_\beta(y^* b_{\phi(\beta)} y).$$

The last equality is true because $\phi(\beta)$ is finite. Set

$$\sigma = \sigma_0 \cap \sigma_1 \cap \sigma_2$$

and take λ_0 in Λ such that, for β in σ and $\lambda \geqq \lambda_0$,

(9) $\quad \|b_\beta^{1/2} x_{\beta\lambda(\beta)} b_\beta^{1/2} y b_\beta^{1/2} x_{\beta\lambda(\beta)} b_\beta^{1/2}\| < \epsilon.$

This choice is possible because of (4) and the fact that each net

$$\{b_\alpha^{1/2} x_{\alpha\lambda(\alpha)} b_\alpha^{1/2}\}$$

excises f_α. With these selections we have for $\lambda \geqq \lambda_0$,

$$(10) \qquad \|p\pi(y)\xi\|^2 = f(y^* p y) \leqq f(y^* x_{\sigma\lambda} y)$$

$$= \lim_{\mathcal{U}} f_\alpha(y^* x_{\sigma\lambda} y) \leqq \sup_{\beta \in \sigma} f_\beta(y^* x_{\sigma\lambda} y)$$

$$= \sup_{\beta \in \sigma} (y^* f_\beta y)(b_\beta^{1/2} x_{\beta\lambda(\beta)} b_\beta^{1/2} + x_{\sigma \setminus \{\beta\}, \lambda}).$$

Now we estimate the two terms within the latter supremum separately. For a fixed β in σ we have for the first term

$$(11) \quad (y^*f_\beta y)(b_\beta^{1/2}x_{\beta\lambda(\beta)}b_\beta^{1/2}) = f_\beta(y^*b_\beta^{1/2}x_{\beta\lambda(\beta)}b_\beta^{1/2}yb_\beta^{1/2}x_{\beta\lambda(\beta)}b_\beta^{1/2}) \leqq \|y\| \, \|b_\beta^{1/2}x_{\beta\lambda(\beta)}b_\beta^{1/2}yb_\beta^{1/2}x_{\beta\lambda(\beta)}b_\beta^{1/2}\| < \epsilon$$

by (9) and the fact that

$$f_\beta(b_\beta^{1/2}x_{\beta\lambda(\beta)}b_\beta^{1/2}) = 1.$$

For the second we have, because $x_{\sigma\lambda} \leqq b_\sigma$ for every σ,

$$\begin{aligned}(12) \quad (y^*f_\beta y)(x_{\sigma\setminus\{\beta\},\lambda}) &\leqq (y^*f_\beta y)(b_{\sigma\setminus\{\beta\}}) \\ &= (y^*f_\beta y)(b_{\sigma\setminus(\theta(\beta)\cup\{\beta\})} + b_{(\sigma\cap\theta(\beta))\setminus\{\beta\}}) \\ &= (y^*f_\beta y)(b_{\sigma\setminus(\theta(\beta)\cup\{\beta\})}) + (y^*f_\beta y)(b_{\sigma\cap\phi(\beta)}) \\ &\leqq (y^*f_\beta y)(b_{\sigma\setminus\theta(\beta)}) + (y^*f_\beta y)(b_{\sigma\cap\theta(\beta)}) \\ &< \epsilon + (y^*f_\beta y)(b_{\sigma\cap\phi(\beta)}),\end{aligned}$$

using the facts $\phi(\beta) = \theta(\beta)\setminus\{\beta\}$, $\sigma \subset \sigma_1$, and (5).

Combining (10), (11), and (12) we conclude that

$$(13) \quad \|p\pi(y)\xi\|^2 \leqq \epsilon + \epsilon + \sup_{\beta\in\sigma}(y^*f_\beta y)(b_{\sigma\cap\phi(\beta)}).$$

For the remaining term, if $\beta \in \sigma$ we have

$$\begin{aligned}(14) \quad (y^*f_\beta y)(b_{\sigma\cap\phi(\beta)}) &= \sum\{t_{\alpha\beta}:\alpha \in \sigma \cap \phi(\beta)\} \\ &\leqq \sum\{t_{\alpha\beta}:\alpha \in \sigma_2 \cap \phi(\beta)\} \\ &< \epsilon f_\beta(y^*b_{\phi(\beta)}y) \leqq \epsilon\end{aligned}$$

using (8) and the fact that $\sigma \subset \sigma_2$. Hence

$$\|p\pi(y)\xi\|^2 \leqq 3\epsilon$$

by (13) and (14). As ϵ was arbitrary, $p\pi(y)\xi = 0$, so (3) follows and the theorem is proved.

THEOREM 3.6. *Assume the Continuum Hypothesis and assume that A has the cardinality of the continuum. If $\{b_n\}$ is an l^∞-embedded sequence in A supporting the pure states $\{f_n\}$, then (in the notation preceeding Lemma 3.4) there is an ultrafilter on* **N** *which is good for $\{f_n\}$. Consequently some weak* limit points of the set $\{f_n\}$ are pure.*

Proof. The proof is broken up into several steps.

Step 1. We shall say that an infinite subset $\sigma = \{n_1, n_2, \dots\}$ (increasing order) of **N** is *good* for an element x of A if for each k in **N** there is an integer $m = m(k, x)$ such that

$$f_{n_k}(x^*b_\sigma x) - \sum_{i=1}^{m} f_{n_k}(x^*b_{n_i}x) < \frac{1}{n_k}.$$

Our goal is to construct an ultrafilter on $\mathcal{U}$ on $\mathbf{N}$ such that for each x in A there is some σ in $\mathcal{U}$ that is good for x.

Step 2. We show that if τ is an infinite subset of $\mathbf{N}$ and if $a \in A$, then there is an infinite subset σ of τ which is good for a. The proof is by induction. Write

$$\tau = \tau_1 = \{n_{11}, n_{21}, n_{31}, \dots\},$$

increasing order. Set $n_1 = n_{11}$. If $\{\rho_1, \dots, \rho_r\}$ is a partition of $\tau_1\backslash\{n_1\}$, then

$$\sum_{i=1}^{r} f_{n_1}(a^*b_{\rho_i}a) \leqq f(a^*a) \leqq \|a^*a\|,$$

so

$$f_{n_1}(a^*b_{\rho_i}a) \leqq \|a^*a\|/r \quad \text{for some } i.$$

Since r was arbitrary there is some infinite subset τ_2 of $\tau_1\backslash\{n_1\}$ such that

$$f_{n_1}(a^*b_{\tau_2}a) < n_1^{-1}.$$

Now suppose that for some $k > 1$ infinite subsets $\tau_1, \tau_2, \dots, \tau_k$ and integers $n_1 < n_2 < \dots < n_{k-1}$ have been chosen as follows:

(i) $\tau_i = \{n_{1i}, n_{2i}, \dots\}$, increasing order.

(ii) $n_i = n_{1i}$ for $i = 1, \dots, k - 1$.

(iii) $\tau_i \subset \tau_{i-1}\backslash\{n_1, \dots, n_{i-1}\}$.

(iv) $f_{n_i}(a^*b_{\tau_{i+1}}a) < n_i^{-1}$ for $i = 1, \dots, k - 1$.

Set $n_k = n_{1k}$; as above there is an infinite subset τ_{k+1} of $\tau_k\backslash\{n_i\}_{i=1}^{k}$ such that

$$f_{n_k}(a^*b_{\tau_{k+1}}a) < n_k^{-1}.$$

This continues the induction.

Now take $\sigma = \{n_1, n_2, \dots\}$. (By construction $n_i < n_{i+1}$.) Also for each $j > k$ we have

$$n_j \in \tau_j \subset \tau_{j-1} \subset \dots \subset \tau_{k+1},$$

so

$$\sigma_k = \sigma\backslash\{n_1, \dots, n_k\} \subset \tau_{k+1}.$$

Hence for each k,

$$f_{n_k}(a^*b_\sigma a) - \sum_{i=1}^{k} f_{n_k}(a^*b_{n_i}a) = f_{n_k}(a^*b_{\sigma_k}a)$$

$$\leqq f_{n_k}(a^* b_{\tau_{k+1}} a) < n_k^{-1}.$$

Thus σ is good for a.

Step 3. We show that there is a free ultrafilter $\mathcal{U}$ on $\mathbf{N}$ such that for each x in A there is a σ in $\mathcal{U}$ which is good for x. The construction is by transfinite induction. First we well-order A as $\{a_\alpha\}_{\alpha<\omega_1}$, where ω_1 is the first uncountable ordinal. For $\alpha = 1$, use Step 2 to get a subset σ_1 of $\mathbf{N}$ which is good for a_1. Suppose that for some ordinal $\alpha < \omega_1$ and all $\beta < \alpha$ we have found infinite subsets σ_β of $\mathbf{N}$ such that

(i) σ_β is good for a_β;

(ii) if $\gamma < \beta$, then $\sigma_\beta \backslash \sigma_\gamma$ is finite.

As α is countable we may enumerate the β's less than α and write

$$\{\beta:\beta < \alpha\} = \{\beta_1, \beta_2, \dots\}.$$

For $j = 1, 2, \dots$ set

$$\rho_j = \sigma_{\beta_1} \cap \dots \cap \sigma_{\beta j}$$

(if $\alpha = n$ is finite put $\rho_{n+k} = \rho_n$). By (ii) of our hypothesis each ρ_j is infinite, so we may select a strictly increasing sequence $n_1 < n_2 < \dots$ with n_j in ρ_j, $j = 1, 2, \dots$. By Step 2 there is an infinite subset σ_α of $\{n_1, n_2, \dots\}$ that is good for a_α. Since

$$\{n_j, n_{j+1}, \dots\} \subset \beta_j,$$

(ii) holds for σ_α, so the induction proceeds.

Recall that if τ is an infinite subset of $\mathbf{N}$ and we write

$$W(\tau) = \{\mathcal{U} \in \beta\mathbf{N}\backslash\mathbf{N}:\tau \in \mathcal{U}\},$$

then $W(\tau)$ is open and closed in $\beta\mathbf{N}\backslash\mathbf{N}$. From (ii) we get that if $\beta < \alpha$, then

$$W(\sigma_\alpha) \subset W(\sigma_\beta).$$

Hence $\{W(\sigma_\alpha)\}_{\alpha<\omega_1}$ is a decreasing net of compact sets in $\beta\mathbf{N}\backslash\mathbf{N}$, so there is an ultrafilter $\mathcal{U}$ in their intersection. Since $\mathcal{U} \in W(\sigma_\alpha)$ for each α, $\sigma_\alpha \in \mathcal{U}$ for each α, and so by (i) $\mathcal{U}$ has the desired property.

Step 4. To see that $\mathcal{U}$ is good for $\{f_n\}$ take x in A and $\epsilon > 0$. By Step 3 there is a set $\tau = \{n_1, n_2, \dots\}$ (increasing order) in $\mathcal{U}$ such that

$$f_{n_k}\left(x^*\left(b_\tau - \sum_{i=1}^{m_k} b_{n_i}\right)x\right) < 1/n_k,$$

$k = 1, 2, \dots$. If we pick k so that $1/n_k < \epsilon$ and put

$$\sigma = \{n_k, n_{k+1}, \dots\},$$

then $\sigma \in \mathcal{U}$ and for $j \geqq k$,

$$f_{n_j}\left(x^*\left(b_\sigma - \sum_{i=k}^{m_j} b_{n_i}\right)x\right) = f_{n_j}\left(x^*\left(b_\tau - \sum_{i=1}^{m_j} b_{n_i}\right)x\right) < 1/n_j < \epsilon.$$

This is exactly what is required to show that $\mathcal{U}$ is good for $\{f_n\}$.

Remark. Robert Solovay has pointed out to us that the construction we use in the above proof could proceed with an axiom which is strictly weaker than the Continuum Hypothesis (but, nonetheless, is not implied by Zermelo-Fraenkel set theory).

COROLLARY 3.7. *If the Continuum Hypothesis holds and if A is a von Neumann algebra on a separable Hilbert space, then every sequence of mutually orthogonal, positive, norm one elements $\{b_n\}$ is l^∞-embedded. If $\{f_n\}$ is a sequence of pure states supported by $\{b_n\}$, then some limit points of $\{f_n\}$ are pure.*

4. Pure states on the multiplier algebra. In Example 3.2 (B) the sequence $\{p_n\}$ is l^∞-embedded in $\mathbf{B}(H)$, because $\mathbf{B}(H)$ is a von Neumann algebra. If we view $\mathbf{B}(H)$ as the multiplier algebra of $\mathcal{K}$, the algebra of compact operators on H, then $\{p_n\}$ is l^∞-embedded in $M(\mathcal{K})$ because every bounded sequence $\{p_na_np_n\}$ gives a strictly convergent series $\sum p_na_np_n$ in $M(\mathcal{K}) = \mathbf{B}(H)$. This point of view leads to a generalization of Example 3.2 (B) in Proposition 4.1 and Theorem 4.2.

PROPOSITION 4.1. *If $\{b_n\}$ is a sequence of mutually orthogonal, positive, norm one elements in A, then $\{b_n\}$ is l^∞-embedded in $M(A)$ if the sum $\sum b_n$ is strictly convergent. Thus if A is σ-unital and b is a strictly positive element, we have an l^∞-embedding if the sum $\sum b_nb$ is norm convergent. This holds in particular if $b_nb = bb_n$ for all n and $\|b_nb\| \to 0$.*

Proof. Define $\Psi:\prod A_n \to A^{**}$ by

$$\Psi(\{x_n\}) = \sum b_n^{1/2}x_nb_n^{1/2},$$

where $A_n = M(A)$ and $\{x_n\}$ is an element in $\prod A_n$. Note that the sum is strong* convergent, since the summands are mutually orthogonal. We must show that

$$\sum b_n^{1/2}x_nb_n^{1/2} \in M(A)$$

which we may identify with the strict closure of A in A^{**}. By assumption $\sum b_n \in M(A)$ which means that $\sum b_na$ converges in norm for every a in A. (Since $b_n = b_n^*$ we need not consider the sums $\sum ab_n$.) If $s = \sup\|x_n\|$ we estimate

$$\left\|\sum_{n>m} b_n^{1/2}x_nb_n^{1/2}a\right\| = \left\|\left(\sum_{n>m} b_n^{1/2}x_nb_n^{1/4}\right)\left(\sum_{n>m} b_n^{1/4}a\right)\right\|$$

$$\leqq s\left|\left|\sum_{n>m} b_n^{1/4}a\right|\right| = s\left|\left|\sum_{n>m} a^*b_n^{1/2}a\right|\right|^{1/2}$$

$$\leqq s||a||^{1/2}\left|\left|\sum_{n>m} b_n^{1/2}a\right|\right|^{1/2}$$

$$= s||a||^{1/2}\left|\left|\sum_{n>m} a^*b_na\right|\right|^{1/4}$$

$$\leqq s||a||^{3/4}\left|\left|\sum_{n>m} b_na\right|\right|^{1/4}.$$

This last quantity tends to zero as $m \to \infty$ which proves that

$$\sum b_n^{1/2}x_nb_n^{1/2} \in M(A).$$

The last part of the proposition follows easily now. If b is strictly positive in A, then bA is dense in A; so (norm) convergence of $\sum b_nb$ entails convergence of $\sum b_na$ for every a in A. Finally, if b commutes with $\{b_n\}$ and $||b_nb|| \to 0$ then

$$\sum b_nb = \sum b_n^{1/2}bb_n^{1/2} \in A,$$

as desired.

THEOREM 4.2. *Let $\{b_n\}$ be a sequence of mutually orthogonal, positive norm one elements in A that supports a sequence $\{f_n\}$ of pure states. If $\sum b_n$ is strictly convergent, then every weak* limit point of $\{f_n\}$ (in $M(A)^*$) is a pure state of $M(A)$.*

Proof. Using the notation from the proof of Proposition 4.1, for each n and for every $\{x_k\}$ in $\prod A_k$ and x in $M(A)$ we have

$$f_n(x^*\Psi(\{x_k\})x) = \sum_k f_n(x^*b_k^{1/2}x_kb_k^{1/2}x),$$

because $\Psi(\{x_k\}) \in A^{**}$ and f_n is normal on A^{**}. Thus each ultrafilter on the positive integers $\mathbf{N}$ is good for the sequence $\{f_n\}$, hence all limit points of $\{f_n\}$ are pure by Theorem 3.5.

From Theorem 4.2 the question naturally arises: which sequences of mutually orthogonal pure states of A are supported by l^∞-embedded sequences? Closely related to this problem is the question: which maximal commutative C^*-subalgebras (MASA's) of A contain an approximate unit for A? (Cf. Proposition 4.1.) We digress briefly from our main theme to show how these concepts relate to MASA's in $M(A)$.

A MASA C in the multiplier algebra $M(A)$ of A is called *atomic* if $C \cap A$ is a MASA in A. The terminology is of course borrowed from the case $A = \mathcal{K}$, where $M(A) = \mathbf{B}(H)$; but in this generality we cannot expect much of the original meaning of atomicity to remain. A more restrictive notion may be needed. We say that C is *strictly atomic* in $M(A)$ if $C \cap A$ is a MASA that contains an approximate unit for A.

LEMMA 4.3. *Let B be a hereditary C^*-subalgebra of A, and C a commutative C^*-subalgebra of B. Denote by C', $C^{\perp}$ and $I(C)$ the commutant, the (two-sided) annihilator and the idealizer of C in A, respectively.*

(i) *If $C^{\perp}$ is commutative, then $I(C)$ is commutative.*

(ii) *If C is a* MASA *in B, then $C' \subset I(C)$.*

(iii) *If $C^{\perp}$ is commutative and C is a* MASA *in B, then $C' = I(C)$ and is a* MASA *in A.*

Proof. (i) Take x, y in $I(C)$ and c in C. Then

$$xyc^2 = x(yc)c = (xc)(yc) = (yc)(xc) = yxc^2.$$

Since $C^2 = C$ it follows that $xy - yx \in C^{\perp}$. Now $C^{\perp}$ is an ideal in $I(C)$, and we have shown that $I(C)/C$ is commutative. If also $C^{\perp}$ is commutative, then so is C.

(ii) If $x \in C'$ and $c \in C_+$, then

$$xc = c^{1/2}xc^{1/2} \in B \cap C' = C,$$

since C is a MASA in B. Thus $C' \subset I(C)$.

(iii) From (i) and (ii) we see that $I(C) \subset C'$ and $C' \subset I(C)$, whence $C' = I(C)$. If D is any commutative subset of A and $C \subset D$, then $D \subset C'$. Thus C' is a MASA in A.

PROPOSITION 4.4. *There is a bijective correspondence between* MASA*'s C in A and atomic* MASA*'s D in $M(A)$, given by $D = C'$ and $C = D \cap A$.*

Proof. If C is a MASA in A, then its annihilator $C^{\perp}$ is an hereditary C^*-subalgebra of $M(A)$ which clearly intersects A in $\{0\}$; hence $C^{\perp} = \{0\}$. It follows from Lemma 4.3 (with A and $M(A)$ in place of B and A) that C' is a MASA in $M(A)$.

LEMMA 4.5. *On bounded subsets of $M(A)$ the weak* topology from A^{**} (i.e., the $\sigma(A^{**}, A^*)$-topology) is weaker than the strict topology; but for every convex subset C of $M(A)$ the relative weak* closure of C in $M(A)$ coincides with the strict closure of C.*

Proof. Let $\{x_\alpha\}$ be a bounded net in $M(A)$ converging strictly to 0, and let f be a state of A. Given $\epsilon > 0$ we can find a in A, $0 \leqq a \leqq 1$, such that $f(a) > 1 - \epsilon$. Consequently,

$$\begin{aligned}|f(x_\alpha)| &\leqq |f(x_\alpha a)| + |f(x_\alpha(1 - \alpha))| \\ &\leqq \|x_\alpha a\| + f(x_\alpha x_\alpha^*)^{1/2} f((1 - a)^2)^{1/2} \leqq \|x_\alpha a\| + \|x_\alpha\|\epsilon^{1/2}.\end{aligned}$$

Since $\|x_\alpha a\| \to 0$ by assumption, it follows that $f(x_\alpha) \to 0$, whence $\{x_\alpha\}$ is weak* convergent to 0.

Conversely, if x is a weak* limit point of C in $M(A)$, then xa is a weak limit point of Ca in A for every a in A. Since Ca is a convex subset of A, it follows (from the Hahn-Banach theorem) that xa belongs to the norm closure of Ca. Thus for every $\epsilon > 0$ there is a c in C with

$$\|xa - ca\| < \epsilon.$$

Since for any $a_1, \ldots, a_n$ in A and $\epsilon > 0$ there exists a in A such that

$$\|aa_j - a_j\| < \epsilon \quad \text{for all } j = 1, \ldots, n,$$

this shows that x belongs to the strict closure of C.

PROPOSITION 4.6. *If C is a* MASA *in A and C' denotes its corresponding atomic* MASA *in $M(A)$, the following conditions are equivalent:*

(i) *C' is strictly atomic*;

(ii) $C = M(C)$;

(iii) *C' is the strict closure of C.*

Proof. (i) $\Rightarrow$ (ii). Since C contains an approximate unit for A we have $M(C) \subset M(A)$ by [**10**, 3.12.12]. In other words $I(C)$ is isomorphic to $M(C)$. However, by Lemma 4.3, $I(C) = C'$.

(ii) $\Rightarrow$ (iii). If $\bar{C}$ denotes the strict closure of C then clearly $\bar{C} \subset C'$, since C is commutative. Assuming that $C' = M(C)$, and using the fact that the embedding $M(C) \subset M(A)$ is weak* continuous (since it arises from the embedding $C^{**} \subset A^{**}$ obtained by double transposition of the embedding map $C \subset A$), we see that each element in C' belongs to the weak* closure of C. By Lemma 4.5 this implies that $C' \subset \bar{C}$.

(iii) $\Rightarrow$ (i). Since $1 \in C'$, there is a net $\{x_\alpha\}$ in C converging strictly to 1. But this means precisely that $\{x_\alpha\}$ is an approximate unit for A.

Example 4.7. Take $A = \mathbf{C}p + \mathscr{K}$, where p is a projection in $\mathbf{B}(H)$ such that both pH and $(1 - p)H$ are infinite dimensional. It is easy to see that

$$M(A) = \mathbf{C}p + (1 - p)\mathbf{B}(H)(1 - p) + \mathscr{K}.$$

Choose now an orthonormal basis $\{\xi_n\}$ for H, such that

$$p(\xi_{2n-1} + \xi_{2n}) = \xi_{2n-1} + \xi_{2n} \quad \text{and}$$

$$p(\xi_{2n-1} - \xi_{2n}) = 0 \quad \text{for all } n.$$

Let C denote the algebra of diagonal operators in $\mathscr{K}$ with respect to $\{\xi_n\}$. Then C is a MASA in A, but it does not contain an approximate unit for A. Indeed,

$$\|p(1 - c)\| \geqq \frac{1}{2}\sqrt{2} \quad \text{for every } c \text{ in } C.$$

From the description of $M(A)$ we see that $C' = C + \mathbf{C}1$ which gives an example of an atomic, but not strictly atomic, MASA in $M(A)$.

Example 4.8. Take $A = C([0, 1]) \otimes \mathcal{K}$ and recall from [**3**] that

$$M(A) = C([0, 1], \mathbf{B}(H)_{s^*})$$

the strong* continuous functions from [0, 1] to $\mathbf{B}(H)$. Choose MASA's C_1 and C_2 in $\mathcal{K}$ (corresponding to orthonormal bases) and set

$$C = \left\{ x \in A : x(t) \in C_1, t < \frac{1}{2}; x(t) \in C_2, t > \frac{1}{2} \right\}.$$

Then C is a MASA in A. It is easy to arrange C_1 and C_2, such that $C_1 \cap C_2 = \{0\}$, but $C_1' \cap C_2'$ (in $\mathbf{B}(H)$) contains many non-trivial projections. Note that C contains an approximate unit for A precisely when $C_1 \cap C_2$ contains an approximate unit for $\mathcal{K}$.

The example above can be elaborated by distributing a whole sequence of MASA's in $\mathcal{K}$ at suitable points in [0, 1]; but this may not exhaust the supply of MASA's in A. Indeed, it is not even known whether a MASA C in A must have a point t in [0, 1] (and therefore a dense set of t's) such that $C(t)$ is a MASA in $\mathcal{K}$.

Returning to our main problem, we now consider a sequence $\{f_n\}$ of mutually orthogonal pure states of A. Even under the assumption that $\{f_n\}$ tends to zero, it is not always true that $\{f_n\}$ is supported by a sequence $\{b_n\}$ of mutually orthogonal, positive, norm one elements in A. Taking the supporting sequence for granted, we can, however, prove that many subsequences of $\{f_n\}$ are supported by l^∞-embedded sequences in the σ-unital case. We say that $\{f_n\}$ *tends rapidly to zero* if $\sum f_n(b) < \infty$ for some strictly positive element b in A. Much of [**1**] is devoted to the question of when this occurs.

PROPOSITION 4.9. *If A is σ-unital, and $\{f_n\}$ is a sequence of mutually orthogonal pure states tending rapidly to zero and supported by a sequence $\{a_n\}$ of mutually orthogonal, positive, norm one elements in A, then $\{f_n\}$ is also supported by an l^∞-embedded sequence, and thus every weak* limit point of $\{f_n\}$ is pure in $M(A)^*$.*

Proof. Choose a strictly positive element b in A such that $\sum f_n(b) < \infty$. By Proposition 2.1 f_1 is excised by a decreasing net $\{x_\lambda\}$ majorized by a_1. Thus

$$\|x_\lambda b x_\lambda\| \leqq \|x_\lambda(b - f_1(b))x_\lambda\| + f_1(b) \leqq 2f_1(b)$$

for a suitable x_λ, which we denote by b_1. Repeating the process with f_2, f_3, et cetera, we obtain a sequence $\{b_n\}$ supporting $\{f_n\}$, such that

$$\|b_n b b_n\| \leqq 2f_n(b) \quad \text{for all } n.$$

Consequently the sum $\sum b_n b^{1/2}$ is norm convergent, because

$$\begin{aligned}\|\sum b_n b^{1/2}\|^2 &= \|\sum b^{1/2} b_n^2 b^{1/2}\| \leqq \sum \|b_n b^{1/2}\|^2 \\ &= \sum \|b_n b b_n\| \leqq 2 \sum f_n(b).\end{aligned}$$

Since $b^{1/2}$ is strictly positive in A, it follows from $\sum b_n \in M(A)$, and the rest follows from Proposition 4.1 and Theorem 4.2.

Recall that a state f is *definite* on an element x in A if

$$f(x^*x) = |f(x)|^2.$$

If (π, H, ξ) is the cyclic representation associated with f via the GNS-construction, this condition means that

$$|\langle \pi(x)\xi, \xi\rangle| = \|\pi(x)\xi\| \|\xi\|,$$

which is equivalent to $\pi(x)\xi = f(x)\xi$. Consequently

$$f(xy) = f(yx) = f(x)f(y) \quad \text{for every } y \text{ in } A.$$

The following result is a simple consequence of the previous concepts, but in the applications it may very well be the case that turns up most frequently.

PROPOSITION 4.10. *If A is σ-unital and $\{f_n\}$ is a sequence of mutually orthogonal pure states tending to zero, such that every f_n is definite on the strictly positive element b and the $\{f_n(b)\}$ are distinct, then $\{f_n\}$ is supported by an l^∞-embedded sequence and every weak* limit point of $\{f_n\}$ is pure in $M(A)^*$.*

Proof. Let $C = C_0(S)$ be the C^*-subalgebra generated by b. Since each f_n is definite on b, it is multiplicative on C. Since moreover $f_n(b) > 0$ for every n and $f_n(b) \to 0$, there is a sequence $\{s_n\}$ in S tending to infinity, such that $f_n(c) = c(s_n)$ for every c in C and all n. Choose by elementary function theory ($S = Sp(b)\setminus\{0\}$) a sequence $\{b_n\}$ in C of mutually, orthogonal, positive, norm one elements, such that $b_n(s_n) = 1$ for every n. Then $\{b_n\}$ supports $\{f_n\}$; and since

$$\sum b_n \in C_b(S) = M(C_0(S)),$$

and $M(C) \subset M(A)$ (cf. [**10**, 3.12.12]), the desired conclusions follow from Proposition 4.1 and Theorem 4.2.

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