ON RELATIONSHIPS AMONGST CERTAIN SPACES OF SEQUENCES IN AN ARBITRARY BANACH SPACE

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1. Introduction. Let X be a Banach space (B-space). A sequence $\{s(i)\}$ in X is *unconditionally summable* if and only if every rearrangement of the series $\sum_i s(i)$ is convergent. The set of unconditionally summable sequences in X will be written as $U(X)$. In this paper several classes of summable sequences in X will be compared with one another. Each class to be considered is identical with $U(X)$ when X has finite dimension.

The following notation will be used. The set of natural numbers will be denoted by N and the collection of non-null finite subsets of N by $\mathscr{F}$. A sequence in X will usually be denoted by the single letter s and its value at $i \in N$ by $s(i)$. If s is a sequence in X and $F \in \mathscr{F}$ the sum of the terms $s(i)$ such that $i \in F$ will be written $\sum_F s(i)$.

A sequence s in X will be called *weakly unconditionally summable* if and only if $\sum_i |f(s(i))| < \infty$ for every $f \in X^*$, the adjoint space of X. Let $B(X)$ stand for the set of weakly unconditionally summable sequences in X. Gelfand **(4)** has shown that $s \in B(X)$ if and only if $\sup[||\sum_F s(i)||: F \in \mathscr{F}] < \infty$. With the usual definitions for addition of sequences and multiplication of a sequence by a scalar $B(X)$ is a vector space. It is known that $B(X)$ is a B-space with the norm of each $s \in B(X)$ defined by $||s|| = \sup[||\sum_F s(i)||: F \in \mathscr{F}]$. This will be the norm intended when $B(X)$ is referred to as a B-space in the sequel. As a consequence of a result of Birkhoff **(2)**, $U(X)$ is a closed linear subspace of $B(X)$.

Following Hadwiger **(5)**, a sequence s in a B-space X has an *invariant sum* if and only if there is an $x \in X$ such that $x = \sum_i s(i)$ and such that x is the sum of each of the convergent rearrangements of $\sum_i s(i)$. Let $IS(X)$ stand for the class of sequences in X with an invariant sum. It is known that if X has finite dimension then $U(X) = IS(X)$. Hadwiger **(5)** has shown that if X is a Hilbert space with infinite dimension then $U(X)$ is a proper subset of $IS(X)$. In this paper Hadwiger's result is sharpened and extended to any B-space with infinite dimension.

If s is a sequence in X and there is $x \in X$ such that $x = \sum_i s(i)$ then x will be called the *sum* of s. In case there is $x \sum X$ such that $f(x) = \sum_i f(s(i))$ for all $f \in X^*$ then x will be called the *weak sum* of s. It follows easily that *a sequence s in a B-space X can have at most one weak sum.* It can be shown that in any B-space X there are sequences which have a sum but are not elements of $B(X)$. Conversely, in some B-spaces, for example, in $X = c_0$, the B-space of real sequences which converge to 0 with $||s|| = \sup[|s(i)|: i \in N]$ for each

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$s \in c_0$, there exist sequences which are elements of $B(X)$ but which do not have sums.

Two new closed linear subspaces of $B(X)$ are introduced in this paper. They are

$B_w(X) = [s \in B(X): s$ has a weak sum$]$, $B_s(X) = [s \in B(X): s$ has a sum$]$.

For any B-space it is true that

$$U(X) \subset B_s(X) = IS(X) \cap B(X) \subset B_w(X) \subset B(X).$$

We show that if $X = c_0$ then all of these containments are proper.

2. Closed linear subspaces of $B(X)$**.** Dunford (**3**) and Gelfand (**4**) have shown that a sequence s in a B-space X is weakly unconditionally summable if and only if there is a real number M such that $\sum_i |f(s(i))| \leqslant M||f||$ for all $f \in X^*$. A norm for the vector space of weakly unconditionally summable sequences in X is defined by setting

$$||s||_1 = \sup[\textstyle\sum_i |f(s(i))|: f \in X^* \text{ and } ||f|| \leqslant 1]$$

for each sequence s of this class. Let $B'(X)$ denote the normed vector space of weakly unconditionally summable sequences in X with the norm of the preceding sentence. As a special case of a result of Dunford (**3**, Theorem 30) we have that $B'(X)$ is a B-space.

The following lemma is essentially given by Pettis (**6**, Theorem 3.2.2.).

LEMMA 2.1. *If s is weakly unconditionally summable then*

$$\sup[||\textstyle\sum_F s(i)||: F \in \mathscr{F}] \leqslant \sup[\sum_i |f(s(i))|: f \in X^* \text{ and } ||f|| \leqslant 1]$$
$$\leqslant 2 \sup[||\textstyle\sum_F s(i)||: F \in \mathscr{F}].$$

LEMMA 2.2. *The normed vector space $B(X)$ is complete.*

Proof. Since $B(X)$ and $B'(X)$ differ only in their norms and $B'(X)$ is complete it is evident from the relationships between their norms given in Lemma 2.1 that $B(X)$ is complete.

THEOREM 2.3. *For any B-space X the spaces $B_w(X)$ and $B_s(X)$ are closed linear subspaces of $B(X)$, and the operation L defined on $B_w(X)$ to X by setting $L(s)$ equal to the weak sum of s for each $s \in B_w(X)$ is linear and has norm* 1.

Proof. To show that $B_w(X)$ is closed in $B(X)$ suppose s_n is a sequence in $B_w(X)$ which converges to $s \in B(X)$. For each $n \in N$ let x_n denote the weak sum of s_n. Since $\{s_n\}$ is a Cauchy sequence in $B(X)$ there is for each $\epsilon > 0$ a natural number n_ϵ such that $||s_n - s_m|| < \epsilon/2$ if $n, m \geqslant n_\epsilon$. For $n, m \geqslant n_\epsilon$ and $f \in X^*$ with $||f|| \leqslant 1$ one has

$$|f(x_m - x_n)| \leqslant \textstyle\sum_i |f(s_n(i) - s_m(i))| \leqslant 2||s_n - s_m|| < \epsilon,$$

the second inequality given by Lemma 2.1. It follows that $\{x_n\}$ is a Cauchy sequence and therefore has a limit x. Again, suppose $\epsilon > 0$ is given and $f \in X^*$ with f non-zero. There is an n_ϵ such that

$$||s_n - s|| < \epsilon/(4||f||) \qquad n \geqslant n_\epsilon,$$

and since x_n converges to x, n_ϵ may be chosen large enough so

$$||x - x_n|| < \epsilon/(2||f||) \qquad n \geqslant n_\epsilon.$$

Hence, if $n \geqslant n_\epsilon$ then

$$|f(x) - \sum_i f(s(i))| \leqslant |f(x) - f(x_n)| + \sum_i |f(s_n(i) - s(i))|$$
$$\leqslant ||f||(\epsilon/(2||f||)) + 2||f||\,||s_n - s|| < \epsilon,$$

using Lemma 2.1 to get the second inequality. This proves that x is the weak sum of s.

To show that $B_s(X)$ is closed in $B(X)$ suppose $\{s_n\}$ is a sequence in $B_s(X)$ which converges to $s \in B(X)$. For each $n \in N$ let x_n denote the sum of s_n. Since $B_s(X) \subset B_w(X)$ and $B_w(X)$ is closed, s has a weak sum x. Also $\{x_n\}$ converges to x. Since $\{x_n\}$ converges to x and $\{s_n\}$ converges to s, if $\epsilon > 0$ is given there is $p \in N$, dependent on ϵ, such that $||x - x_p|| < \epsilon/3$ and $||s_p - s|| < \epsilon/3$. Also since $x_p = \sum_i s_p(i)$, there is a $q \in N$ such that if $r \geqslant q$ then

$$\left|\left|x_p - \sum_{i=1}^{r} s_p(i)\right|\right| < \epsilon/3.$$

Hence if $r \geqslant q$, then

$$\left|\left|x - \sum_{i=1}^{r} s(i)\right|\right| \leqslant ||x - x_p|| + \left|\left|x_p - \sum_{i=1}^{r} s_p(i)\right|\right| + \left|\left|\sum_{i=1}^{r} s_p(i) - \sum_{i=1}^{r} s(i)\right|\right| < \epsilon.$$

This shows that x is the sum of s.

It remains to show that L is a linear operation with norm 1. Let

$$E = [f\colon f \in X^* \text{ and } ||f|| = 1].$$

Fix $s \in B_w(X)$ and let $x = L(s)$. Then

$$||x|| = \sup[|f(x)|\colon f \in E] = \sup\left[\lim_{n\to\infty}\left|\sum_{i=1}^{n} f(s(i))\right|\colon f \in E\right]$$
$$\leqslant \sup\left[\sup\left\{\left|f\left(\sum_{i=1}^{n} s(i)\right)\right|\colon n \in N\right\}\colon f \in E\right]$$
$$= \sup\left[\sup\left\{\left|f\left(\sum_{i=1}^{n} s(i)\right)\right|\right\}\colon f \in E\colon n \in N\right]$$
$$= \sup\left[\left|\left|\sum_{i=1}^{n} s(i)\right|\right|\colon n \in N\right] \leqslant ||s||.$$

Hence L, which is obviously additive, is continuous and $||L|| \leqslant 1$. Since for any $x_0 \in X$ the sequence $\{x_0, \theta, \theta, \ldots, \theta, \ldots\}$ is in $B_w(X)$ and has x_0 for its norm, clearly $||L|| = 1$.

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3. Extension of a theorem of Hadwiger to B-spaces. The following theorem is obtained by applying a modification of Hadwiger's argument **(5)** to the general case.

THEOREM 3.1. *If X is a B-space the following are equivalent:*

(i) *X has infinite dimension.*

(ii) *the difference $IS(X) \sim B(X)$ is non-void.*

(iii) *$U(X)$ is a proper subset of $IS(X)$.*

Proof. Because of the well-known fact that $U(X) \subset IS(X) \cap B(X)$ for all X, it is evident that (ii) implies (iii). Since $U(X) = IS(X)$ if X has finite dimension, (iii) implies (i). It will now be shown that (i) implies (ii). By a remark of Banach's **(1**, p. 238**)**, X contains a closed infinite dimensional linear subspace X_0 which has a basis $\{x(i)\}$ with $||x(i)|| = 1$, $i \in N$. Using a result of Banach **(1**, pp. 110–111**)**, there is a sequence $\{f_i\}$ in X^* such that $f_i(x(j)) = \delta_{ij}$ and for each $x \in X_0$, $x = \sum_i f_i(x)x(i)$.

Consider the sequence of finite blocks

$$B_k = \{x(k)/k, -x(k)/k, \ldots, x(k)/k, -x(k)/k\}, \qquad k = 1, 2, 3, \ldots$$

where B_k consists of $2k^2$ terms each of which is either $x(k)/k$ or $-x(k)/k$ according as it is in an odd or an even place in B_k. Note that $x(k)/k$ occurs k^2 times in each B_k so the sum of the odd place terms in B_k has norm k. Construct a sequence s in X by adjoining the second block of terms to the first, the third block to this, etc. Since the norm of the sum of the odd place terms in each block is k, $s \notin B(X)$. Clearly $\sum_i s(i) = \theta$. It remains to show that s has an invariant sum. Suppose that s' is a rearrangement of s and that $y = \sum_i s'(i)$. Since X_0 is closed, $y \in X_0$. Express y by its biorthogonal development $y = \sum_i f_i(y)x(i)$. For arbitrary $i \in N$, we have $f_i(y) = \sum_j f_i(s'(j))$. Take n_0 large enough so that all terms in the block B_i occur in the sum

$$s'(1) + s'(2) + \ldots + s'(n_0).$$

If $n \geqslant n_0$ then

$$\sum_{j=1}^{n} f_i(s'(j)) = f_i\Big(\sum_{j \in F} s'(j)\Big) + \sum_{j \in F'} f_i(s'(j))$$

where $F = [j : j \leqslant n$ and $s'(j)$ is a term of $B_i]$ and

$$F' = [j : j \leqslant n \text{ and } j \notin F].$$

Now $\sum_F s'(j) = \theta$, and by biorthogonality $f_i(s'(j)) = 0$ if $j \in F'$, so $f_i(y) = 0$. Since $f_i(y) = 0$ for all i it follows that $y = \theta$.

4. Comparison of subspaces of $B(X)$. For any B-space X, $U(X) \subset B(X)$ so clearly $U(X) \subset B_s(X)$. Also $B_s(X) \subset IS(X)$ for any B-space X, because if $s \in B_s(x)$ and s has the sum x and if s' is a rearrangement of s with sum x' it follows that $f(x) = f(x')$ for all $f \in X^*$ so $x = x'$. With these observations the following lemma is obvious.

LEMMA 4.1. *For any B-space X,* $U(X) \subset B_s(X) = IS(X) \cap B(X) \subset B_w(X) \subset B(x)$.

A B-space X is *weakly complete* if and only if every weakly convergent sequence in X is weakly convergent to an element of X.

THEOREM 4.2. *If X is weakly complete then*

$$U(X) = B_s(X) = IS(X) \cap B(X) = B_w(X) = B(X) \subset IS(X).$$

The containment is proper if and only if X has infinite dimension.

Proof. For any B-space, $U(X) \subset IS(X)$ and it is well known that when X is weakly complete that $U(X) = B(X)$. Hence $B(X) \subset IS(X)$ when X is weakly complete. The theorem then follows by Lemma 4.1 and Theorem 3.1.

LEMMA 4.3. *If for a B-space X, U(X) is a proper subspace of B(X), then U(X) is a proper subspace*[1] *of* $B_s(X)$.

Proof. Suppose $s \in B(X) \sim U(X)$. For each $k \in N$ let B_k denote a block of $2k$ terms as follows:

$$B_k = \{s(k)/k, \; -s(k)k/, \ldots, s(k)/k, \; -s(k)/k\}.$$

that is, the even place terms in B_k are $s(k)/k$ and the odd place terms are $-s(k)/k$. We construct $s' \in B_s(X) \sim U(X)$ by adjoining the terms of the block B_2 to those of B_1 and then adjoining the terms of B_3 to these, etc. Clearly $\theta = \sum_i s'(i)$ and for each $f \in X^*$,

$$\sum_i |f(s'(i))| = 2\sum_i |f(s(i))| < \infty,$$

so $s' \in B_s(X)$. Finally, since $s \notin U(X)$ it follows that the series $\sum_i s'(i)$ has a subseries, namely, $\sum_i s'(2i - 1)$ which does not converge unconditionally. Hence $s' \notin U(X)$.

COROLLARY 4.4. *The B-space* $U(c_0)$ *is a proper subspace of* $B_s(c_0)$.

Proof. Consider the sequence $\{s_n\}$ in c_0 where for each n, $s_n(i) = 1$ if $i = n$ and $s_n(i) = 0$ if $i \neq n$. The sequence $\{s_n\}$ is an element of $B(c_0)$ but it does not have a sum so is not an element of $U(c_0)$. The corollary follows by Lemma 4.3.

LEMMA 4.5. *If for a B-space X, U(X) is a proper subspace of* $B_s(X)$ *then* $B_s(X)$ *is a proper subspace of* $B_w(X)$.

Proof. If $s \in B_s(X) \backsim U(X)$ then there is a permutation t of N such that the sequence $\{s(t(i))\}$ does not have a sum. Let x denote the sum of s. Then x is the weak sum of s and since $s \in B(X)$ it follows that x is the weak sum of $\{s(t(i))\}$.

By Corollary 4.4 and Lemma 4.5 we have the next corollary.

COROLLARY 4.6. *The space* $B_s(c_0)$ *is a proper subspace of* $B_w(c_0)$.

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LEMMA 4.7. *If for a B-space X, $U(X)$ is a proper subset of $B(X)$ then $B_w(X)$ is a proper subset of $B(X)$.*

Proof. By hypothesis there exists an $s \in B(X) \backsim U(X)$. Using a result of Orlicz (**1**, (3) on p. 270), there is a strictly increasing sequence t of natural numbers such that the sequence $\{s(t(i))\}$ does not have a weak sum. However it obviously inherits the property of belonging to $B(X)$ from s.

COROLLARY 4.8. *The space $B_w(c_0)$ is a proper subspace of $B(c_0)$.*

Proof. Since $B(c_0) \backsim U(c_0)$ is non-void the conclusion follows by Lemma 4.7.

Putting together the preceding corollaries we have the following

THEOREM 4.9. *For the B-space c_0, $U(c_0) \subset B_s(c_0) \subset B_w(c_0) \subset B(c_0)$, and* each containment is proper.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).
2. G. Birkhoff, *Integration of functions with values in a Banach space*, Trans. Amer. Math. Soc., *38* (1935), 357–378.
3. N. Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc., *44* (1938), 305–356.
4. I. Gelfand, *Abstrakte Funktionen und lineare Operatoren*, Mat. Sbornik, N.S., *46* (1938), 235–284.
5. H. Hadwiger, *Über die konvergenzarten unendlicher reihen in Hilbertschen raum*, Math. Zeit., *47* (1941), 325–329.
6. B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc., *44* (1938), 277–304.

[1]The author is indebted to the referee for the present form of Lemma 4.3 which is simpler and more general than the original.

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