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COMPARABLE DIFFERENTIABILITY CHARACTERISATIONS OF TWO CLASSES OF BANACH SPACES

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We characterise Banach spaces not containing ℓ_1 by a differentiability property of each equivalent norm and show that a slightly stronger differentiability property characterises Asplund spaces.

A continuous convex function ϕ on an open convex subset A of a normed linear space X is *Gâteaux differentiable at* $x \in A$ *in the direction* $y \in X$ if

$$\phi'(x)(y) \equiv \lim_{\lambda \to 0} \frac{\phi(x+\lambda y) - \phi(x)}{\lambda}$$

exists, and is *Gâteaux differentiable at* x if $\phi'(x)(y)$ exists for all $y \in X$. Further ϕ is *Fréchet differentiable* at x if the limit $\phi'(x)(y)$ is approached uniformly for all $y \in X$, $\|y\| = 1$

An *Asplund* space is a Banach space X where every continuous convex function ϕ on an open convex subset A of X is Fréchet differentiable on a dense G_δ subset of A. The theory of Asplund spaces is by now well established, (see [**6**]). Our first interest is in the following characterisations.

PROPOSITION 1. *For a Banach space X, the following are equivalent.*

(i) *X is an Asplund space,*
(ii) *every nonempty bounded subset K of X^* has weak* slices of arbitrarily small diameter,* [**6**, p.31],
(iii) *every continuous weak* lower semi-continuous convex function ϕ on an open convex subset A of X^{**} is Fréchet differentiable at the points of a dense G_δ subset of A,* [**6**, p.94],
(iv) *every equivalent norm p on X is Fréchet differentiable at some point of X,* [**6**, p.33].

We note that given a nonempty bounded subset K of X^*, a *weak* slice* of K is a nonempty subset of K of the form

$$S(K, \widehat{x}, \delta) \equiv \{f \in K : f(x) > \sup \widehat{x}(K) - \delta\}$$

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for some $x \in X \setminus \{0\}$ and $\delta > 0$.

The study of Banach spaces not containing ℓ_1 is also well established, (see [**8**]). However, a fascinating characterisation for such spaces was given in [**7**, p.422].

PROPOSITION 2. *A Banach space X does not contain a subspace topologically isomorphic to ℓ_1 if and only if, given $F \in X^{**}$, every nonempty bounded subset K of X^* has weak* slices over which the oscillation of F is arbitrarily small.*

We recall that, given $F \in X^{**}$ and a nonempty bounded set K in X^*, the *oscillation* of F over K is

$$\omega(F(K)) \equiv \sup\{|F(f-g)| : f, g \in K\}.$$

Because the characterisation in Proposition 2 is comparable to that given in Proposition 1 (i) $\Longleftrightarrow$ (ii), it suggests that we investigate a differentiability characterisation comparable to Proposition 1 (i) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv).

A set-valued mapping Φ from a topological space A into subsets of the dual X^* of a Banach space X is said to be *minimal* if given any open set $U \subseteq A$ and a weak* open half-space W in X^* such that $\Phi(U) \cap W \neq \emptyset$ there exists a nonempty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$. Further Φ is said to be *locally bounded* if for every $x \in A$ there exists a neighbourhood U of x such that $\Phi(U)$ is bounded in X^*.

Given a continuous convex function ϕ on an open convex subset A of a Banach space X, the *subdifferential* of ϕ at $x \in A$ is the set

$$\partial\phi(x) \equiv \left\{f \in X^* : f(y) \leqslant \phi'_+(x)(y) \ \text{ for all } y \in X\right\}.$$

Given a separated locally convex topology τ on the dual X^*, the *subdifferential mapping* $x \mapsto \partial\phi(x)$ is *τ-upper semi-continuous* at $x \in A$ if given W a τ-open subset of X^* such that $\partial\phi(x) \subseteq W$ there exists a $\delta > 0$ such that $\partial\phi(y) \subseteq W$ for all $y \in A$, $\|x - y\| < \delta$. The subdifferential mapping $x \mapsto \partial\phi(x)$ is a *minimal weak* cusco* on A; that is, given $x \in A$, $\partial\phi(x)$ is nonempty, weak* compact and convex and the mapping is weak* upper semi-continuous and minimal on A. It is also locally bounded. Now ϕ is Gâteaux differentiable at $x \in A$ if and only if $\partial\phi(x)$ is singleton and is Fréchet differentiable at x if and only if $\partial\phi(x)$ is singleton and the subdifferential mapping $x \mapsto \partial\phi(x)$ is norm upper semi-continuous at x, [**6**, p.19].

Given a continuous convex function ϕ on an open convex subset A of a Banach space X we can extend ϕ as a lower semi-continuous convex function $\overline{\phi}$ on X by defining

$$\overline{\phi}(x) = \begin{cases} \liminf\limits_{y \to x} \phi(y) & \text{for } x \in \overline{A} \\ +\infty & \text{otherwise.} \end{cases}$$

The *subdifferential* of $\overline{\phi}$ at $x \in A$ is the set $\partial\phi(x)$. The lower semi-continuous convex function ϕ^* on X^*, the *Fenchel conjugate* of ϕ on A is defined by

$$\phi^*(f) = \sup\left\{f(x) - \overline{\phi}(x) : x \in \overline{A}\right\}.$$

Now $f \in \partial\phi(x)$ if and only if $\widehat{x} \in \partial\phi^*(f)$. Also $\phi^{**}|_{\widehat{A}} = \phi$, [**6. p.42**].

Theorem 1. *For a Banach space X, the following are equivalent.*

(i) *X does not contain a subspace topologically isomorphic to ℓ_1,*

(ii) *for every nonempty bounded subset K of X^*, given $\varepsilon > 0$ and $F \in X^{**}$ there exists $x \in X\backslash\{0\}$ and $\delta(\varepsilon, F, x) > 0$ such that*

$$\omega\left(F\left(S(K, \widehat{x}, \delta)\right)\right) < \varepsilon$$

(iii) *for every continuous convex function ϕ on an open convex subset A of X, given $F \in X^{**}$ the real set-valued mapping $x \mapsto F(\partial\phi(x))$ is single-valued and upper semi-continuous at the points of a dense G_δ subset D_F of A.*

(iv) *for every continuous convex function ϕ on an open convex subset A of X, given $F \in X^{**}\backslash\{0\}$, ϕ^{**}, the second Fenchel conjugate of ϕ on X^{**}, is Gâteaux differentiable in the direction F at the points of a dense G_δ subset D_F of $\widehat{A}$.*

(v) *for every equivalent norm p on X, given $F \in X^{**}\backslash\{0\}$ the norm p^{**} on X^{**} induced by p is Gâteaux differentiable in the direction F at some point of $\widehat{X}$.*

Proof:

(i) $\iff$ (ii) is Proposition 2.

(ii) $\implies$ (iii) Given $F \in X^{**}$ and $\varepsilon > 0$ consider the set

$$O_\varepsilon \equiv \bigcup \{\text{ open } U \subseteq A : \omega\left(F(\partial\phi(U))\right) < \varepsilon\}.$$

Now O_ε is open in A; we show that it is dense in A. Consider open $U \subseteq A$ such that $\partial\phi(U)$ is bounded. By (ii) there exists a weak* slice S of $\partial\phi(U)$ such that $\omega\left(F(S)\right) < \varepsilon$. Since the subdifferential mapping $x \mapsto \partial\phi(x)$ is a minimal weak* cusco on A, there exists a nonempty open set $V \subseteq U$ such that $\partial\phi(V) \subseteq S$. But then $\omega\left(F(\partial\phi(V))\right) < \varepsilon$. We conclude that $D_F \equiv \bigcap_{\varepsilon>0} O_\varepsilon$, the set where the mapping $x \mapsto F\left(\partial\phi(x)\right)$ is single-valued and upper semi-continuous, is a dense G_δ subset of A.

(iii) $\implies$ (iv) Suppose that ϕ^{**} is not Gâteaux differentiable in the direction F at $\widehat{x}_0 \in \widehat{A}$.

Then there exist $\widehat{f}_0, \mathcal{F} \in \partial\phi^{**}(\widehat{x}_0)$, $\mathcal{F} \neq \widehat{f}_0$ and $r > 0$ such that $\left(\mathcal{F} - \widehat{f}_0\right)(F) > r$. Since $B\left(\widehat{X}^*\right)$ is weak* dense in $B(X^{***})$, for each $n \in \mathbb{N}$ there exists $g_n \in \widehat{X}^*$, $\|g_n\| \leqslant \|\mathcal{F}\|$ such that

$$|(\mathcal{F} - \widehat{g}_n)(F)| < \frac{1}{n} \text{ and } |(\mathcal{F} - \widehat{g}_n)(\widehat{x}_0)| < \frac{1}{n}.$$

Since $$\mathcal{F}(F) - \mathcal{F}(\widehat{x}_0) \leqslant \phi^{**}(F) - \phi^{**}(\widehat{x}_0) \text{ for all } F \in \widehat{A}$$

then $$g_n(x) - g_n(x_0) \leqslant \phi(x) - \phi(x_0) + \frac{2}{n} \text{ for all } x \in A.$$

By the Brøndsted–Rockafeller Theorem, [**6**, p.48] there exists $x_n \in A$ and $f_n \in \partial\phi(x_n)$ such that

$$\|x_0 - x_n\| < \frac{2}{\sqrt{n}} \quad \text{and} \quad \|g_n - f_n\| < \frac{1}{\sqrt{n}}.$$

Now $|F(g_n - f_0)| > r - 1/n$ so $\|x_0 - x_n\| < 2/\sqrt{n}$ but $|F(f_n - f_0)| > r - 1/n - 1/\sqrt{n}$. We conclude that the mapping $x \mapsto F(\partial\phi(x))$ cannot be both singleton and upper semi-continuous at x_0.

(iv) $\Longrightarrow$ (v) is obvious.

(v) $\Longrightarrow$ (ii) Suppose that there exists a nonempty bounded set A in X^* and $F \in X^{**}$ and $r > 0$ such that every weak* slice S of A has $\omega(F(S)) > r$. Write $C \equiv co\,(A \cup (-A))$ and $K \equiv C + B(X^*)$. Now every weak* slice S of K has $\omega(F(S)) > r$. The functional p on X defined by

$$p(x) = \sup\{f(x) : f \in K\}$$

is an equivalent norm on X. Given $x \in X\backslash\{0\}$, for all $n \in \mathbb{N}$

$$\omega\left(F\left(\{f \in X^* : f(x) > p(x) - r/3n\}\right)\right) > r$$

so there exist $f_n, g_n \in K$ such that

$$f_n(x) > p(x) - \frac{r}{3n}\ ,\ g_n(x) > p(x) - \frac{r}{3n} \text{ and } |F(f_n - g_n)| > r - \frac{1}{n}.$$

Therefore for p^{**} on X^{**} where $p^{**}(F) = \sup\{F(f) : f \in K\}$ we have

$$\begin{aligned} p^{**}\Big(\widehat{x} + \frac{1}{n}\,F\Big) &+ p^{**}\Big(\widehat{x} - \frac{1}{n}\,F\Big) - 2p^{**}(\widehat{x}) \\ &\geqslant \widehat{f}_n\Big(\widehat{x} + \frac{1}{n}\,F\Big) + \widehat{g}_n\Big(\widehat{x} - \frac{1}{n}\,F\Big) - (f_n + g_n)(x) - \frac{2r}{3n} \\ &= \frac{1}{n}\,F(f_n - g_n) - \frac{2r}{3n} \\ &> \frac{r}{3n} - \frac{1}{n^2}. \end{aligned}$$

Then $$n\,\{p^{**}(\widehat{x} + (1/n)\,F) + p^{**}(\widehat{x} - (1/n)\,F) - 2p^{**}(\widehat{x})\} > r/3 - 1/n$$

and so p^{**} is not Gâteaux differentiable at $\widehat{x}$ in the direction F. We conclude that if, given $F \in X^{**}\backslash\{0\}$, p^{**} is Gâteaux differentiable at some $\widehat{x} \in \widehat{X}$ in the direction F, then (ii) holds. ∎

We note that a similar characterisation was proved by Gilles Godefroy [**3**, p.8]

It is clear that the proof of Theorem 1 (i) $\Longleftrightarrow$ (iii) can be generalised.

Corollary 1. *A Banach space X does not contain a subspace topologically isomorphic to ℓ_1, if and only if for every locally bounded minimal set-valued mapping Φ from a Baire space A into subsets of the dual X^* with its weak* topology, given $F \in X^{**}$ the mapping $t \mapsto F(\Phi(t))$ is single-valued and upper semi-continuous at the points of a dense G_δ subset D_F of A.*

Proposition 1 (i) $\Longleftrightarrow$ (iv) implies that a Banach space which is not an Asplund space has an equivalent norm which is nowhere Fréchet differentiable. Similarly, Theorem 1 (i) $\Leftrightarrow$ (v) implies that a Banach space which contains a subspace topologically isomorphic to ℓ_1 has an equivalent norm and $F \in X^{**}\backslash\{0\}$ such that the norm on X^{**} is nowhere Gâteaux differentiable in the direction F. On $(\ell_1, \|\cdot\|_1)$, the norm $\|\cdot\|_1$ is nowhere Fréchet differentiable, [**6**, p.8], but also the norm on ℓ_1^{**} exhibits this other property.

Proof: The norm $\|\cdot\|_1$ is Gâteaux differentiable only at those points $f \equiv \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$ where $\lambda_n \neq 0$ for all $n \in \mathbb{N}$, [**6**, p.3]. So it is sufficient to consider the differentiability of the norm $\|\cdot\|$ on ℓ_1^{**} at such points $\widehat{f}$ in $\widehat{\ell}_1$. The norm $\|\cdot\|$ on ℓ_1^{**} is

$$\|\mathcal{F}\| = \|\widehat{g}\| + \left\|x^\perp\right\| \quad \text{where } \mathcal{F} = \widehat{g} + x^\perp \in \ell_1^{**} \quad \text{and } x \in c_0 \quad \text{and } g \in \ell_1.$$

Now

$$\begin{aligned} \frac{\|\widehat{f} + \lambda\mathcal{F}\| - \|\widehat{f}\|}{\lambda} &= \frac{\|\widehat{f} + \lambda(\widehat{g} + x^\perp)\| - \|\widehat{f}\|}{\lambda} \\ &= \frac{\|f + \lambda g\| - \|f\|}{\lambda} + \frac{|\lambda| \left\|x^\perp\right\|}{\lambda} \to \|f\|'(g) \pm \left\|x^\perp\right\| \text{ as } \lambda \to 0. \end{aligned}$$

So we conclude that for any $x^\perp \in c_0^\perp \setminus \{0\}$ the norm on ℓ_1^{**} is nowhere Gâteaux differentiable on $\widehat{\ell}_1$ in the direction $x^\perp$. ∎

We should note that Theorem 1 (i) $\Longleftrightarrow$ (v) implies that ℓ_∞ has an equivalent norm which is nowhere Gâteaux differentiable. For if every equivalent norm on ℓ_∞ had a point of Gâteaux differentiability then the fact that weak* convergent sequences are weakly convergent in ℓ_∞^* would imply that Theorem 1(v) would be satisfied. But that would contradict the fact that ℓ_∞ contains a subspace isometrically isomorphic to ℓ_1.

It is known that any Banach space X which contains a subspace topologically isomorphic to ℓ_1 has the property that there exists $F \in X^{**} \setminus \{0\}$ and an equivalent norm p on X such that

$$p^{**}(\widehat{x} + F) = p^{**}(\widehat{x}) + p^{**}(F) \quad \text{for all } x \in X,$$

[**1,** p.107]. So Theorem 1 (v) $\Longrightarrow$ (i) could be deduced from this renorming property.

Using the Bishop–Phelps Theorem, [**6,** p.49], it is not difficult to show that given $F \in X^{**}$ and an equivalent norm p on X with closed unit ball $B_p(X)$, the mapping $x \mapsto F(\partial p(x))$ is single–valued and upper semi–continuous at $x \in X, p(x) = 1$ if and only if given $\varepsilon > 0$ there exists a $\delta(\varepsilon, F) > 0$ such that $\omega\left(F\left(S\left(B_p^*(X), \widehat{x}, \delta\right)\right)\right) < \varepsilon$. So using this and the previous comment we can deduce a result similar to that given in [**1,** p.112].

Corollary 2. *A Banach space X does not contain a subspace topologically isomorphic to ℓ_1 if and only if given $F \in X^{**}$ and an equivalent norm p on X there exists a point in $B_p^*(X)$ where F restricted to $B_p^*(X)$ is weak* continuous.*

It is interesting to compare the characterisations given in Theorem 1 with a generalisation of the characterisations given in Proposition 1.

Theorem 2. *For a Banach space X the following are equivalent.*

(i) *X is an Asplund space,*

(ii) *for every nonempty bounded subset K of X^*, there exists $x \in X\backslash\{0\}$ such that, given $\varepsilon > 0$ and $F \in X^{**}$ there exists $\delta(\varepsilon, F) > 0$ such that*

$$\omega\left(F\left(S(K, \widehat{x}, \delta)\right)\right) < \varepsilon.$$

(iii) *for every continuous convex function ϕ on an open convex subset A of X, the subdifferential mapping $x \mapsto \partial\phi(x)$ is single–valued and weak upper semi–continuous at the points of a dense G_δ subset of A,*

(iv) *for every continuous convex function ϕ on on an open convex subset A of X, ϕ^{**}, the second Fenchel conjugate of ϕ on X^{**}, is Gâteaux differentiable at the points of a dense G_δ subset of $\widehat{A}$.*

(v) *for every equivalent norm p on X, the norm p^{**} on X^{**} induced by p is Gâteaux differentiable at some point of $\widehat{X}$.*

Proof:

(i) $\Longrightarrow$ (ii) follows from Proposition 1 (i) $\Longrightarrow$ (ii).

(i) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (v) are obvious.

(iii) $\Longrightarrow$ (iv) follows as in Theorem 1 (iii) $\Longrightarrow$ (iv).

(v) $\Longrightarrow$ (ii). Suppose that there exists a nonempty bounded set A in X^* such that given $x \in X\backslash\{0\}$ there exists $F \in X^{**}$ and $r > 0$ such that every weak* slice S of A

generated by x has $\omega(F(S)) > r$. Writing $C \equiv$ co $(A \cup (-A))$ and $K \equiv C + B(X^*)$ and considering the equivalent norm p on X defined by $p(x) = \sup\{f(x) : f \in K\}$, we have as in Theorem 1 (v) $\Longrightarrow$ (ii) that p^{**} on X^{**} where $p^{**}(F) = \sup\{F(f) : f \in K\}$, is not Gâteaux differentiable at $\widehat{x}$ in the direction F. We conclude that if p^{**} is Gâteaux differentiable at some point of $\widehat{X}$, then (ii) holds.

(ii) $\Longrightarrow$ (i). Consider a nonempty closed bounded convex set K in X^*. From (ii) there exists $x \in X\backslash\{0\}$ such that given $\varepsilon > 0$ and $F \in X^{**}$ there exists $\delta(\varepsilon, F) > 0$ such that $\omega\big(F\big(S(\overline{K}^{w^*}, \widehat{x}, \delta)\big)\big) < \varepsilon$. So for $C \equiv \{f \in \overline{K}^{w^*} : f(x) = \sup \widehat{x}(\overline{K}^{w^*})\} = \bigcap_{\delta>0} S(\overline{K}^{w^*}, \widehat{x}, \delta)$ we have $\omega(F(C)) = 0$ for all $F \in X^{**}$. But this implies that C is a singleton and an extreme point of $\overline{K}^{w^*}$. Write $C \equiv \{f_0\}$. However, (ii) implies that given $\varepsilon > 0$ and $F \in X^{**}$ there exists $\delta(\varepsilon, F) > 0$ such that

$$|F(f - f_0)| < \varepsilon \quad \text{when } f \in S(\overline{K}^{w^*}, \widehat{x}, \delta).$$

But $S(\overline{K}^{w^*}, \widehat{x}, \delta) \cap K \neq \emptyset$ for all $\delta > 0$ and we deduce that $f_0 \in \overline{K}^{w} = K$. Therefore f_0 is an extreme point of K. This is sufficient to prove that X^* has the Krein–Milman property, [**2**, p.190], which in turn implies that X is an Asplund space, [**4**]. ∎

We note that the proof (ii) $\Longrightarrow$ (i) is due to Isaac Namioka, [private communication].

We are able to use Theorem 2 to deduce the following condition for a Banach space to be Asplund.

Corollary 3. [**5**, p.501]. *A Banach space X where $X^{**}/\widehat{X}$ is separable is an Asplund space.*

Proof: If $X \supseteq \ell_1$ then $\ell_1^{**}/\widehat{\ell_1}$ is topologically isomorphic to a subspace of $X^{**}/\widehat{X}$, but then $X^{**}/\widehat{X}$ is not separable. So $X \not\supseteq \ell_1$. Given an equivalent norm p on X and $x \in X$, $p_+^{**'}(\widehat{x})(F)$ is continuous in F so from Theorem 1 we have that p^{**} is Gâteaux differentiable at the points of a dense G_δ subset D of $\widehat{X}$ in all directions $X^{**}\backslash\widehat{X}$ and since $X^{**}\backslash\widehat{X}$ is dense in X^{**} we deduce that p^{**} is Gâteaux differentiable at each point of D. We conclude that X is an Asplund space. ∎

An Asplund space is also characterised by the structure of the weak* compact convex subsets in its dual, [**6**, p.86].

Given a nonempty weak* closed convex subset K of the dual X^* of a Banach space X, we say that $f \in K$ is a *weak* exposed point* of K if there exists an $x \in X \setminus \{0\}$ such that

$$f(x) = \sup \widehat{x}(K) > g(x) \quad \text{for all } g \in K, f \neq g.$$

We say that x *weak* exposes* K at f. If, when $g_n(x) \to f(x)$ as $n \to \infty$ for $\{g_n\} \subseteq K$ we have that $\{g_n\}$ is norm convergent to f, then we say that f is a *weak* strongly exposed* point of K.

PROPOSITION 3. *For a Banach space X the following are equivalent.*

(i) *X is an Asplund space,*
(ii) *every nonempty weak* closed convex subset of X^* is the weak* closed convex hull of its weak* strongly exposed points,*
(iii) *every nonempty weak* closed convex subset of X^* has at least one weak* strongly exposed point.*

A comparable characterisation can be given for a Banach space which does not contain a subspace topologically isomorphic to ℓ_1.

For a nonempty subset A of the dual X^* of a Banach space X we say that, given $F \in X^{**}\backslash\{0\}$, an element $x \in X$ *weak* F-exposes* A if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega(F(S(A, \widehat{x}, \delta))) < \varepsilon$ and in this case if $\bigcap_{\delta>0} S(A, \widehat{x}, \delta)$ is nonempty we call this set a *weak* F-exposed face* of A.

LEMMA. *Consider a continuous positive sublinear functional p on a Banach space X and the set $C \equiv \{x \in X : p(x) \leqslant 1\}$. Given $F \in X^{**}\backslash\{0\}$, the real set-valued mapping $x \mapsto F(\partial p(x))$ is single-valued and upper semi-continuous at $x_0 \in X$ if and only if $\partial p(x_0)$ is a weak* F-exposed face of $C^0 \equiv \{f \in X^* : f(x) \leqslant 1$ for all $x \in C\}$.*

PROOF: Suppose that given $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega\left(F\left(S\left(C^0, \widehat{x}_0, \delta\right)\right)\right) < \varepsilon$. Recall that $\partial p(x) \subseteq C^0$ for all $x \in X$. Now

$$\partial p(x_0) \subseteq W \equiv \{f \in X^* : f(x_0) > p(x_0) - \delta\}.$$

Since the subdifferential mapping $x \mapsto \partial p(x)$ is weak* upper semi-continuous at x_0 there exists a neighbourhood N of x_0 such that

$$\partial p(x) \subseteq W \cap C^0 = S\left(C^0, \widehat{x}_0, \delta\right) \text{ for all } x \in N.$$

So then $\omega(F(\partial p(N))) < \varepsilon$ implying that the mapping $x \mapsto F(\partial p(x_0))$ is single-valued and upper semi-continuous at x_0.

For the converse, we may assume that $\|F\| = 1$. Consider the mapping $x \mapsto F(\partial p(x))$ single-valued at x_0. Suppose that there exists a sequence $\{f_n\}$ in C^0 and $r > 0$ such that

$$f_n(x_0) \to \partial p(x_0)(x_0) = p(x_0) \text{ as } n \to \infty$$

but

$$F(f_n - \partial p(x_0)) > r \quad \text{for all } n \in \mathbb{N}.$$

Then there exists a subsequence $\{f_{n_k}\}$ such that

$$f_{n_k}(x - x_0) \leqslant p(x) - p(x_0) + \frac{1}{k^2} \text{ for all } x \in X \text{ and } k \in \mathbb{N}.$$

Then by the Brøndsted–Rockafeller Theorem, [**6**, p.48], for each $k \in \mathbb{N}$ there exist $x_k \in X$ and $f_k \in \partial p(x_k)$ such that $\|x_k - x_0\| \leqslant 1/k$ and $\|f_{n_k} - f_k\| < 1/k$. So $x_k \to x_0$ as $k \to \infty$ but

$$F(f_k - \partial p(x_0)) > \frac{r}{2} \text{ for all } k > \frac{2}{r}$$

and we conclude that the mapping $x \mapsto F(\partial p(x))$ is not upper semi-continuous at x_0. ∎

Theorem 3. *For a Banach space X, the following are equivalent.*

(i) *X does not contain a subspace topologically isomorphic to ℓ_1,*

(ii) *given $F \in X^{**}\backslash\{0\}$, every nonempty weak* compact convex subset of X^* is the weak* closed convex hull of its weak* F–exposed faces,*

(iii) *given $F \in X^{**}\backslash\{0\}$, every nonempty weak* compact convex subset of X^* has at least one weak* F–exposed face.*

Proof:

(iii) $\Rightarrow$ (i). Given any nonempty bounded set K in X^*, the weak* closed convex hull of K has weak* F–exposed faces. So K has weak* slices over which the oscillation of F is arbitrarily small. By Proposition 2 we have that X does not contain a subspace topologically isomorphic to ℓ_1.

(i) $\Rightarrow$ (ii). Consider A a nonempty weak* compact convex subset of X^*. We may assume that $0 \in A$ and we define

$$p(x) = \sup\{f(x) : x \in A\} \equiv M(x, A).$$

Then p is a continuous positive sublinear functional on X and is the gauge of $C \equiv \{x \in X : p(x) \leqslant 1\}$ and $C^0 = A$. Consider K, the weak* closed convex hull of the weak* F–exposed faces of A, and suppose that $K \neq A$. Then there exists $x \in X$ such that $M(x, K) < M(x, A)$. Since both $M(x, K)$ and $M(x, A)$ are continuous on X then $\{x \in X : M(x, K) < M(x, A)\}$ is open in X. By Theorem 1 (i) $\Leftrightarrow$ (iii), this set contains a point x_0 where the real set–valued mapping $x \mapsto F(\partial p(x))$ is single–valued and upper semi–continuous. Then by the Lemma, $\partial p(x_0)$ is a weak* F–exposed face of $C^0 = A$. Then $\partial p(x_0)(x_0) = M(x_0, A)$. But this contradicts our supposition about K.

(ii) $\Rightarrow$ (iii) is obvious. ∎

Again it is interesting to compare the characterisation given in Theorem 3 with a generalisation of the characterisation given in Proposition 3.

For a nonempty weak* closed convex subset K of the dual X^* of a Banach space X and $x \in X$ where x weak* exposes K at $f \in K$, we say that f is a *weak* weak exposed* point of K if given $\varepsilon > 0$ and $F \in X^{**}\backslash\{0\}$ there exists a $\delta(\varepsilon, F) > 0$ such that $\omega\left(F\left(S(A, \widehat{x}, \delta)\right)\right) < \varepsilon$. Clearly $f \in K$ is a weak* weak exposed point of K if and only if x weak* F exposes K at f for every $F \in X^{**}$.

From Theorem 2 and the Lemma we have the following characterisation.

THEOREM 4. *For a Banach space X the following are equivalent.*

(i) *X is an Asplund space,*

(ii) *every nonempty weak* compact convex subset of X^* is the weak* closed convex hull of its weak* weak exposed points,*

(iii) *every nonempty weak* compact convex subset of X^* has at least one weak* weak exposed point.*

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