J. Aust. Math. Soc. **95** (2013), 241–265
doi:10.1017/S1446788713000219

C^*-ALGEBRAS ASSOCIATED WITH LAMBDA-SYNCHRONIZING SUBSHIFTS AND FLOW EQUIVALENCE

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(Received 15 October 2012; accepted 10 April 2013; first published online 7 August 2013)

Communicated by A. Sims

Abstract

The class of λ-synchronizing subshifts generalizes the class of irreducible sofic shifts. A λ-synchronizing subshift can be presented by a certain λ-graph system, called the λ-synchronizing λ-graph system. The λ-synchronizing λ-graph system of a λ-synchronizing subshift can be regarded as an analogue of the Fischer cover of an irreducible sofic shift. We will study algebraic structure of the C^*-algebra associated with a λ-synchronizing λ-graph system and prove that the stable isomorphism class of the C^*-algebra with its Cartan subalgebra is invariant under flow equivalence of λ-synchronizing subshifts.

2010 *Mathematics subject classification*: primary 46L35; secondary 37B10, 46L80.

Keywords and phrases: subshifts, C^*-algebras, λ-graph systems, flow equivalence, K-groups, Bowen–Franks groups, λ-synchronization.

1. Introduction

Let Σ be a finite set with its discrete topology. We call it an alphabet and each member of it a symbol or a label. Let $\Sigma^{\mathbb{Z}}$, $\Sigma^{\mathbb{N}}$ respectively be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, $\prod_{i=1}^{\infty} \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ for $(x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ is called the full shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$, that is, $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_{\Lambda})$ is called a subshift or a symbolic dynamical system, and written simply as Λ. The theory of symbolic dynamical systems forms a basic ingredient in the theory of topological dynamical systems (see [16, 24]).

The author has introduced the notion of the λ-graph system, that is, a labeled Bratteli diagram with an additional structure called an ι-map [27]. A λ-graph system $\mathfrak{L}$ presents a subshift and yields a C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ [30]. For a subshift Λ, one may construct a λ-graph system $\mathfrak{L}^{\Lambda}$ called the canonical λ-graph system for Λ in a canonical way. It is a left Krieger cover version for a subshift. The C^*-algebra $\mathcal{O}_{\mathfrak{L}^\Lambda}$ for $\mathfrak{L}^\Lambda$ coincides with the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ ([25]; see [6]). It has been proved that the stable isomorphism class of the C^*-algebra $\mathcal{O}_\Lambda$ is invariant under not only the topological conjugacy class of Λ but also the flow equivalence class of Λ, so that the K-groups $K_i(\mathcal{O}_\Lambda)$, $i = 0, 1$, and the Ext-groups $\mathrm{Ext}^i(\mathcal{O}_\Lambda)$, $i = 0, 1$, are invariant under flow equivalence of subshifts [8, 28, 29]. The latter groups $\mathrm{Ext}^i(\mathcal{O}_\Lambda)$, $i = 0, 1$, have been defined as the Bowen–Franks groups for Λ [28, 29] (see [4, 10]). For an irreducible sofic shift, there is another important cover called the (left or right) Fischer cover. The (left) Fischer cover is an irreducible labeled graph, that is, a minimal (left)-resolving presentation, whereas the (left) Krieger cover is not necessarily irreducible.

In [23], a certain synchronizing property for subshifts called λ-synchronization was introduced. The λ-synchronizing property is weaker than the usual synchronizing property, so that irreducible sofic shifts are λ-synchronizing just as Dyck shifts, β-shifts, Morse shifts, etc. are λ-synchronizing. Many irreducible subshifts have this property. For a λ-synchronizing subshift Λ there exists a λ-graph system called the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$. The λ-synchronizing λ-graph system for an irreducible sofic shift is the λ-graph system associated with the left Fischer cover. Hence the λ-synchronizing λ-graph system of a λ-synchronizing subshift can be regarded as an analogue of the left Fischer cover of an irreducible sofic shift.

In [36], it was proved that the K-groups $K_i^\lambda(\Lambda)$, $i = 0, 1$, and the Bowen–Franks groups $BF_\lambda^i(\Lambda)$, $i = 0, 1$, for the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ of a λ-synchronizing subshift Λ are invariant under not only the topological conjugacy class but also the flow equivalence class of Λ. The groups are called the λ-synchronizing K-groups and the λ-synchronizing Bowen–Franks groups, respectively. Hence they yield flow equivalence invariants of λ-synchronizing subshifts.

In this paper, we will study the algebraic structure of the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\lambda(\Lambda)}}$ associated with the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ for Λ. The algebra is denoted by $\mathcal{O}_{\lambda(\Lambda)}$. We will first show the following theorem.

Theorem 1.1 (Theorem 3.8). *Suppose that the right one-sided subshift of a λ-synchronizing subshift Λ is homeomorphic to the Cantor set. If Λ is λ-synchronizingly transitive, the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ is simple.*

For an irreducible sofic shift Λ, the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ is always simple (Section 5), whereas the C^*-algebra $\mathcal{O}_\Lambda (= \mathcal{O}_{\mathfrak{L}^\Lambda})$ is not simple in many cases unless the sofic shift Λ is a shift of finite type (see [1]). The λ-synchronization is invariant under not only topological conjugacy but also flow equivalence [23, 36]. We will next prove the following theorem.

Theorem 1.2 (Theorem 4.17). *The stable isomorphism class of the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ with its Cartan subalgebra $\mathcal{D}_{\lambda(\Lambda)}$ is invariant under flow equivalence of λ-synchronizing subshifts.*

Therefore the stable isomorphism class of the pair $(\mathcal{O}_{\lambda(\Lambda)}, \mathcal{D}_{\lambda(\Lambda)})$ is a new invariant for flow equivalence of λ-synchronizing subshifts. Since

$$K_i^\lambda(\Lambda) = K_i(\mathcal{O}_{\lambda(\Lambda)}), \quad BF_\lambda^i(\Lambda) = \mathrm{Ext}_i(\mathcal{O}_{\lambda(\Lambda)}), \quad i = 0, 1,$$

we have a C^*-algebraic proof for the above-mentioned fact as its corollary.

Corollary 1.3 (Corollary 4.18). *The λ-synchronizing K-groups $K_i^\lambda(\Lambda)$, $i = 0, 1$, and the λ-synchronizing Bowen–Franks groups $BF_\lambda^i(\Lambda)$, $i = 0, 1$, for a λ-synchronizing subshift Λ are invariant under flow equivalence.*

Throughout the paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_+$ the set of nonnegative integers.

2. λ-synchronizing λ-graph systems

Let Λ be a subshift over Σ. We denote by $X_\Lambda (\subset \Sigma^{\mathbb{N}})$ the set of all right one-sided sequences appearing in Λ,

$$X_\Lambda = \{(x_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}} \mid (x_n)_{n\in\mathbb{Z}} \in \Lambda\},$$

which is called the right one-sided subshift for Λ. For a natural number $l \in \mathbb{N}$, we denote by $B_l(\Lambda)$ the set of all words appearing in Λ with length equal to l. Put $B_*(\Lambda) = \bigcup_{l=0}^\infty B_l(\Lambda)$ where $B_0(\Lambda) = \{\emptyset\}$ the empty word. For a word $\mu = \mu_1 \cdots \mu_k \in B_*(\Lambda)$, a right infinite sequence $x = (x_i)_{i\in\mathbb{N}} \in X_\Lambda$ and $l \in \mathbb{Z}_+$, put

$$\begin{aligned}
\Gamma_l^-(\mu) &= \{\nu_1 \cdots \nu_l \in B_l(\Lambda) \mid \nu_1 \cdots \nu_l \mu_1 \cdots \mu_k \in B_*(\Lambda)\},\\
\Gamma_l^-(x) &= \{\nu_1 \cdots \nu_l \in B_l(\Lambda) \mid (\nu_1, \ldots, \nu_l, x_1, x_2, \ldots) \in X_\Lambda\},\\
\Gamma_l^+(\mu) &= \{\omega_1 \cdots \omega_l \in B_l(\Lambda) \mid \mu_1 \cdots \mu_k \omega_1 \cdots \omega_l \in B_*(\Lambda)\},\\
\Gamma_*^+(\mu) &= \bigcup_{l=0}^\infty \Gamma_l^+(\mu).
\end{aligned}$$

A word $\mu = \mu_1 \cdots \mu_k \in B_*(\Lambda)$ for $l \in \mathbb{Z}_+$ is said to be *l-synchronizing* if for all $\omega \in \Gamma_*^+(\mu)$ the equality $\Gamma_l^-(\mu) = \Gamma_l^-(\mu\omega)$ holds. Denote by $S_l(\Lambda)$ the set of all l-synchronizing words of Λ. We say that an irreducible subshift Λ is *λ-synchronizing* if for any $\eta \in B_l(\Lambda)$ and $k \ge l$ there exists $\nu \in S_k(\Lambda)$ such that $\eta\nu \in S_{k-l}(\Lambda)$. Irreducible sofic shifts are λ-synchronizing. More generally, synchronizing subshifts are λ-synchronizing (see [3] for synchronizing subshifts). Many irreducible subshifts including Dyck shifts, β-shifts and Morse shifts are λ-synchronizing. There exists a concrete example of an irreducible subshift that is not λ-synchronizing (see [23]).

Proposition 2.1 ([36, Theorem 4.4]; see [20, 23]). *λ-synchronization is invariant under not only topological conjugacy but also flow equivalence of subshifts.*

For $\mu, \nu \in B_*(\Lambda)$, we say that μ is l-past equivalent to ν if $\Gamma_l^-(\mu) = \Gamma_l^-(\nu)$. We write this as $\mu \underset{l}{\sim} \nu$. The following lemma is straightforward.

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LEMMA 2.2 [23, 36]. *Let Λ be a λ-synchronizing subshift. Then:*

(i) *for $\mu \in S_l(\Lambda)$, there exists $\mu' \in S_{l+1}(\Lambda)$ such that $\mu \underset{l}{\sim} \mu'$;*

(ii) *for $\mu \in S_l(\Lambda)$, there exist $\beta \in \Sigma$ and $\nu \in S_{l+1}(\Lambda)$ such that $\mu \underset{l}{\sim} \beta\nu$.*

A λ-graph system is a graphical object presenting a subshift [27]. It is a generalization of a finite labeled graph and yields a C^*-algebra [30]. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ with vertex set $V = \bigcup_{l \in \mathbb{Z}_+} V_l$ and edge set $E = \bigcup_{l \in \mathbb{Z}_+} E_{l,l+1}$ with a labeling map $\lambda : E \to \Sigma$, and supplied with surjective maps $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here the vertex sets V_l, $l \in \mathbb{Z}_+$, are finite disjoint sets. Also $E_{l,l+1}$, $l \in \mathbb{Z}_+$, are finite disjoint sets. Each edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1}, respectively. Every vertex in V has a successor and every vertex in V_l for $l \in \mathbb{N}$ has a predecessor. It is then required that there exists an edge in $E_{l,l+1}$ with label α and its terminal vertex is $v \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,l}$ with label α and its terminal vertex is $\iota(v) \in V_l$. For $u \in V_{l-1}$ and $v \in V_{l+1}$, put

$$E^{\iota}_{l,l+1}(u, v) = \{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\},$$
$$E_{\iota}^{l-1,l}(u, v) = \{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}.$$

Then we require a bijective correspondence preserving their labels between $E^{\iota}_{l,l+1}(u, v)$ and $E_{\iota}^{l-1,l}(u, v)$ for each pair of vertices u, v. We call this property the local property of a λ-graph system. We call an edge in E a labeled edge and a finite sequence of connecting labeled edges a labeled path. If a labeled path γ labeled ν starts at a vertex $v \in V_l$ and ends at a vertex $u \in V_{l+n}$, we say that ν leaves v and write $s(\gamma) = v$, $t(\gamma) = u$, $\lambda(\gamma) = \nu$. We henceforth assume that $\mathfrak{L}$ is left-resolving, which means that $t(e) \neq t(f)$ whenever $\lambda(e) = \lambda(f)$ for $e, f \in E$. For a vertex $v \in V_l$ denote by $\Gamma_l^-(v)$ the predecessor set of v which is defined by the set of words with length l appearing as labeled paths from a vertex in V_0 to the vertex v. $\mathfrak{L}$ is said to be predecessor-separated if $\Gamma_l^-(v) \neq \Gamma_l^-(u)$ whenever $u, v \in V_l$ are distinct. Two λ-graph systems $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ and $\mathfrak{L}' = (V', E', \lambda', \iota')$ over Σ are said to be isomorphic if there exist bijections $\Phi_V : V \longrightarrow V'$ and $\Phi_E : E \longrightarrow E'$ satisfying $\Phi_V(V_l) = V'_l$ and $\Phi_E(E_{l,l+1}) = E'_{l,l+1}$ such that they give rise to a labeled graph isomorphism compatible to ι and ι'. We note that any essential finite directed labeled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \lambda)$ over Σ with vertex set $\mathcal{V}$, edge set $\mathcal{E}$ and labeling map $\lambda : \mathcal{E} \longrightarrow \Sigma$ gives rise to a λ-graph system $\mathfrak{L}_{\mathcal{G}} = (V, E, \lambda, \iota)$ by setting $V_l = \mathcal{V}$, $E_{l,l+1} = \mathcal{E}$, $\iota = \mathrm{id}$ for all $l \in \mathbb{Z}_+$ (see [30]).

For a λ-synchronizing subshift Λ over Σ, we have introduced a λ-graph system

$$\mathfrak{L}^{\lambda(\Lambda)} = (V^{\lambda(\Lambda)}, E^{\lambda(\Lambda)}, \lambda^{\lambda(\Lambda)}, \iota^{\lambda(\Lambda)})$$

defined by λ-synchronization of Λ as follows [23, 36]. Let $V_l^{\lambda(\Lambda)}$ be the l-past equivalence classes of $S_l(\Lambda)$. We denote by $[\mu]_l$ the equivalence class of $\mu \in S_l(\Lambda)$. For $\nu \in S_{l+1}(\Lambda)$ and $\alpha \in \Gamma_1^-(\nu)$, define a labeled edge from $[\alpha\nu]_l \in V_l^{\lambda(\Lambda)}$ to $[\nu]_l \in V_{l+1}^{\lambda(\Lambda)}$ labeled α. Such labeled edges are denoted by $E_{l,l+1}^{\lambda(\Lambda)}$. Denote by $\lambda^{\lambda(\Lambda)} : E_{l,l+1}^{\lambda(\Lambda)} \longrightarrow \Sigma$

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the labeling map. Since $S_{l+1}(\Lambda) \subset S_l(\Lambda)$, we have a natural map $[\mu]_{l+1} \in V_{l+1}^{\lambda(\Lambda)} \longrightarrow [\mu]_l \in V_l^{\lambda(\Lambda)}$ that we denote by $\iota_{l,l+1}^{\lambda(\Lambda)}$. Then $\mathfrak{L}^{\lambda(\Lambda)} = (V^{\lambda(\Lambda)}, E^{\lambda(\Lambda)}, \lambda^{\lambda(\Lambda)}, \iota^{\lambda(\Lambda)})$ defines a predecessor-separated, left-resolving λ-graph system that presents Λ. We call $\mathfrak{L}^{\lambda(\Lambda)}$ the canonical λ-synchronizing λ-graph system of Λ. The canonical λ-synchronizing λ-graph system may be characterized in an intrinsic way. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a predecessor-separated, left-resolving λ-graph system over Σ that presents a subshift Λ. Denote by $\{v_1^l, \ldots, v_{m(l)}^l\}$ the vertex set V_l at level l. For an admissible word $\nu \in B_n(\Lambda)$ and a vertex $v_i^l \in V_l$, we say that v_i^l *launches* ν if the following two conditions hold.

(i) There exists a path labeled ν in $\mathfrak{L}$ leaving the vertex v_i^l and ending at a vertex in V_{l+n}.

(ii) The word ν does not leave any other vertex in V_l than v_i^l.

We call the vertex v_i^l the launching vertex for ν. We set

$$S_{v_i^l}(\Lambda) = \{\nu \in B_*(\Lambda) \mid v_i^l \text{ launches } \nu\}.$$

Definition 2.3. A λ-graph system $\mathfrak{L}$ is said to be λ-*synchronizing* if for any $l \in \mathbb{N}$ and any vertex $v_i^l \in V_l$, there exists a word $\nu \in B_*(\Lambda)$ such that v_i^l launches ν.

In the following lemma we retain the above notation.

Lemma 2.4 [36, Lemma 3.4]. *Assume that $\mathfrak{L} = (V, E, \lambda, \iota)$ is λ-synchronizing. Then:*

(i) $\bigsqcup_{i=1}^{m(l)} S_{v_i^l}(\Lambda) = S_l(\Lambda)$;

(ii) *the l-past equivalence classes of $S_l(\Lambda)$ are $S_{v_i^l}(\Lambda)$, $i = 1, \ldots, m(l)$;*

(iii) *for any l-synchronizing word $w \in S_l(\Lambda)$, there exists a vertex $v_{i(\omega)}^l \in V_l$ such that $v_{i(\omega)}^l$ launches ω and $\Gamma_l^-(\omega) = \Gamma_l^-(v_{i(\omega)}^l)$.*

Definition 2.5. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ is said to be ι-*irreducible* if for any two vertices $v, u \in V_l$ and a labeled path γ starting at u, there exist a labeled path from v to a vertex $u' \in V_{l+n}$ such that $\iota^n(u') = u$, and a labeled path γ' starting at u' such that $\iota^n(t(\gamma')) = t(\gamma)$ and $\lambda(\gamma') = \lambda(\gamma)$, where $t(\gamma'), t(\gamma)$ denote the terminal vertices of γ', γ respectively and $\lambda(\gamma'), \lambda(\gamma)$ the words labeled by γ', γ respectively.

We denote by Λ the subshift presented by a λ-graph system $\mathfrak{L}$. It has been proved that if $\mathfrak{L}$ is ι-irreducible, then Λ is irreducible [36, Lemma 3.5]. If, in particular, $\mathfrak{L}$ is λ-synchronizing, the subshift Λ is irreducible if and only if $\mathfrak{L}$ is ι-irreducible [36, Proposition 3.7]. We then have the following proposition.

Proposition 2.6 [36, Proposition 3.8]. *A subshift Λ is λ-synchronizing if and only if there exists a left-resolving, predecessor-separated, ι-irreducible, λ-synchronizing λ-graph system that presents Λ.*

Theorem 2.7 [36, Theorem 3.9]. *For a λ-synchronizing subshift Λ, there exists a unique left-resolving, predecessor-separated, ι-irreducible, λ-synchronizing λ-graph system that presents Λ. The unique λ-synchronizing λ-graph system is the canonical λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ for Λ.*

As in Theorem 2.7, the canonical λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ has a unique property. We henceforth call $\mathfrak{L}^{\lambda(\Lambda)}$ the λ-synchronizing λ-graph system for Λ. We say that a λ-graph system $\mathfrak{L}$ is *minimal* if there is no proper λ-graph subsystem of $\mathfrak{L}$ that presents Λ. This means that if $\mathfrak{L}'$ is a λ-graph subsystem of $\mathfrak{L}$ and presents the same subshift as the subshift presented by $\mathfrak{L}$, then $\mathfrak{L}'$ coincides with $\mathfrak{L}$. The λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ of a λ-synchronizing subshift Λ is minimal [36, Proposition 3.10].

3. λ-synchronizing C^*-algebras

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a left-resolving predecessor-separated λ-graph system over Σ and Λ the presented subshift by $\mathfrak{L}$. We denote by $\{v_1^l, \ldots, v_{m(l)}^l\}$ the vertex set V_l. Define the transition matrices $A_{l,l+1}, I_{l,l+1}$ of $\mathfrak{L}$ by setting, for $i = 1, 2, \ldots, m(l)$, $j = 1, 2, \ldots, m(l+1)$, $\alpha \in \Sigma$,

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise.} \end{cases}$$

The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is realized as the universal unital C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \ldots, m(l), \ l \in \mathbb{Z}_+$, subject to the following operator relations called $(\mathfrak{L})$:

$$\sum_{\beta \in \Sigma} S_\beta S_\beta^* = 1,$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \quad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) E_j^{l+1},$$

$$S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*,$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) E_j^{l+1},$$

for $\alpha \in \Sigma, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+$. It is nuclear and belongs to the UCT class [30, Proposition 5.6]. For a word $\mu = \mu_1 \cdots \mu_k \in B_k(\Lambda)$, we set $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$. The algebra of all finite linear combinations of the elements of the form

$$S_\mu E_i^l S_\nu^* \quad \text{for } \mu, \nu \in B_*(\Lambda), i = 1, \ldots, m(l), l \in \mathbb{Z}_+$$

is a dense $*$-subalgebra of $\mathcal{O}_{\mathfrak{L}}$. Let us denote by $\mathcal{A}_{\mathfrak{L}}$ the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the projections $E_i^l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+$, which is a commutative AF-algebra. For a vertex $v_i^l \in V_l$, put

$$\Gamma_\infty^+(v_i^l) = \{(\alpha_1, \alpha_2, \ldots,) \in \Sigma^{\mathbb{N}} \mid \text{there exists an edge } e_{n,n+1} \in E_{n,n+1} \text{ for } n \geq l \\ \text{such that } v_i^l = s(e_{l,l+1}), t(e_{n,n+1}) = s(e_{n+1,n+2}), \lambda(e_{n,n+1}) = \alpha_{n-l+1}\},$$

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the set of all label sequences in $\mathfrak{L}$ starting at v_i^l. We say that $\mathfrak{L}$ satisfies condition (I) if for each $v_i^l \in V$, the set $\Gamma_\infty^+(v_i^l)$ contains at least two distinct sequences. Under condition (I), the algebra $\mathcal{O}_\mathfrak{L}$ can be realized as the unique C^*-algebra subject to the relations $(\mathfrak{L})$ [30, Theorem 4.3]. A λ-graph system $\mathfrak{L}$ is said to λ-irreducible if for an ordered pair of vertices $u, v \in V_l$, there exists a number $L_l(u, v) \in \mathbb{N}$ such that for a vertex $w \in V_{l+L_l(u,v)}$ with $\iota^{L_l(u,v)}(w) = u$, there exists a path γ in $\mathfrak{L}$ such that $s(\gamma) = v$, $t(\gamma) = w$, where $\iota^{L_l(u,v)}$ means the $L_l(u, v)$-times compositions of ι, and $s(\gamma), t(\gamma)$ denote the source vertex and the terminal vertex of γ, respectively [33]. If $\mathfrak{L}$ is λ-irreducible with condition (I), then the C^*-algebra $\mathcal{O}_\mathfrak{L}$ is simple ([30, Theorem 4.7], [33]).

Proposition 3.1. *Let Λ be a λ-synchronizing subshift over Σ and $\mathfrak{L}^{\lambda(\Lambda)}$ the λ-synchronizing λ-graph system for Λ. Then the right one-sided subshift X_Λ of Λ is homeomorphic to the Cantor set if and only if $\mathfrak{L}^{\lambda(\Lambda)}$ satisfies condition* (I).

Proof. Assume that the right one-sided subshift X_Λ of Λ is homeomorphic to the Cantor set. For a vertex $v_i^l \in V_l^{\lambda(\Lambda)}$, take a l-synchronizing word $\mu = \mu_1 \cdots \mu_k \in S_l(\Lambda)$ such that v_i^l launches μ. Take an infinite sequence $x \in X_\Lambda$ such that $\mu \in \Gamma_k^-(x)$. Since X_Λ is homeomorphic to the Cantor set, any neighborhood of μx in X_Λ contains an element that is different from μx. Hence there exists an infinite sequence $x' \in X_\Lambda$ such that $\mu x' \in X_\Lambda$ and $x \neq x'$. As μ must leave the vertex v_i^l, both the sequences μx and $\mu x'$ are contained in $\Gamma_\infty^+(v_i^l)$ so that $\mathfrak{L}^{\lambda(\Lambda)}$ satisfies condition (I).

Conversely, assume that $\mathfrak{L}^{\lambda(\Lambda)}$ satisfies condition (I). Since X_Λ is a compact, totally disconnected metric space, it suffices to show that X_Λ is perfect. For any $x = (x_1, x_2, \ldots) \in X_\Lambda$ and a word $\mu_1 \cdots \mu_k$ with $\mu_1 = x_1, \ldots, \mu_k = x_k$, consider a cylinder set $U_\mu = \{(y_n)_{n\in\mathbb{N}} \in X_\Lambda \mid y_1 = \mu_1, \ldots, y_k = \mu_k\}$. Take an infinite path $(e_n)_{n\in\mathbb{N}}$ in $\mathfrak{L}^{\lambda(\Lambda)}$ labeled x such that $\lambda(e_n) = x_n$, $t(e_n) = s(e_{n+1})$, $n \in \mathbb{N}$. Let us denote by $v_i^k \in V_k^{\lambda(\Lambda)}$ the terminal vertex of the edge e_k. Since the follower set $\Gamma_\infty^+(v_i^k)$ of v_i^k has at least two distinct sequences, there exists $x' = (x'_{k+1}, x'_{k+2}, \ldots) \in \Gamma_\infty^+(v_i^k)$ such that $x' \neq (x_{k+1}, x_{k+2}, \ldots)$. As x' starts at v_i^k, the right one-sided sequence $\mu x' = (\mu_1, \ldots, \mu_k, x'_{k+1}, x'_{k+2}, \ldots)$ is contained in X_Λ and hence in U_μ. One then sees that x is a cluster point in X_Λ. □

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a left-resolving, predecessor-separated λ-graph system over Σ that presents a λ-synchronizing subshift Λ. Let $S_\alpha, \alpha \in \Sigma$ and $E_i^l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+$, be the generating partial isometries and the projections in $\mathcal{O}_\mathfrak{L}$ satisfying the relations $(\mathfrak{L})$. If $\mathfrak{L}$ is the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ for Λ, the algebra $\mathcal{O}_\mathfrak{L}$ is denoted by $\mathcal{O}_{\lambda(\Lambda)}$. We will study the algebraic structure of the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ of a λ-synchronizing subshift Λ.

Lemma 3.2. *If $\mathfrak{L}$ is the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$, then:*

(i) *for a vertex $v_i^l \in V_l$, there exists a word $\mu \in S_l(\Lambda)$ such that $E_i^l \geq S_\mu S_\mu^*$;*

(ii) *for a word $\mu \in S_l(\Lambda)$, there exists a unique vertex $v_i^l \in V_l^{\lambda(\Lambda)}$ such that $E_i^l \geq S_\mu S_\mu^*$.*

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Proof. (i) For a vertex $v_i^l \in V_l$, take a word $\mu \in S_l(\Lambda)$ such that v_i^l launches μ. Since the word μ does not leave any other vertex in V_l than v_i^l, we have $S_\mu^* E_j^l S_\mu = 0$ for $j \neq i$ so that $S_\mu S_\mu^* E_j^l = 0$ for $j \neq i$. Let $n = |\mu|$. It then follows that

$$E_i^l = \sum_{\nu \in B_n(\Lambda)} S_\nu S_\nu^* E_i^l \geq S_\mu S_\mu^* E_i^l = \sum_{j=1}^{m(l)} S_\mu S_\mu^* E_j^l = S_\mu S_\mu^*.$$

(ii) For a word $\mu \in S_l(\Lambda)$, put $v_i^l = [\mu]_l \in V_l^{\lambda(\Lambda)}$. Since v_i^l launches μ, we have $S_\mu^* E_j^l S_\mu = 0$ for $j \neq i$ so that $S_\mu S_\mu^* E_j^l = 0$ for $j \neq i$. As in the above discussions, we have $E_i^l \geq S_\mu S_\mu^*$. If there exists $j = 1, \ldots, m(l)$ such that $E_j^l \geq S_\mu S_\mu^*$, we have $S_\mu^* E_j^l S_\mu \geq S_\mu^* S_\mu \neq 0$ so that $S_\mu^* E_j^l S_\mu \neq 0$. Hence there exists a path in $\mathfrak{L}^{\lambda(\Lambda)}$ labeled μ that leaves v_j^l. Since v_i^l launches μ, we have $j = i$. □

The following proposition describes a C^*-algebraic characterization for λ-synchronization of a λ-graph system.

Proposition 3.3. *A λ-graph system $\mathfrak{L}$ is λ-synchronizing if and only if for every $v_i^l \in V_l$, there exists a word $\mu \in S_l(\Lambda)$ such that $E_i^l \geq S_\mu S_\mu^*$ in $O_{\mathfrak{L}}$.*

Proof. Since the λ-synchronizing λ-graph system for Λ is unique and it is $\mathfrak{L}^{\lambda(\Lambda)}$, the only if part has been proved in the preceding lemma. We will prove the if part. For a vertex $v_i^l \in V_l$, there exists a word $\mu = \mu_1 \ldots \mu_n \in S_l(\Lambda)$ such that $E_i^l \geq S_\mu S_\mu^*$. Hence we have $S_\mu^* E_i^l S_\mu \neq 0$ so that the word μ leaves the vertex v_i^l and hence $\Gamma_l^-(v_i^l) \subset \Gamma_l^-(\mu)$. For $\xi \in \Gamma_l^-(\mu)$ we have $S_\xi E_i^l S_\xi^* \geq S_\xi S_\mu S_\mu^* S_\xi^* \neq 0$ so that $\xi \in \Gamma_l^-(v_i^l)$. This implies that $\Gamma_l^-(\mu) \subset \Gamma_l^-(v_i^l)$, so that

$$\Gamma_l^-(v_i^l) = \Gamma_l^-(\mu). \tag{3.1}$$

Suppose that μ leaves v_j^l. Take a path labeled μ in $\mathfrak{L}$ from v_j^l to $v_{j'}^{l+n} \in V_{l+n}$. By the hypothesis, there exists $\nu \in S_{l+n}(\Lambda)$ for the vertex $v_{j'}^{l+n}$ such that $E_{j'}^{l+n} \geq S_\nu S_\nu^*$. By a similar argument to the above, we know that

$$\Gamma_{l+n}^-(v_{j'}^{l+n}) = \Gamma_{l+n}^-(\nu). \tag{3.2}$$

One then sees that

$$\Gamma_l^-(v_j^l) = \Gamma_l^-(\mu\nu). \tag{3.3}$$

One indeed sees that $\xi\mu \in \Gamma_{l+n}^-(v_{j'}^{l+n})$ for $\xi \in \Gamma_l^-(v_j^l)$. By (3.2), we have $\xi\mu \in \Gamma_{l+n}^-(\nu)$ so that $\xi \in \Gamma_l^-(\mu\nu)$. Conversely, for $\eta \in \Gamma_l^-(\mu\nu)$, we have $\eta\mu \in \Gamma_{l+n}^-(\nu)$ so that by (3.2) $\eta\mu \in \Gamma_{l+n}^-(v_{j'}^{l+n})$. As $\mathfrak{L}$ is left-resolving, we have $\eta \in \Gamma_l^-(v_j^l)$. Hence we have (3.3). Now we know that $\Gamma_l^-(\mu\nu) = \Gamma_l^-(\mu)$, so that

$$\Gamma_l^-(v_j^l) = \Gamma_l^-(\mu). \tag{3.4}$$

By (3.1) and (3.4), we have

$$\Gamma_l^-(v_i^l) = \Gamma_l^-(v_j^l).$$

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Since $\mathfrak{L}$ is left-resolving, we obtain that $v_i^l = v_j^l$ and hence v_i^l launches μ. Thus $\mathfrak{L}$ is λ-synchronizing. □

The following lemmas are stated in terms of the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ associated with the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ of a λ-synchronizing subshift Λ.

LEMMA 3.4. *For $\xi, \eta \in B_*(\Lambda)$, we have $\Gamma_*^+(\xi) = \Gamma_*^+(\eta)$ if and only if $S_\xi^* S_\xi = S_\eta^* S_\eta$.*

PROOF. Let $p = |\xi|$, $q = |\eta|$. We may assume that $p \le q$. Let $V_{t(\xi)}^p$ be the set of all terminal vertices in V_p of paths in $\mathfrak{L}^{\lambda(\Lambda)}$ labeled ξ, that is,

$$V_{t(\xi)}^p = \{v_j^p \in V_p \mid \xi \in \Gamma_p^-(v_j^p)\}.$$

Denote by $\xi(p)$ the cardinal number of $V_{t(\xi)}^p$. We write $V_{t(\xi)}^p = \{v_{j_1}^p, \ldots, v_{j_{\xi(p)}}^p\}$. Similarly, let us denote by $V_{t(\eta)}^q$ the set of all terminal vertices in V_q of paths in $\mathfrak{L}^{\lambda(\Lambda)}$ labeled η. Denote by $\eta(q)$ the cardinal number of $V_{t(\eta)}^q$. We write $V_{t(\eta)}^q = \{v_{k_1}^q, \ldots, v_{k_{\eta(q)}}^q\}$. By the relations $(\mathfrak{L})$, we see that

$$S_\xi^* S_\xi = E_{j_1}^p + \cdots + E_{j_{\xi(p)}}^p, \quad S_\eta^* S_\eta = E_{k_1}^q + \cdots + E_{k_{\eta(q)}}^q.$$

We set

$$\iota^{q-p}(V_{t(\eta)}^q) = \{\iota^{q-p}(v_{k_1}^q), \ldots, \iota^{q-p}(v_{k_{\eta(q)}}^q)\} \subset V_p,$$
$$\iota^{p-q}(V_{t(\xi)}^p) = \{v_k^q \in V_q \mid \iota^{q-p}(v_k^q) \in V_{t(\xi)}^p\} \subset V_q.$$

We then have $S_\xi^* S_\xi = S_\eta^* S_\eta$ if and only if $\iota^{p-q}(V_{t(\xi)}^p) = V_{t(\eta)}^q$.

Now assume that $\Gamma_*^+(\xi) = \Gamma_*^+(\eta)$. For $v_k^q \in V_{t(\eta)}^q$, take $\nu(k) \in S_q(\Lambda)$ such that v_k^q launches $\nu(k)$. It is easy to see that $\iota^{q-p}(v_k^q)$ launches $\nu(k)$. Since $\nu(k) \in \Gamma_*^+(\eta)$, we have $\nu(k) \in \Gamma_*^+(\xi)$ so that $\nu(k)$ leaves a vertex in $V_{t(\xi)}^p$. As $\iota^{q-p}(v_k^q)$ is the only vertex which $\nu(k)$ leaves, we have $\iota^{q-p}(v_k^q) \in V_{t(\xi)}^p$. Hence we have $\iota^{q-p}(V_{t(\eta)}^q) \subset V_{t(\xi)}^p$ so that $V_{t(\eta)}^q \subset \iota^{p-q}(V_{t(\xi)}^p)$. For the other inclusion relation, take an arbitrary vertex $v_k^p \in \iota^{p-q}(V_{t(\xi)}^p)$ and $\mu(q) \in S_q(\Lambda)$ such that v_k^p launches $\mu(q)$. The word $\mu(q)$ leaves $\iota^{q-p}(v_k^q)$ and $\iota^{q-p}(v_k^q)$ launches $\mu(q)$. As $\mu(q) \in \Gamma_*^+(\xi)$, we have $\mu(q) \in \Gamma_*^+(\eta)$ so that there exists a vertex $v_{k_n}^q \in V_{t(\eta)}^q$ such that $\mu(q)$ leaves $v_{k_n}^q$. Therefore we have $v_k^q = v_{k_n}^q$ and hence $v_k^q \in V_{t(\eta)}^q$ so that $\iota^{p-q}(V_{t(\xi)}^p) \subset V_{t(\eta)}^q$. This implies that $S_\xi^* S_\xi = S_\eta^* S_\eta$.

Conversely, assume that the equality $S_\xi^* S_\xi = S_\eta^* S_\eta$ holds so that $\iota^{p-q}(V_{t(\xi)}^p) = V_{t(\eta)}^q$. By the local property of λ-graph system, we can easily see that the set of followers of $V_{t(\xi)}^p$ coincides with the set of followers of $V_{t(\eta)}^q$. This implies that $\Gamma_*^+(\xi) = \Gamma_*^+(\eta)$. □

For $\mu, \nu \in B_*(\Lambda)$, we write $\mu \succ \nu$ if there exists a word $\eta \in B_*(\Lambda)$ such that $\Gamma_*^+(\nu) = \Gamma_*^+(\mu\eta\nu)$. The following lemma follows from the preceding lemma.

LEMMA 3.5. *For $\mu, \nu \in B_*(\Lambda)$, the following three conditions are equivalent.*

(i) $\mu \succ \nu$.
(i) *There exists a word $\eta \in B_*(\Lambda)$ such that $S_\nu^* S_\nu = S_\nu^* S_\eta^* S_\mu^* S_\mu S_\eta S_\nu$ in $\mathcal{O}_{\lambda(\Lambda)}$.*
(iii) *There exists a word $\eta \in B_*(\Lambda)$ such that $S_\nu S_\nu^* \le S_\eta^* S_\mu^* S_\mu S_\eta$ in $\mathcal{O}_{\lambda(\Lambda)}$.*

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Proof. The equivalence between (i) and (ii) follows from Lemma 3.4. It is clear that the equality $S_\nu^* S_\nu = S_\nu^* S_\eta^* S_\mu^* S_\mu S_\eta S_\nu$ is equivalent to the inequality $S_\nu S_\nu^* \leq S_\eta^* S_\mu^* S_\mu S_\eta$. □

Definition 3.6. A λ-synchronizing subshift Λ is said to be *synchronizingly transitive* if for any two words $\mu, \nu \in B_*(\Lambda)$, both the relations $\mu > \nu$ and $\nu > \mu$ hold.

We note that the λ-irreduciblity for $\mathfrak{L}$ is rephrased in terms of the algebra $\mathcal{O}_{\mathfrak{L}}$ as the property that for any E_i^l, $i = 1, \ldots, m(l)$, there exists $n \in \mathbb{N}$ such that $\sum_{k=1}^n \lambda_{\mathfrak{L}}^k(E_i^l) \geq 1$, where $\lambda_{\mathfrak{L}}^k(X) = \sum_{\mu \in B_k(\Lambda)} S_\mu^* X S_\mu$ for $X \in \mathcal{A}_{\mathfrak{L}}$ [33].

Lemma 3.7. *If Λ is synchronizingly transitive, then $\mathfrak{L}^{\lambda(\Lambda)}$ is λ-irreducible.*

Proof. Take an ordered pair $v_i^l, v_j^l \in V_l$ of vertices. Since Λ is λ-synchronizing, by Lemma 3.2, there exists $\mu \in S_l(\Lambda)$ such that v_i^l launches μ so that $E_i^l \geq S_\mu S_\mu^*$. For the vertex v_j^l, take a word $\nu \in B_l(\Lambda)$ such that $\nu \in \Gamma_l^-(v_j^l)$ so that $S_\nu^* S_\nu \geq E_j^l$. Now Λ is synchronizingly transitive so that

$$S_\nu^* S_\eta^* S_\mu^* S_\mu S_\eta S_\nu = S_\nu^* S_\nu$$

for some $\eta \in B_*(\Lambda)$, and hence

$$S_\nu^* S_\eta^* S_\mu^* E_i^l S_\mu S_\eta S_\nu \geq S_\nu^* S_\nu \geq E_j^l.$$

Put $k = |\mu\eta\nu|$. Then $\lambda_{\mathfrak{L}^{\lambda(\Lambda)}}^k(E_i^l) \geq E_j^l$. Thus we may find $n \in \mathbb{N}$ such that

$$\sum_{k=1}^n \lambda_{\mathfrak{L}^{\lambda(\Lambda)}}^k(E_i^l) \geq 1. \qquad \square$$

Theorem 3.8. *Let Λ be a λ-synchronizing subshift over Σ. Assume that the right one-sided subshift X_Λ of Λ is homeomorphic to the Cantor set. If Λ is synchronizingly transitive, then the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ associated with the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ for Λ is simple.*

Proof. Since X_Λ is homeomorphic to the Cantor set, the λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ satisfies condition (I). By the preceding proposition, the synchronizing transitivity of Λ implies that $\mathfrak{L}^{\lambda(\Lambda)}$ is λ-irreducible so that the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ is simple by [30, Theorem 4.7]. □

4. Flow equivalence and λ-synchronizing C^*-algebras

It has been proved that λ-synchronization is invariant under flow equivalence [36]. The proof uses Parry and Sullivan's result [37] which says that the flow equivalence relation on homeomorphisms of the Cantor set is generated by topological conjugacy and expansion of symbols. Let Λ be a subshift over the alphabet $\Sigma = \{1, 2, \ldots, N\}$. A new subshift $\widetilde{\Lambda}$ over the alphabet $\widetilde{\Sigma} = \{0, 1, 2, \ldots, N\}$ is defined as the subshift

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consisting of all bi-infinite sequences of $\widetilde{\Sigma}$ obtained by replacing the symbol 1 in a bi-infinite sequence in the subshift Λ by the word 01. This operation is called expansion. Parry and Sullivan's result, stated above, is the following lemma.

Lemma 4.1 [37]. *The flow equivalence relation of subshifts is generated by topological conjugacy and expansion* $\Lambda \to \widetilde{\Lambda}$.

In [36], it has been proved that the λ-synchronizing K-groups $K_0^\lambda(\Lambda)$, $K_1^\lambda(\Lambda)$ and the λ-synchronizing Bowen–Franks groups $BF_\lambda^0(\Lambda)$, $BF_\lambda^1(\Lambda)$ for a λ-synchronizing subshift Λ are invariant under flow equivalence of subshifts. The groups $K_0^\lambda(\Lambda)$, $K_1^\lambda(\Lambda)$ and the Bowen–Franks groups $BF_\lambda^0(\Lambda)$, $BF_\lambda^1(\Lambda)$ are realized as the K-groups $K_0(\mathcal{O}_{\lambda(\Lambda)})$, $K_1(\mathcal{O}_{\lambda(\Lambda)})$ and the Ext-groups $\mathrm{Ext}^0(\mathcal{O}_{\lambda(\Lambda)})$, $\mathrm{Ext}^1(\mathcal{O}_{\lambda(\Lambda)})$ for the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ associated with the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$. If the algebra $\mathcal{O}_{\lambda(\Lambda)}$ is simple and purely infinite, the K-groups $K_0(\mathcal{O}_{\lambda(\Lambda)})$, $K_1(\mathcal{O}_{\lambda(\Lambda)})$ determine the stable isomorphism class of $\mathcal{O}_{\lambda(\Lambda)}$ by the structure theorem of purely infinite simple C^*-algebras [14, 15, 38].

In this section, we will prove that the stable isomorphism class of the pair $(\mathcal{O}_{\lambda(\Lambda)}, \mathcal{D}_{\lambda(\Lambda)})$ of $\mathcal{O}_{\lambda(\Lambda)}$ with its Cartan subalgebra $\mathcal{D}_{\lambda(\Lambda)}$ is invariant under flow equivalence of λ-synchronizing subshifts. The outline of the proof essentially follows the proof of [28, Theorem 9.3]. As there are many technical differences between the proofs, we will give a complete proof. We will not assume simplicity of the algebra $\mathcal{O}_{\lambda(\Lambda)}$. As a result, we also give a C^*-algebraic proof of the above invariance of the groups $K_0^\lambda(\Lambda)$, $K_1^\lambda(\Lambda)$ and the Bowen–Franks groups $BF_\lambda^0(\Lambda)$, $BF_\lambda^1(\Lambda)$ under flow equivalence.

Let Λ be a λ-synchronizing subshift over $\Sigma = \{1, 2, \ldots, N\}$. Let S_i, $i \in \Sigma$, and E_i^l, $i = 1, \ldots, m(l)$, $l \in \mathbb{Z}_+$, be the generating partial isometries and the projections in the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ satisfying the relations $(\mathfrak{L}^{\lambda(\Lambda)})$. The Cartan subalgebra $\mathcal{D}_{\lambda(\Lambda)}$ is defined to be the C^*-subalgebra of $\mathcal{O}_{\lambda(\Lambda)}$ generated by the projections of the form $S_\mu E_i^l S_\mu^*$, $i = 1, \ldots, m(l)$, $\mu \in B_*(\Lambda)$, which is a regular maximal abelian subalgebra in $\mathcal{O}_{\lambda(\Lambda)}$ if the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ satisfies condition (I). Consider the subshift $\widetilde{\Lambda}$ over $\widetilde{\Sigma} = \{0, 1, \ldots, N\}$ that is obtained from Λ by replacing 1 in Λ by 01. It has been proved in [36] that $\widetilde{\Lambda}$ is λ-synchronizing. Denote by $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ the C^*-algebra associated with the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\widetilde{\Lambda})}$ for $\widetilde{\Lambda}$. Similarly, let $\widetilde{S}_i$, $i \in \widetilde{\Sigma}$, and $\widetilde{E}_i^l$, $i = 1, \ldots, \tilde{m}(l)$, $l \in \mathbb{Z}_+$, be the generating partial isometries and the projections in the C^*-algebra $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ satisfying the relations $(\mathfrak{L}^{\lambda(\widetilde{\Lambda})})$. We set the partial isometries

$$s_1 = \widetilde{S}_0 \widetilde{S}_1, \quad s_i = \widetilde{S}_i, \quad \text{for } i = 2, \ldots, N,$$

and the projection

$$P = \widetilde{S}_0 \widetilde{S}_0^* + \widetilde{S}_2 \widetilde{S}_2^* + \widetilde{S}_3 \widetilde{S}_3^* + \cdots + \widetilde{S}_N \widetilde{S}_N^* = 1 - \widetilde{S}_1 \widetilde{S}_1^*$$

in $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$.

Lemma 4.2. $\widetilde{S}_0^* \widetilde{S}_0 = \widetilde{S}_1 \widetilde{S}_1^*$ *and hence* $s_1 s_1^* = \widetilde{S}_0 \widetilde{S}_0^*$, $s_1^* s_1 = \widetilde{S}_1^* \widetilde{S}_1$.

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PROOF. We note that the set $V_0^{\lambda(\widetilde{\Lambda})}$ is a singleton. There exists a unique vertex $v_{j_0}^1$ in $V_1^{\lambda(\widetilde{\Lambda})}$ such that the symbol 0 goes to $v_{j_0}^1$ from $V_0^{\lambda(\widetilde{\Lambda})}$. The vertex $v_{j_0}^1$ is the 1-past equivalence class $[1\mu]_1$ for a word $1\mu \in B_*(\widetilde{\Lambda})$. It launches the symbol 1. Since 1 is the only symbol which leaves $v_{j_0}^1$, we see that $\widetilde{S}_\alpha^* \widetilde{E}_{j_0}^1 \widetilde{S}_\alpha \neq 0$ if and only if $\alpha = 1$. It then follows that

$$\widetilde{E}_{j_0}^1 = \sum_{\alpha \in \widetilde{\Sigma}} \widetilde{S}_\alpha \widetilde{S}_\alpha^* \widetilde{E}_{j_0}^1 = \widetilde{S}_1 \widetilde{S}_1^* \widetilde{E}_{j_0}^1.$$

Hence, $\widetilde{E}_{j_0}^1 \leq \widetilde{S}_1 \widetilde{S}_1^*$. Since the inequality $\widetilde{E}_{j_0}^1 \geq \widetilde{S}_1 \widetilde{S}_1^*$ is clear,

$$\widetilde{E}_{j_0}^1 = \widetilde{S}_1 \widetilde{S}_1^*.$$

As $v_{j_0}^1$ is the unique vertex in $V_1^{\lambda(\widetilde{\Lambda})}$ such that the symbol 0 goes to $v_{j_0}^1$, we have $\widetilde{S}_0^* \widetilde{S}_0 = \widetilde{E}_{j_0}^1$. The equalities $s_1 s_1^* = \widetilde{S}_0 \widetilde{S}_0^*$, $s_1^* s_1 = \widetilde{S}_1^* \widetilde{S}_1$ are obvious. □

LEMMA 4.3. (i) $P = \sum_{j=1}^N s_j s_j^*$.
(ii) $P \geq s_\mu^* s_\mu$ *for all* $\mu \in B_l(\Lambda)$, $l \in \mathbb{N}$.
(iii) $\sum_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu \geq P$ *for all* $l \in \mathbb{N}$.

PROOF. (i) Since $\widetilde{S}_0 \widetilde{S}_0^* = s_1 s_1^*$, the assertion is clear.

(ii) Since $P = 1 - \widetilde{E}_{j_0}^1$, it suffices to show that $\widetilde{E}_{j_0}^1 \perp s_\mu^* s_\mu$ for $\mu = \mu_1 \cdots \mu_l \in B_l(\Lambda)$. If $\mu_l \neq 1$, then $s_{\mu_l} = \widetilde{S}_{\mu_l}$ so that $s_{\mu_l} \widetilde{S}_1 = \widetilde{S}_{\mu_l} \widetilde{S}_1 = 0$. If $\mu_l = 1$, then $s_{\mu_l} = \widetilde{S}_0 \widetilde{S}_1$ so that $s_{\mu_l} \widetilde{S}_1 = \widetilde{S}_0 \widetilde{S}_1 \widetilde{S}_1 = 0$. In any case we have $s_{\mu_l} \widetilde{S}_1 = 0$ so that $s_\mu^* s_\mu \widetilde{E}_{j_0}^1 = 0$.

(iii) We will first prove that $\sum_{i=1}^N s_i^* s_i \geq P$. We know that $\widetilde{S}_i^* \widetilde{S}_i = s_i^* s_i$ for $i = 1, \ldots, N$ and $\widetilde{S}_0^* \widetilde{S}_0 = \widetilde{S}_1 \widetilde{S}_1^* = 1 - P$. Since $\sum_{i=0}^N \widetilde{S}_i^* \widetilde{S}_i \geq 1$ in $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$, one obtains

$$\sum_{i=0}^N \widetilde{S}_i^* \widetilde{S}_i = 1 - P + \sum_{i=1}^N s_i^* s_i \geq 1$$

so that $\sum_{i=1}^N s_i^* s_i \geq P$. Suppose that the inequality $\sum_{\mu \in B_k(\Lambda)} s_\mu^* s_\mu \geq P$ holds for some $k \in \mathbb{N}$. It then follows that

$$\begin{aligned}\sum_{\nu \in B_{k+1}(\Lambda)} s_\nu^* s_\nu &= \sum_{i=1}^N s_i^* \Big(\sum_{\mu \in B_k(\Lambda)} s_\mu^* s_\mu \Big) s_i \\ &\geq \sum_{i=1}^N s_i^* P s_i = \sum_{i,j=1}^N s_i^* s_j s_j^* s_i = \sum_{i=1}^N s_i^* s_i \geq P.\end{aligned}$$

Hence we have the desired inequalities. □

In the λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$, recall that the set $\Gamma_l^-(v_i^l)$ for a vertex v_i^l in V_l denotes the predecessor of v_i^l which is the set of words of $B_l(\Lambda)$ presented by labeled paths

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terminating at v_i^l. Put the projections for $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$,

$$e_i^l = \prod_{\mu \in \Gamma_l^-(v_i^l)} s_\mu^* s_\mu \prod_{\nu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l)} (P - s_\nu^* s_\nu).$$

For $\mu \in B_*(\Lambda)$, put

$$s_\mu^* s_\mu^1 = s_\mu^* s_\mu, \quad s_\mu^* s_\mu^{-1} = P - s_\mu^* s_\mu.$$

For $v_i^l \in V_l^{\lambda(\Lambda)}$, define a function $f_i^l : B_l(\Lambda) \longrightarrow \{1, -1\}$ by setting

$$f_i^l(\mu) = \begin{cases} 1 & \text{if } \mu \in \Gamma_l^-(v_i^l), \\ -1 & \text{if } \mu \notin \Gamma_l^-(v_i^l), \end{cases}$$

so that

$$e_i^l = \prod_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu^{f_i^l(\mu)}.$$

Denote by $\{1, -1\}^{B_l(\Lambda)}$ the set of all functions from $B_l(\Lambda)$ to $\{1, -1\}$.

Lemma 4.4. *For $\epsilon \in \{1, -1\}^{B_l(\Lambda)}$, we have $\prod_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu^{\epsilon(\mu)} \neq 0$ if and only if $\epsilon = f_i^l$ for some $i = 1, \ldots, m(l)$. In this case $\prod_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu^{\epsilon(\mu)} = e_i^l$.*

Proof. Suppose that $\epsilon = f_i^l$ for some $i = 1, \ldots, m(l)$. Since Λ is λ-synchronizing, there exists $\nu \in S_l(\Lambda)$ such that v_i^l launches ν so that

$$s_\mu^* s_\mu \geq s_\nu s_\nu^* \quad \text{for } \mu \in \Gamma_l^-(v_i^l),$$
$$P - s_\mu^* s_\mu \geq s_\nu s_\nu^* \quad \text{for } \mu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l).$$

Hence, $\prod_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu^{f_i^l(\mu)} \geq s_\nu s_\nu^* \neq 0$.

Conversely, suppose that $\prod_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu^{\epsilon(\mu)} \neq 0$. Since $\prod_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu^{\epsilon(\mu)} \in \mathcal{A}_{\lambda(\widetilde{\Lambda})}$, there exist $k \geq l$ and $i_1 = 1, 2, \ldots, \tilde{m}(k)$ such that $\prod_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu^{\epsilon(\mu)} \geq \widetilde{E}_{i_1}^k \in \mathcal{A}_{\lambda(\widetilde{\Lambda})}$. Take $\omega \in S_k(\widetilde{\Lambda})$ such that $v_{i_1}^k$ launches ω. Since $\sum_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu \geq P$, there exists $\mu \in B_l(\Lambda)$ such that $s_\mu^* s_\mu \geq \widetilde{E}_{i_1}^k$. Hence we see that $\mu\omega \in B_*(\Lambda)$. As the rightmost letter of μ is not 0, the leftmost letter of ω is not 1. Let $\bar{\omega}$ be the word in $B_*(\Lambda)$ obtained from ω by putting 1 in place of 01 in ω. Since $\widetilde{E}_{i_1}^k \geq \widetilde{S}_\omega \widetilde{S}_\omega^*$, we see that

$$\prod_{\mu \in B_l(\Lambda)} s_\mu^* s_\mu^{\epsilon(\mu)} \geq s_{\bar{\omega}} s_{\bar{\omega}}^*.$$

As $[\bar{\omega}]_l \in V_l^{\lambda(\Lambda)}$, we have $[\bar{\omega}]_l = v_i^l$ for some $i = 1, \ldots, m(l)$. The vertex v_i^l launches $\bar{\omega}$ so that $\epsilon = f_i^l$. □

Lemma 4.5. *For $\mu, \nu \in B_l(\Lambda)$ and $\alpha, \beta \in \Sigma$, we have:*

(i) $s_\mu^*(P - s_\alpha^* s_\alpha) s_\mu \cdot s_\mu^* s_\beta^* s_\beta s_\mu = (P - s_{\alpha\mu}^* s_{\alpha\mu}) s_{\beta\mu}^* s_{\beta\mu}$;
(ii) $s_\alpha^* \cdot s_\mu^* s_\mu (P - s_\nu^* s_\nu) s_\alpha = s_{\mu\alpha}^* s_{\mu\alpha} (P - s_{\nu\alpha}^* s_{\nu\alpha})$.

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PROOF. (i) Since $Ps_\beta^* s_\beta = s_\beta^* s_\beta$ and hence $s_\mu^* P s_\beta^* s_\beta s_\mu = s_{\beta\mu}^* s_{\beta\mu}$,

$$\begin{aligned} s_\mu^*(P - s_\alpha^* s_\alpha)s_\mu \cdot s_\mu^* s_\beta^* s_\beta s_\mu &= s_\mu^* P s_\beta^* s_\beta s_\mu - s_\mu^* s_\alpha^* s_\alpha s_\beta^* s_\beta s_\mu \\ &= P s_{\beta\mu}^* s_{\beta\mu} - s_{\alpha\mu}^* s_{\alpha\mu} s_{\beta\mu}^* s_{\beta\mu} \\ &= (P - s_{\alpha\mu}^* s_{\alpha\mu}) s_{\beta\mu}^* s_{\beta\mu}. \end{aligned}$$

(ii) Since $Ps_\alpha = s_\alpha$ and $s_{\mu\alpha}^* s_{\mu\alpha} = s_{\mu\alpha}^* s_{\mu\alpha} P$,

$$\begin{aligned} s_\alpha^* \cdot s_\mu^* s_\mu (P - s_\nu^* s_\nu) s_\alpha &= s_{\mu\alpha}^* s_{\mu\alpha} - s_{\mu\alpha}^* s_{\mu\alpha} s_{\nu\alpha}^* s_{\nu\alpha} \\ &= s_{\mu\alpha}^* s_{\mu\alpha}(P - s_{\nu\alpha}^* s_{\nu\alpha}). \end{aligned}$$

□

LEMMA 4.6. *The partial isometries $s_\alpha, \alpha \in \Sigma$ and the projections $e_i^l, i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$, satisfy the following operator relations:*

$$\sum_{\beta\in\Sigma} s_\beta s_\beta^* = P, \tag{4.1}$$

$$\sum_{i=1}^{m(l)} e_i^l = P, \quad e_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) e_j^{l+1}, \tag{4.2}$$

$$s_\alpha s_\alpha^* e_i^l = e_i^l s_\alpha s_\alpha^*,$$

$$s_\alpha^* e_i^l s_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) e_j^{l+1}, \tag{4.3}$$

for $\alpha \in \Sigma$, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$, where $I_{l,l+1}, A_{l,l+1}$ denote the transition matrices for the λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$.

PROOF. Equality (4.1) has been proved in Lemma 4.3(i).

It follows that

$$P = \prod_{\mu\in B_l(\Lambda)} (s_\mu^* s_\mu + P - s_\mu^* s_\mu) = \sum_{\epsilon\in\{-1,1\}^{B_l(\Lambda)}} \prod_{\mu\in B_l(\Lambda)} s_\mu^* s_\mu^{\epsilon(\mu)}.$$

By Lemma 4.4, the nonzero $\prod_{\mu\in B_l(\Lambda)} s_\mu^* s_\mu^{\epsilon(\mu)}$ is of the form $\prod_{\mu\in B_l(\Lambda)} s_\mu^* s_\mu^{f_i^l(\mu)}$ for some $i = 1, \ldots, m(l)$ so that we have $P = \sum_{i=1}^{m(l)} e_i^l$.

We will next show equality (4.3). It follows that

$$\begin{aligned} s_\alpha^* e_i^l s_\alpha &= s_\alpha^* \Big(\prod_{\mu\in\Gamma_l^-(v_i^l)} s_\mu^* s_\mu \prod_{\nu\in B_l(\Lambda)\setminus\Gamma_l^-(v_i^l)} (P - s_\nu^* s_\nu) \Big) s_\alpha \\ &= \prod_{\mu\in\Gamma_l^-(v_i^l)} s_{\mu\alpha}^* s_{\mu\alpha} \prod_{\nu\in B_l(\Lambda)\setminus\Gamma_l^-(v_i^l)} (P - s_{\nu\alpha}^* s_{\nu\alpha}). \end{aligned}$$

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Hence $s_\alpha^* e_i^l s_\alpha$ is written as a finite sum of e_j^{l+1}, $j = 1, \dots, m(l+1)$. If $s_\alpha^* e_i^l s_\alpha \geq e_j^{l+1}$, then

$$s_\alpha^*(s_\mu^* s_\mu) s_\alpha \geq e_j^{l+1} \quad \text{for } \mu \in \Gamma_l^-(v_i^l),$$

$$s_\alpha^*(P - s_\nu^* s_\nu) s_\alpha \geq e_j^{l+1} \quad \text{for } \nu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l).$$

Since

$$e_j^{l+1} = \prod_{\xi \in \Gamma_{l+1}^-(v_j^{l+1})} s_\xi^* s_\xi \prod_{\eta \in B_{l+1}(\Lambda) \setminus \Gamma_{l+1}^-(v_j^{l+1})} (P - s_\eta^* s_\eta)$$

and Λ is λ-synchronizing, there exists $\zeta(j) \in S_{l+1}(\Lambda)$ such that $[\zeta(j)]_{l+1} = v_j^{l+1}$. Hence, $e_j^{l+1} \geq s_{\zeta(j)} s_{\zeta(j)}^*$. As $s_\alpha^* e_i^l s_\alpha \geq e_j^{l+1} \geq s_{\zeta(j)} s_{\zeta(j)}^*$, we have $e_i^l \geq s_{\alpha\zeta(j)} s_{\alpha\zeta(j)}^* \neq 0$. Hence

$$\mu\alpha\zeta(j) \in B_*(\Lambda) \quad \text{for } \mu \in \Gamma_l^-(v_i^l),$$

$$\nu\alpha\zeta(j) \notin B_*(\Lambda) \quad \text{for } \nu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l)$$

so that $[\alpha\zeta(j)]_l = v_i^l$. Since $[\zeta(j)]_{l+1} = v_j^{l+1}$, we have $A_{l,l+1}(i, \alpha, j) = 1$. Therefore the condition $s_\alpha^* e_i^l s_\alpha \geq e_j^{l+1}$ implies that $A_{l,l+1}(i, \alpha, j) = 1$. We thus obtain

$$s_\alpha^* e_i^l s_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) e_j^{l+1}.$$

We will next prove the second equality of (4.2). By the equalities

$$e_i^l = \prod_{\mu \in \Gamma_l^-(v_i^l)} s_\mu^* s_\mu \prod_{\nu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l)} (P - s_\nu^* s_\nu)$$

$$= \prod_{\mu \in \Gamma_l^-(v_i^l)} \Big(\sum_{k=1}^{m(1)} s_\mu^* e_k^1 s_\mu \Big) \prod_{\nu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l)} \Big(P - \sum_{h=1}^{m(1)} s_\nu^* e_h^1 s_\nu \Big)$$

we know that e_i^l is a finite sum of $e_1^l, \dots, e_{m(l+1)}^{l+1}$. Suppose that $e_i^l \geq e_j^{l+1}$. Since $v_j^{l+1} = [\zeta(j)]_{l+1}$ for some $\zeta(j) \in S_{l+1}(\Lambda)$, we have $e_j^{l+1} \geq s_{\zeta(j)} s_{\zeta(j)}^*$ and hence $e_i^l \geq s_{\zeta(j)} s_{\zeta(j)}^*$. This implies that

$$\prod_{\mu \in \Gamma_l^-(v_i^l)} s_\mu^* s_\mu \prod_{\nu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l)} (P - s_\nu^* s_\nu) \geq s_{\zeta(j)} s_{\zeta(j)}^*$$

so that

$$s_\mu^* s_\mu \geq s_{\zeta(j)} s_{\zeta(j)}^* \quad \text{and hence } s_{\mu\zeta(j)} \neq 0 \text{ for } \mu \in \Gamma_l^-(v_i^l),$$

$$P - s_\nu^* s_\nu \geq s_{\zeta(j)} s_{\zeta(j)}^* \quad \text{and hence } s_{\nu\zeta(j)} = 0 \text{ for } \nu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l).$$

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Hence

$$\mu\zeta(j) \in B_*(\Lambda) \quad \text{for } \mu \in \Gamma_l^-(v_i^l),$$
$$\nu\zeta(j) \notin B_*(\Lambda) \quad \text{for } \nu \in B_l(\Lambda) \setminus \Gamma_l^-(v_i^l).$$

Thus we have $[\zeta(j)]_l = v_i^l$. As $[\zeta(j)]_{l+1} = v_j^{l+1}$, we obtain that $I_{l,l+1}(i, j) = 1$. We conclude that the second equality of (4.2) holds.

The projections e_i^l and $s_\alpha^* s_\alpha$ all belong to the commutative C^*-subalgebra of $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ generated by the projections $\widetilde{S}_\mu \widetilde{S}_{\xi_1}^* \widetilde{S}_{\xi_1} \cdots \widetilde{S}_{\xi_n}^* \widetilde{S}_{\xi_n} \widetilde{S}_\mu^*, \mu, \xi_1 \cdots \xi_n \in B_*(\widetilde{\Lambda})$. The commutativity between e_i^l and $s_\alpha^* s_\alpha$ is obvious. Thus we complete the proof. □

Therefore we have the following corollary.

Corollary 4.7. *Suppose that the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ of a λ-synchronizing subshift Λ satisfies condition* (I). *Then the C^*-subalgebra of $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ generated by the partial isometries $s_\alpha, \alpha \in \Sigma$ and the projections $e_i^l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+$, is canonically isomorphic to the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ associated with the λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$.*

Proof. Since the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ satisfies condition (I), the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\lambda(\Lambda)}}$, which is $\mathcal{O}_{\lambda(\Lambda)}$, is the unique C^*-algebra subject to the relations $(\mathfrak{L}^{\lambda(\Lambda)})$ by [30, Theorem 4.3]. Therefore the assertion follows from the preceding lemma. □

We identify the algebra $\mathcal{O}_{\lambda(\Lambda)}$ with the above C^*-subalgebra of $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ generated by the partial isometries $s_\alpha, \alpha \in \Sigma$ and the projections $e_i^l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+$. We note that the projections $e_i^l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+$, and P are written by $s_\alpha, s_\alpha^*, \alpha \in \Sigma$, so that the subalgebra $\mathcal{O}_{\lambda(\Lambda)}$ is generated by $s_\alpha, \alpha \in \Sigma$.

We will henceforth prove that the C^*-subalgebra $P\mathcal{O}_{\lambda(\widetilde{\Lambda})}P$ is generated by $s_\alpha, \alpha \in \Sigma$, that is, $P\mathcal{O}_{\lambda(\widetilde{\Lambda})}P = \mathcal{O}_{\lambda(\Lambda)}$. Let $\mathcal{A}_{\lambda(\widetilde{\Lambda})}$ be the C^*-subalgebra of $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ generated by the projections $\widetilde{E}_i^l, i = 1, \ldots, \tilde{m}(l), l \in \mathbb{Z}_+$, similarly $\mathcal{A}_{\lambda(\Lambda)}$ the C^*-subalgebra of $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ generated by the projections $e_i^l, i = 1, \ldots, \tilde{m}(l), l \in \mathbb{Z}_+$. The subalgebra $\mathcal{A}_{\lambda(\Lambda)}$ is naturally regarded as a corresponding subalgebra of $\mathcal{O}_{\lambda(\Lambda)}$ through the canonical isomorphism in the above corollary.

For a word $\nu = \nu_1 \cdots \nu_l \in B_l(\widetilde{\Lambda})$ satisfying $\nu_1 \neq 1, \nu_l \neq 0$, we define the word $\overline{\nu} \in B_*(\Lambda)$ by putting 1 in place of 01 in ν. Since $s_1 = \widetilde{S}_0 \widetilde{S}_1$, the following lemma is straightforward.

Lemma 4.8. *For any $\mu = \mu_1 \cdots \mu_k \in B_k(\widetilde{\Lambda})$, the partial isometry $\widetilde{S}_\mu$ is of the form:*

$$\widetilde{S}_\mu = \begin{cases} s_{\overline{\mu}} & \text{if } \mu_1 \neq 1, \mu_k \neq 0, \\ \widetilde{S}_1 s_{\overline{\mu_2 \cdots \mu_k}} & \text{if } \mu_1 = 1, \mu_k \neq 0, \\ s_{\overline{\mu_1 \cdots \mu_{k-1}}} \widetilde{S}_0 & \text{if } \mu_1 \neq 1, \mu_k = 0, \\ \widetilde{S}_1 s_{\overline{\mu_2 \cdots \mu_{k-1}}} \widetilde{S}_0 & \text{if } \mu_1 = 1, \mu_k = 0. \end{cases}$$

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Lemma 4.9. *For any* $\mu = \mu_1 \cdots \mu_k \in B_k(\widetilde{\Lambda})$,

$$\widetilde{S}_\mu P = \begin{cases} s_{\overline{\mu}}P & \text{if } \mu_1 \neq 1, \mu_k \neq 0, \\ \widetilde{S}_1 s_{\overline{\mu_2\cdots\mu_k}}P & \text{if } \mu_1 = 1, \mu_k \neq 0, \\ 0 & \text{if } \mu_1 \neq 1, \mu_k = 0, \\ 0 & \text{if } \mu_1 = 1, \mu_k = 0. \end{cases}$$

Proof. By the preceding lemma, it suffices to show that $\widetilde{S}_0 P = 0$ for both the third and fourth cases. As $\widetilde{S}_0^*\widetilde{S}_0 = \widetilde{S}_1\widetilde{S}_1^*$,

$$\widetilde{S}_0 P = \widetilde{S}_0\widetilde{S}_1\widetilde{S}_1^* P = \widetilde{S}_0\widetilde{S}_1\widetilde{S}_1^*(1 - \widetilde{S}_1\widetilde{S}_1^*) = 0. \qquad \square$$

Lemma 4.10. *For any* $\mu = \mu_1 \cdots \mu_k \in B_k(\widetilde{\Lambda})$,

$$P\widetilde{S}_\mu^*\widetilde{S}_\mu P = \begin{cases} Ps_{\overline{\mu}}^* s_{\overline{\mu}}P & \text{if } \mu_1 \neq 1, \mu_k \neq 0, \\ Ps_{\overline{\mu_2\cdots\mu_k}}^* s_1^* s_1 s_{\overline{\mu_2\cdots\mu_k}}P & \text{if } \mu_1 = 1, \mu_k \neq 0, \\ 0 & \text{if } \mu_1 \neq 1, \mu_k = 0, \\ 0 & \text{if } \mu_1 = 1, \mu_k = 0. \end{cases}$$

Proof. By the preceding lemma, it suffices to show the equality for the second case. For $\mu_1 = 1, \mu_k \neq 0$, we have $\widetilde{S}_\mu P = \widetilde{S}_1 s_{\overline{\mu_2\cdots\mu_k}}P$ so that

$$P\widetilde{S}_\mu^*\widetilde{S}_\mu P = Ps_{\overline{\mu_2\cdots\mu_k}}^*\widetilde{S}_1^*\widetilde{S}_1 s_{\overline{\mu_2\cdots\mu_k}}P = Ps_{\overline{\mu_2\cdots\mu_k}}^* s_1^* s_1 s_{\overline{\mu_2\cdots\mu_k}}P. \qquad \square$$

Corollary 4.11. *Therefore we have* $P\mathcal{A}_{\lambda(\widetilde{\Lambda})}P = \mathcal{A}_{\lambda(\Lambda)}$.

Proof. By the previous lemma, we see that for $\mu \in B_*(\widetilde{\Lambda})$, the element $P\widetilde{S}_\mu^*\widetilde{S}_\mu P$ belongs to $P\mathcal{A}_{\lambda(\Lambda)}P$. As P is the unit of $\mathcal{A}_{\lambda(\Lambda)}$, we know that $P\widetilde{S}_\mu^*\widetilde{S}_\mu P \in \mathcal{A}_{\lambda(\Lambda)}$. Since $\mathcal{A}_{\lambda(\widetilde{\Lambda})}$ is generated by the projections $\widetilde{S}_\mu^*\widetilde{S}_\mu$, $\mu \in B_*(\widetilde{\Lambda})$, we have $P\mathcal{A}_{\lambda(\widetilde{\Lambda})}P \subset \mathcal{A}_{\lambda(\Lambda)}$. The converse inclusion relation $P\mathcal{A}_{\lambda(\widetilde{\Lambda})}P \supset \mathcal{A}_{\lambda(\Lambda)}$ is clear. $\square$

Lemma 4.12. *For any* $\mu = \mu_1 \cdots \mu_k \in B_k(\widetilde{\Lambda})$,

$$(1 - P)\widetilde{S}_\mu^*\widetilde{S}_\mu(1 - P) = \begin{cases} \widetilde{S}_1 s_{\overline{\mu_1\cdots\mu_k 1}}^* s_{\overline{\mu_1\cdots\mu_k 1}}\widetilde{S}_1^* & \text{if } \mu_1 \neq 1, \\ \widetilde{S}_1 s_{\overline{\mu_2\cdots\mu_k 1}}^* s_1^* s_1 s_{\overline{\mu_2\cdots\mu_k 1}}\widetilde{S}_1^* & \text{if } \mu_1 = 1. \end{cases}$$

Proof. Since $1 - P = \widetilde{S}_1\widetilde{S}_1^*$, it follows that

$$(1 - P)\widetilde{S}_\mu^*\widetilde{S}_\mu(1 - P) = \widetilde{S}_1\widetilde{S}_{\mu 1}^*\widetilde{S}_{\mu 1}\widetilde{S}_1^* = \begin{cases} \widetilde{S}_1 s_{\overline{\mu_1\cdots\mu_k 1}}^* s_{\overline{\mu_1\cdots\mu_k 1}}\widetilde{S}_1^* & \text{if } \mu_1 \neq 1, \\ \widetilde{S}_1 s_{\overline{\mu_2\cdots\mu_k 1}}^*\widetilde{S}_1^*\widetilde{S}_1 s_{\overline{\mu_2\cdots\mu_k 1}}\widetilde{S}_1^* & \text{if } \mu_1 = 1. \end{cases}$$

As $\widetilde{S}_1^*\widetilde{S}_1 = s_1^* s_1$, the desired equalities follow. $\square$

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COROLLARY 4.13. *Therefore we have* $(1-P)\mathcal{A}_{\lambda(\widetilde{\Lambda})}(1-P) \subset \widetilde{S}_1\mathcal{A}_{\lambda(\Lambda)}\widetilde{S}_1^*$.

PROOF. By the previous lemma, we see that for $\mu \in B_*(\widetilde{\Lambda})$, the element $(1-P)\widetilde{S}_\mu^*\widetilde{S}_\mu(1-P)$ belongs to $\widetilde{S}_1\mathcal{A}_{\lambda(\Lambda)}\widetilde{S}_1^*$, so that $(1-P)\mathcal{A}_{\lambda(\widetilde{\Lambda})}(1-P) \subset \widetilde{S}_1\mathcal{A}_{\lambda(\Lambda)}\widetilde{S}_1^*$. □

PROPOSITION 4.14. $P\mathcal{O}_{\lambda(\widetilde{\Lambda})}P \subset \mathcal{O}_{\lambda(\Lambda)}$.

PROOF. The C^*-algebra $P\mathcal{O}_{\lambda(\widetilde{\Lambda})}P$ is generated by the elements of the form:

$$P\widetilde{S}_\mu\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n}\widetilde{S}_\nu^*P, \quad \mu, \xi_1, \ldots, \xi_n, \nu \in B_*(\widetilde{\Lambda}).$$

Suppose that $P\widetilde{S}_\mu\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n}\widetilde{S}_\nu^*P \neq 0$. Let $\mu = \mu_1\cdots\mu_k, \nu = \nu_1\cdots\nu_h$. Since $P\widetilde{S}_\mu = \widetilde{S}_\mu \neq 0$ and $\widetilde{S}_\nu^*P = \widetilde{S}_\nu^* \neq 0$, we have $\mu_1 \neq 1, \nu_1 \neq 1$. Hence the words μ, ν satisfy the first condition or the third condition in Lemma 4.8. We have then the following four cases in which the rightmost letters of μ, ν are zero or not.

Case 1: $\mu_k \neq 0, \nu_h \neq 0$. Since $\widetilde{S}_{\mu_k}\widetilde{S}_1\widetilde{S}_1^* = 0$, we have $\widetilde{S}_{\mu_k}(1-P) = 0$ so that $\widetilde{S}_\mu P = \widetilde{S}_\mu$. Hence $\widetilde{S}_\mu$ commutes with P. Similarly, $\widetilde{S}_\nu$ commutes with P. By Lemma 4.8, one sees that $\widetilde{S}_\mu = s_{\overline{\mu}}, \widetilde{S}_\nu = s_{\overline{\nu}}$. It then follows that

$$P\widetilde{S}_\mu\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n}\widetilde{S}_\nu^*P = s_{\overline{\mu}}P\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n}Ps_{\overline{\nu}}^*.$$

Since $\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n} \in \mathcal{A}_{\lambda(\widetilde{\Lambda})}$ and $P\mathcal{A}_{\lambda(\widetilde{\Lambda})}P = \mathcal{A}_{\lambda(\Lambda)}$, the element

$$P\widetilde{S}_\mu\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n}\widetilde{S}_\nu^*P$$

belongs to $s_{\overline{\mu}}\mathcal{A}_{\lambda(\Lambda)}s_{\overline{\nu}}^*$ and hence to $\mathcal{O}_{\lambda(\Lambda)}$.

Case 2: $\mu_k \neq 0, \nu_h = 0$. As in the above discussion, $\widetilde{S}_\mu$ commutes with P. Since $P\widetilde{S}_0^*\widetilde{S}_0 = 0$, we have

$$\begin{aligned}P\widetilde{S}_\mu\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n}\widetilde{S}_\nu^*P &= \widetilde{S}_\mu P\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n}\widetilde{S}_0^*\widetilde{S}_{\nu_1\cdots\nu_{h-1}}^*P \\ &= \widetilde{S}_\mu P\widetilde{S}_0^*\widetilde{S}_0\widetilde{S}_{\xi_1}^*\widetilde{S}_{\xi_1}\cdots\widetilde{S}_{\xi_n}^*\widetilde{S}_{\xi_n}\widetilde{S}_0^*\widetilde{S}_{\nu_1\cdots\nu_{h-1}}^*P = 0,\end{aligned}$$

a contradiction.

Case 3: $\mu_k = 0, \nu_h \neq 0$. This case is similar to Case 2.

Case 4: $\mu_k = 0, \nu_h = 0$. Since $\widetilde{S}_0P = 0$, we have $\widetilde{S}_\mu = \widetilde{S}_\mu(1-P)$ and similarly $\widetilde{S}_\nu^* = (1-P)\widetilde{S}_\nu^*$. As both words μ, ν satisfy the third condition in Lemma 4.8, one sees that

$$\widetilde{S}_\mu = s_{\overline{\mu_1\cdots\mu_{k-1}}}\widetilde{S}_0, \quad \widetilde{S}_\nu = s_{\overline{\nu_1\cdots\nu_{h-1}}}\widetilde{S}_0.$$

It then follows that

$$P\widetilde{S}_\mu = \widetilde{S}_\mu = s_{\overline{\mu_1\cdots\mu_{k-1}}}\widetilde{S}_0(1-P), \quad \widetilde{S}_\nu^*P = \widetilde{S}_\nu^* = (1-P)\widetilde{S}_0^*s_{\overline{\nu_1\cdots\nu_{h-1}}}^*.$$

Hence

$$
\begin{aligned}
&P\widetilde{S}_\mu \widetilde{S}^*_{\xi_1}\widetilde{S}_{\xi_1}\cdots \widetilde{S}^*_{\xi_n}\widetilde{S}_{\xi_n}\widetilde{S}^*_\nu P\\
&\quad = s_{\overline{\mu_1\cdots\mu_{k-1}}}\widetilde{S}_0(1-P)\widetilde{S}^*_{\xi_1}\widetilde{S}_{\xi_1}\cdots \widetilde{S}^*_{\xi_n}\widetilde{S}_{\xi_n}(1-P)\widetilde{S}^*_0 s^*_{\overline{\nu_1\cdots\nu_{h-1}}}.
\end{aligned}
$$

By the preceding lemma, one knows that $(1-P)\mathcal{A}_{\lambda(\widetilde{\Lambda})}(1-P)\subset \widetilde{S}_1\mathcal{A}_{\lambda(\Lambda)}\widetilde{S}^*_1$ so that the element $\widetilde{S}_0(1-P)\widetilde{S}^*_{\xi_1}\widetilde{S}_{\xi_1}\cdots \widetilde{S}^*_{\xi_n}\widetilde{S}_{\xi_n}(1-P)\widetilde{S}^*_0$ belongs to $\widetilde{S}_0\widetilde{S}_1\mathcal{A}_{\lambda(\Lambda)}\widetilde{S}^*_1\widetilde{S}^*_0$ which is $s_1\mathcal{A}_{\lambda(\Lambda)}s^*_1$. Then the element $P\widetilde{S}_\mu \widetilde{S}^*_{\xi_1}\widetilde{S}_{\xi_1}\cdots \widetilde{S}^*_{\xi_n}\widetilde{S}_{\xi_n}\widetilde{S}^*_\nu P$ belongs to $s_1\mathcal{A}_{\lambda(\Lambda)}s^*_1$ and hence to $\mathcal{O}_{\lambda(\Lambda)}$.

Therefore in all cases $P\widetilde{S}_\mu \widetilde{S}^*_{\xi_1}\widetilde{S}_{\xi_1}\cdots \widetilde{S}^*_{\xi_n}\widetilde{S}_{\xi_n}\widetilde{S}^*_\nu P$ belongs to $\mathcal{O}_{\lambda(\Lambda)}$ so that we conclude that $P\mathcal{O}_{\lambda(\widetilde{\Lambda})}P\subset \mathcal{O}_{\lambda(\Lambda)}$. □

Let $\mathcal{D}_{\lambda(\widetilde{\Lambda})}$ be the C^*-subalgebra of $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ generated by the projections $\widetilde{S}_\mu\widetilde{E}^l_i\widetilde{S}^*_\mu$, $\mu\in B_*(\widetilde{\Lambda})$, $i=1,\ldots,\tilde{m}(l)$, $l\in\mathbb{Z}_+$, and similarly $\mathcal{D}_{\lambda(\Lambda)}$ the C^*-subalgebra of $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$ generated by the projections $s_\nu e^l_i s^*_\nu$, $\nu\in B_*(\Lambda)$, $i=1,\ldots,\tilde{m}(l)$, $l\in\mathbb{Z}_+$. The subalgebra $\mathcal{D}_{\lambda(\Lambda)}$ is naturally regarded as a corresponding subalgebra of $\mathcal{O}_{\lambda(\Lambda)}$ through the canonical isomorphism in Corollary 4.7.

Proposition 4.15.

(i) $P\mathcal{O}_{\lambda(\widetilde{\Lambda})}P=\mathcal{O}_{\lambda(\Lambda)}$.
(ii) $\mathcal{O}_{\lambda(\widetilde{\Lambda})}P\mathcal{O}_{\lambda(\widetilde{\Lambda})}=\mathcal{O}_{\lambda(\widetilde{\Lambda})}$.
(iii) $P\mathcal{D}_{\lambda(\widetilde{\Lambda})}P=\mathcal{D}_{\lambda(\Lambda)}$.

Proof. (i) The inclusion relation $P\mathcal{O}_{\lambda(\widetilde{\Lambda})}P\supset \mathcal{O}_{\lambda(\Lambda)}$ is obvious so that, by the preceding proposition, $P\mathcal{O}_{\lambda(\widetilde{\Lambda})}P=\mathcal{O}_{\lambda(\Lambda)}$.

(ii) Since $\widetilde{S}^*_0\widetilde{S}_0=\widetilde{S}_1\widetilde{S}^*_1$ we have $\widetilde{S}^*_0P\widetilde{S}_0=\widetilde{S}^*_0\widetilde{S}_0=\widetilde{S}_1\widetilde{S}^*_1$. It follows that

$$
\widetilde{S}^*_0P\widetilde{S}_0+P=\sum_{j=0}^{N}\widetilde{S}_j\widetilde{S}^*_j=1.
$$

This means that P is a full projection in $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$.

(iii) In the proof of Proposition 4.14, the projection $P\widetilde{S}_\mu \widetilde{S}^*_{\xi_1}\widetilde{S}_{\xi_1}\cdots \widetilde{S}^*_{\xi_n}\widetilde{S}_{\xi_n}\widetilde{S}^*_\mu P$ belongs to $\mathcal{D}_{\lambda(\Lambda)}$ so that $P\mathcal{D}_{\lambda(\widetilde{\Lambda})}P\subset\mathcal{D}_{\lambda(\Lambda)}$. The other inclusion relation $P\mathcal{D}_{\lambda(\widetilde{\Lambda})}P\supset \mathcal{D}_{\lambda(\Lambda)}$ is clear. □

Let $K(H)$ be the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space H and $C(H)$ a maximal commutative C^*-subalgebra of $K(H)$.

Theorem 4.16. *Assume that the right one-sided subshift of a λ-synchronizing subshift Λ is homeomorphic to the Cantor set. Then*

$$
(\mathcal{O}_{\lambda(\widetilde{\Lambda})}\otimes K(H),\mathcal{D}_{\lambda(\widetilde{\Lambda})}\otimes C(H))\cong(\mathcal{O}_{\lambda(\Lambda)}\otimes K(H),\mathcal{D}_{\lambda(\Lambda)}\otimes C(H)).
$$

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In particular,

$$\mathcal{O}_{\lambda(\widetilde{\Lambda})} \otimes \mathcal{K}(H) \cong \mathcal{O}_{\lambda(\Lambda)} \otimes \mathcal{K}(H).$$

Proof. Proposition 4.15(ii) shows that the projection P is full in $\mathcal{O}_{\lambda(\widetilde{\Lambda})}$. By [5], we have the desired assertions. □

Therefore we conclude the following theorem.

Theorem 4.17. *Assume that the right one-sided subshifts of λ-synchronizing subshifts Λ_1 and Λ_2 are both homeomorphic to the Cantor set. Suppose that Λ_1 is flow equivalent to Λ_2. Then*

$$(\mathcal{O}_{\lambda(\Lambda_1)} \otimes \mathcal{K}(H), \mathcal{D}_{\lambda(\Lambda_1)} \otimes \mathcal{C}(H)) \cong (\mathcal{O}_{\lambda(\Lambda_2)} \otimes \mathcal{K}(H), \mathcal{D}_{\lambda(\Lambda_2)} \otimes \mathcal{C}(H)).$$

In particular,

$$\mathcal{O}_{\lambda(\Lambda_1)} \otimes \mathcal{K}(H) \cong \mathcal{O}_{\lambda(\Lambda_2)} \otimes \mathcal{K}(H).$$

Proof. The flow equivalence relation of subshifts is generated by topological conjugacy and expansion $\Lambda \longrightarrow \widetilde{\Lambda}$. Suppose that λ-synchronizing subshifts Λ_1 and Λ_2 are topologically conjugate. By [23, Proposition 3.5], their symbolic matrix systems $(\mathcal{M}^{\lambda(\Lambda_1)}, I^{\lambda(\Lambda_1)})$ and $(\mathcal{M}^{\lambda(\Lambda_2)}, I^{\lambda(\Lambda_2)})$ are strong shift equivalence. Then

$$(\mathcal{O}_{\lambda(\Lambda_1)} \otimes \mathcal{K}(H), \mathcal{D}_{\lambda(\Lambda_1)} \otimes \mathcal{C}(H)) \cong (\mathcal{O}_{\lambda(\Lambda_2)} \otimes \mathcal{K}(H), \mathcal{D}_{\lambda(\Lambda_2)} \otimes \mathcal{C}(H))$$

by [31, Theorem 4.4]. Hence by the above theorem, we have the desired assertions. □

Corollary 4.18 [36]. *Assume that the right one-sided subshifts of λ-synchronizing subshifts Λ_1 and Λ_2 are both homeomorphic to the Cantor set. Suppose that Λ_1 is flow equivalent to Λ_2. Then their λ-synchronizing K-groups and their λ-synchronizing Bowen–Franks groups are isomorphic, that is,*

$$K_i^\lambda(\Lambda_1) \cong K_i^\lambda(\Lambda_2) \quad \text{and} \quad BF_\lambda^i(\Lambda_1) \cong BF_\lambda^i(\Lambda_2), \quad i = 0, 1.$$

Proof. The λ-synchronizing K-groups $K_i^\lambda(\Lambda)$ and the λ-synchronizing Bowen–Franks groups $BF_\lambda^i(\Lambda)$ for a λ-synchronizing subshift Λ are isomorphic to the K-groups and the Ext-groups for the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ respectively:

$$K_i^\lambda(\Lambda) = K_i(\mathcal{O}_{\lambda(\Lambda)}), \quad BF_\lambda^i(\Lambda) = \mathrm{Ext}_i(\mathcal{O}_{\lambda(\Lambda)}), \quad i = 0, 1.$$

Hence the assertion is direct from the above theorem. □

5. Examples

5.1. Sofic shifts. Let Λ be an irreducible sofic shift which is homeomorphic to the Cantor set. Let $\mathcal{G}_{F(\Lambda)}$ be a finite directed labeled graph of the minimal left-resolving presentation of Λ. Such a labeled graph is unique up to graph isomorphism and is called the left Fischer cover [9, 18, 19, 40]. Let $\mathfrak{L}_{\mathcal{G}_{F(\Lambda)}}$ be the λ-graph system associated with the finite labeled graph $\mathcal{G}_{F(\Lambda)}$ (see [30, Proposition 8.2]). Then the

λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ for the sofic shift Λ is nothing but the λ-graph system $\mathfrak{L}_{\mathcal{G}_{F(\Lambda)}}$. Let N be the number of the vertices of the graph $\mathcal{G}_{F(\Lambda)}$. Let $\mathcal{M}_{F(\Lambda)}$ be the $N \times N$ symbolic matrix of the graph $\mathcal{G}_{F(\Lambda)}$. Let $A_{F(\Lambda)}$ be the $N \times N$ nonnegative matrix defined from $\mathcal{M}_{F(\Lambda)}$ by setting all symbols equal to 1 in each component of $\mathcal{M}_{F(\Lambda)}$. Then the C^*-algebra $\mathcal{O}_{\lambda(\Lambda)}$ of the λ-graph system $\mathfrak{L}^{\lambda(\Lambda)}$ is simple and purely infinite. The algebra $\mathcal{O}_{\lambda(\Lambda)}$ is also realized as the labeled graph C^*-algebra $\mathcal{O}_{\mathcal{G}_{F(\Lambda)}}$ for the labeled graph $\mathcal{G}_{F(\Lambda)}$ (see [2]). It is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{A_{F(\Lambda)}}$. The λ-synchronizing K-groups and Bowen–Franks groups are as follows:

$$K_0^\lambda(\Lambda) = \mathbb{Z}^N/(I_N - A^t_{F(\Lambda)})\mathbb{Z}^N, \quad K_1^\lambda(\Lambda) = \mathrm{Ker}(I_N - A^t_{F(\Lambda)}) \quad \text{in } \mathbb{Z}^N$$

and

$$BF_\lambda^0(\Lambda) = \mathbb{Z}^N/(I_N - A_{F(\Lambda)})\mathbb{Z}^N, \quad BF_\lambda^1(\Lambda) = \mathrm{Ker}(I_N - A_{F(\Lambda)}) \quad \text{in } \mathbb{Z}^N.$$

They are all invariant under flow equivalence of Λ (see [11]).

5.2. Dyck shifts. Let $N > 1$ be a fixed positive integer. We consider the Dyck shift D_N with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1, \ldots, \alpha_N\}$, $\Sigma^+ = \{\beta_1, \ldots, \beta_N\}$. The symbols α_i, β_i correspond to the brackets $(_i,)_i$ respectively. The Dyck inverse monoid for Σ has the relations

$$\alpha_i\beta_j = \begin{cases} \mathbf{1} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

for $i, j = 1, \ldots, N$ ([17, 22]; see [7]). A word $\omega_1 \cdots \omega_n$ of Σ is admissible for D_N precisely if $\prod_{m=1}^n \omega_m \neq 0$. For a word $\omega = \omega_1 \cdots \omega_n$ of Σ, we denote by $\tilde{\omega}$ its reduced form. That is, $\tilde{\omega}$ is a word of $\Sigma \cup \{0, \mathbf{1}\}$ obtained after the operations (5.1). Hence a word ω of Σ is forbidden for D_N if and only if $\tilde{\omega} = 0$.

Let us describe the Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ of D_N introduced in [22]. Let Λ_N be the full N-shift $\{1, \ldots, N\}^{\mathbb{Z}}$. We denote by $B_l(D_N)$ and by $B_l(\Lambda_N)$ the set of admissible words of length l of D_N and that of Λ_N, respectively. The vertices V_l of $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ at level l are given by the words of length l consisting of the symbols of Σ^+. That is,

$$V_l = \{\beta_{\mu_1} \cdots \beta_{\mu_l} \in B_l(D_N) \mid \mu_1 \cdots \mu_l \in B_l(\Lambda_N)\}.$$

It is easy to see that each word of V_l is l-synchronizing in D_N such that V_l represent the all l-past equivalence classes of D_N. Hence we know that $V_l = V_l^{\lambda(D_N)}$. The cardinal number of V_l is N^l. The mapping $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ deletes the rightmost symbol of a word such as

$$\iota(\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) = \beta_{\mu_1} \cdots \beta_{\mu_l}, \quad \beta_{\mu_1} \cdots \beta_{\mu_{l+1}} \in V_{l+1}. \tag{5.2}$$

There exists an edge labeled α_j from $\beta_{\mu_1} \cdots \beta_{\mu_l} \in V_l$ to $\beta_{\mu_0}\beta_{\mu_1} \cdots \beta_{\mu_l} \in V_{l+1}$ precisely if $\mu_0 = j$, and there exists an edge labeled β_j from $\beta_j\beta_{\mu_1} \cdots \beta_{\mu_{l-1}} \in V_l$ to $\beta_{\mu_1} \cdots \beta_{\mu_{l+1}} \in V_{l+1}$. The resulting labeled Bratteli diagram with ι-map is the Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ of D_N.

https://doi.org/10.1017/S1446788713000219 Published online by Cambridge University Press

Proposition 5.1. *The Dyck shift D_N is λ-synchronizing, and the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(D_N)}$ is the Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_N)}$.*

The Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ gives rise to a purely infinite simple C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}$ [22, 34]. The K-groups of the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_N)}}$ are realized as the K-groups of the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_N)}$ so that [22, 34]

$$K_0(\mathcal{O}_{\lambda(D_N)}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{K}, \mathbb{Z}), \qquad K_1(\mathcal{O}_{\lambda(D_N)}) \cong 0,$$

where $C(\mathfrak{K}, \mathbb{Z})$ denotes the abelian group of all $\mathbb{Z}$-valued continuous functions on the Cantor set $\mathfrak{K}$. The Ext-groups for $\mathcal{O}_{\lambda(D_N)}$ are computed from the universal coefficient theorem for K-theory [39] so that we know [22] that

$$K_0^\lambda(D_N) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{K}, \mathbb{Z}), \qquad K_1^\lambda(D_N) \cong 0,$$
$$BF_\lambda^0(D_N) \cong \mathbb{Z}/N\mathbb{Z}, \qquad BF_\lambda^1(D_N) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{K}, \mathbb{Z}), \mathbb{Z}).$$

5.3. Topological Markov–Dyck shifts. We consider a generalization of the above discussions for the Dyck shifts. Let $A = [A(i, j)]_{i,j=1,\ldots,N}$ be an $N \times N$ matrix with entries in $\{0, 1\}$. Consider the Dyck inverse monoid for the alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1, \ldots, \alpha_N\}$ and $\Sigma^+ = \{\beta_1, \ldots, \beta_N\}$ satisfy relations (5.1). Let $\mathcal{O}_A$ be the Cuntz–Krieger algebra of the matrix A that is the universal C^*-algebra generated by N partial isometries $t_1, \ldots, t_N$ subject to the following relations:

$$\sum_{j=1}^N t_j t_j^* = 1, \qquad t_i^* t_i = \sum_{j=1}^N A(i, j) t_j t_j^*, \qquad \text{for } i = 1, \ldots, N$$

[8]. Define a correspondence $\varphi_A : \Sigma \longrightarrow \{t_i^*, t_i \mid i = 1, \ldots, N\}$ by setting

$$\varphi_A(\alpha_i) = t_i^*, \qquad \varphi_A(\beta_i) = t_i, \qquad i = 1, \ldots, N.$$

We denote by Σ^* the set of all words $\gamma_1 \cdots \gamma_n$ of elements of Σ. Define the set

$$\mathfrak{F}_A = \{\gamma_1 \cdots \gamma_n \in \Sigma^* \mid \varphi_A(\gamma_1) \cdots \varphi_A(\gamma_n) = 0 \text{ in } \mathcal{O}_A\}.$$

Let D_A be the subshift over Σ whose forbidden words are $\mathfrak{F}_A$. The subshift is called the topological Markov–Dyck shift defined by A [35]. These kinds of subshifts first appeared in [21] in a semigroup setting and in [12] in a more general setting without using C^*-algebras (see [35]). If all entries of A are 1, the subshift becomes the Dyck shift D_N with $2N$ brackets, because the partial isometries $\{\varphi_A(\alpha_i), \varphi(\beta_i) \mid i = 1, \ldots, N\}$ yield the Dyck inverse monoid. Consider the following subsystem of D_A:

$$D_A^+ = \{(\gamma_i)_{i\in\mathbb{Z}} \in D_A \mid \gamma_i \in \Sigma^+, i \in \mathbb{Z}\},$$

which is identified with the topological Markov shift

$$\Lambda_A = \{(x_i)_{i\in\mathbb{Z}} \in \{1, \ldots, N\}^{\mathbb{Z}} \mid A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}\}$$

defined by the matrix A. If A satisfies condition (I) in the sense of [8], the subshift D_A is not sofic [35, Proposition 2.1]. Similarly to the Dyck shifts, one may consider the Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ for the topological Markov–Dyck shift D_A, as studied in [35]. We denote by $B_l(D_A^+)$ the set of admissible words of length l of D_A^+. The vertices V_l, $l \in \mathbb{Z}_+$, of $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ are given by the admissible words of length l consisting of the symbols of Σ^+. They are l-synchronizing words of D_A such that their l-past equivalence classes coincide with the l-past equivalence classes of the set of all l-synchronizing words of D_A. Hence $V_l = V_l^{\lambda(D_A)}$. Since V_l is identified with $B_l(\Lambda_A)$, we may write V_l as

$$V_l = \{\beta_{\mu_1} \cdots \beta_{\mu_l} \mid \mu_1 \cdots \mu_l \in B_l(\Lambda_A)\}.$$

The mapping $\iota(=\iota_{l,l+1}) : V_{l+1} \to V_l$ is defined by deleting the rightmost symbol of a corresponding word as in (5.2). There exists an edge labeled α_j from $\beta_{\mu_1} \cdots \beta_{\mu_l} \in V_l$ to $\beta_{\mu_1} \cdots \beta_{\mu_{l+1}} \in V_{l+1}$ precisely if $\mu_0 = j$, and there exists an edge labeled β_j from $\beta_j \beta_{\mu_1} \cdots \beta_{\mu_{l-1}} \in V_l$ to $\beta_{\mu_1} \cdots \beta_{\mu_{l+1}} \in V_{l+1}$. It is easy to see that the resulting labeled Bratteli diagram with ι-map becomes a λ-graph system written $\mathfrak{L}^{\mathrm{Ch}(D_A)}$ called the Cantor horizon λ-graph system for the topological Markov–Dyck shifts D_A.

Proposition 5.2. *The subshift D_A is λ-synchronizing, and the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(D_A)}$ is the Cantor horizon λ-graph system $\mathfrak{L}^{\mathrm{Ch}(D_A)}$.*

Hence the C^*-algebra $\mathcal{O}_{\lambda(D_A)}$ coincides with the algebra $\mathcal{O}_{\mathfrak{L}^{\mathrm{Ch}(D_A)}}$. By [35, Lemma 2.5], if A satisfies condition (I) in the sense of [8], the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(\Lambda_A)}$ satisfies λ-condition (I) in the sense of [33]. If A is irreducible, the λ-graph system $\mathfrak{L}^{\mathrm{Ch}(\Lambda_A)}$ is λ-irreducible. We have the following proposition.

Proposition 5.3. *Suppose that A is an irreducible matrix with entries in $\{0, 1\}$ satisfying condition (I). Then the C^*-algebra $\mathcal{O}_{\lambda(D_A)}$ associated with the λ-synchronizing λ-graph system $\mathfrak{L}^{\lambda(D_A)}$ for the topological Markov–Dyck shift D_A is simple and purely infinite.*

One knows that β-shifts for $1 < \beta \in \mathbb{R}$, a synchronizing counter-shift called the context-free shift in [24, Example 1.2.9], and Motzkin shifts are all λ-synchronizing. Their C^*-algebras for the λ-synchronizing λ-graph systems have been studied in the papers [13, 26, 32] respectively.

Acknowledgements

The author would like to thank the referee for careful reading of the first draft and useful suggestions and comments. The author also would like to thank Wolfgang Krieger for various discussions and constant encouragement. This work was supported by JSPS Grant-in-Aid for Scientific Reserch ((C), No. 20540215) and ((C), No. 23540237).

https://doi.org/10.1017/S1446788713000219 Published online by Cambridge University Press

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