MIXED ABELIAN GROUPS

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Introduction. The difficulties encountered in the theory of mixed Abelian groups can become decidedly less complex, if it is possible to reduce the question to mixed groups whose torsion subgroup is p-primary. Call such a group a *p-mixed group*. In §1 we show that the splitting problem for a mixed group is reducible to the same problem for certain associated p-mixed groups. In §2 we look at groups which are a direct sum of p-mixed groups.

1. The splitting problem. When G is an extension of T by J we have the short exact sequence

$$E: 0 \to T \xrightarrow{\alpha} G \xrightarrow{\beta} J \to 0. \tag{1.1}$$

Now if T is the direct sum

$$T = \sum_{i=1}^{n} T_i,$$

let $T^i = \sum_{j \neq i} T_j$. Then identifying T with $\alpha(T)$ we get the n short exact sequences

$$E_i: 0 \to T_i \xrightarrow{\alpha_i} G/T^i \xrightarrow{\beta_i} J \to 0 \tag{1.2}$$

where $\alpha_i(T_i) = t_i + T^i$ and $\beta_i(g + T^i) = \beta(g)$ for $t_i \in T_i$, $g \in G$.

THEOREM 1. *The short exact sequence* (1) *splits if and only if each of the n short exact sequences* (2) *splits.*

Proof. The isomorphism

$$\text{Ext}(J, T) \cong \sum_{i=1}^{n} \text{Ext}(J, T_i)$$

takes the class of E to the class of $\pi_1 E \oplus \ldots \oplus \pi_n E$ where $\pi_i: T \to T_i$ are the projection maps; cf. (**2**, Chapter 3). So E splits if and only if $\pi_i E$ splits for each i. Hence we need only show that $\pi_i E$ and E_i are in the same class; this is accomplished by the following commutative diagram

$$\begin{array}{rccccccccc}
E_i: & 0 & \longrightarrow & T_i & \longrightarrow & G/T^i & \longrightarrow & J & \longrightarrow & 0 \\
 & & & \Vert & & \downarrow \psi_i & & \Vert & & \\
\pi_i E: & 0 & \longrightarrow & T_i & \xrightarrow{\alpha'_i} & G_i & \xrightarrow{\beta'_i} & J & \longrightarrow & 0
\end{array}$$

where $G_i = (T_i + G)/N$, $N = \{(-t_i, t_i) | t_i \in T_i\}$, $\alpha'_i(t_i) = (t_i, 0) + N$, $\beta'_i((t_i, g) + N) = \beta(g)$, $\psi_i(g + T^i) = (0, g) + N$.

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In the special case when T is a torsion group, α the inclusion map, and J is torsion-free G is a mixed group with maximal torsion subgroup $tG = T$.

COROLLARY. *Suppose that the maximal torsion subgroup T of the mixed group G has the primary decomposition*

$$T = \sum_{i=1}^{n} T_i,$$

where T_i is the p_i component of T. Then G splits if, and only if, the p_i-mixed groups G/T^i split for each i.

It should be noted that the corollary cannot be extended to the infinite case, i.e. when tG contains an infinite number of non-zero primary components. For example, let G be the unrestricted direct sum of the cyclic groups Z_p of order p, one for each prime. Then $tG = \sum Z_p$. Now G is reduced but G/tG is divisible so G does not split. However each of the p-mixed groups G/T^p do split.

2. Direct sums of p-mixed groups. We investigate the following question: If G has a direct sum decomposition into p-mixed groups, does a direct summand of G have the same decomposition?

First we mention some properties of torsion-free groups J. The rank of J is denoted by $r(J)$ and if $r(J) = 1$ its type **(1)** is denoted by $\tau(J)$. J is completely decomposable if it is a direct sum of groups of rank one. If $J = \sum J_i$, $r(J_i) = 1$ for each $i \in I$, then $r(\alpha)$ for a given type α denotes the rank of that subgroup $\sum J_k$ of J, where the sum is taken over those $k \in I$ such that $\tau(J_k) = \alpha$.

Definition. Suppose G is the direct sum of the groups $\{G_i | i \in I\}$ and K is a subgroup of G. Then K is an *n-diagonal subgroup* of G if

$$K \subseteq \sum_{j=1}^{n} G_{i(j)},$$

and if $0 \neq g \in K$, then $g = g_1 + \ldots + g_n$ where $0 \neq g_j \in G_{i(j)}$ for $1 \leqslant j \leqslant n$.

The proofs of the following are left to the reader:

(i) Suppose that $G = \sum G_i$ satisfies $r(G_i) \leqslant n$ for all $i \in I$. If K is a diagonal subgroup of G, then $r(K) \leqslant n$.

(ii) K is a diagonal subgroup of $\sum G_i$, $r(G_i) = 1$ for all $i \in I$, if and only if, $r(K) \leqslant 1$.

(iii) Suppose that the completely decomposable group G satisfies $r(\alpha) = 1$ for all types α. Then a rank one direct summand of G of maximal type (if one exists) appears in all internal direct decompositions of G into groups of rank one.

Call G a *simple p-mixed* group if G is a p-mixed group and $r(G/tG) \leqslant 1$. So if G is a direct sum of simple p-mixed groups, G/tG will be completely decomposable.

PROPOSITION 1. *Let G be a direct sum of a finite number of simple p-mixed groups for different primes p. Suppose also that G/tG satisfies $r(\alpha) = 1$ for all types α. Then any direct summand of G is a direct sum of simple p-mixed groups.*

Proof. Let

$$G = \sum_{i=1}^{n} G_i = S \oplus K$$

where G_i is a simple p_i-mixed group. We can assume K is torsion-free and proceed by induction on n and $r(K)$. If some G_i is a torsion group the argument is easy, so we have $r(G/tG) = n$.

$r(K) = 1$: Let $\pi: G \to G/tG$ be the natural quotient map and $\pi(X) = \bar{X}$ for any subgroup X of G. Then we have two decompositions of $\bar{G}$ into groups of rank 1, namely $\bar{G} = \sum \bar{G}_i = \sum L_i$, where $L_1 = \bar{K}$ and $\bar{S} = L_2 + \ldots + L_n$. Now, since the types appearing must be the same we assume that $\tau(\bar{G}_i) = \tau(L_i)$. By (ii) and (iii) some L_j is 1-diagonal in $\sum \bar{G}_i$. If $j = 1$, then $\mathrm{L}_1 = \bar{G}_1$ and

$$G = K \oplus tG_1 \oplus \sum_{i=2}^{n} G_i.$$

Hence

$$S \cong G/K \cong tG_1 \oplus \sum_{i=1}^{n} G_i,$$

which is a direct sum of p-mixed groups.

If L_2 is 1-diagonal, then $L_2 = \bar{G}_2$ and $G_2 \subseteq S$. So $S = G_2 \oplus S'$ for some subgroup S' of G. But then $H = G/G_2 \cong K \oplus S'$. Since $r(H/tH) = n - 1$ we get, by induction, that S' is a direct sum of simple p-mixed groups and then so also is $S = G_2 \oplus S'$.

$r(K) > 1$: Then $G = L_1 \oplus (L + S)$, where $r(L_1) = 1$ and $K = L_1 \oplus L$.

But then $L \oplus S$ is a direct sum of simple p-mixed groups by the above case. Hence, by induction, S has the desired decomposition.

THEOREM 2. *Let G be a direct sum of simple p-mixed groups for different primes p and suppose that G/tG satisfies $r(\alpha) = 1$ for all types α. If $G = S_1 \oplus S_2$ and S_1/tS_1 has finite rank, then both S_1 and S_2 are a direct sum of simple p-mixed groups.*

Proof. Let $G = \sum G_i$, $i \in I$, where G_i is a simple p-mixed group. Now

$$\bar{S}_1 = \sum_{k=1}^{n} J_k,$$

$r(J_k) = 1$ for $1 \leqslant k \leqslant n$. So each J_k is an n_k-diagonal subgroup of $\sum \bar{G}_i = \bar{G}$. It follows that

$$\bar{S}_1 \subseteq \sum_{u=1}^{s} \bar{G}_{i(u)}$$

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and thus

$$S_1 = \sum_{u=1}^{s} G_{i(u)} \oplus T',$$

where T' is a torsion subgroup of S_1. Letting

$$H = \sum_{u=1}^{s} G_{i(u)},$$

we have $H = (H \cap S_1) \oplus (H \cap (T' \oplus S_2))$. So by Proposition 1 $H \cap S_1$ and thus S_1 is a direct sum of simple p-mixed groups, say $S_1 = \sum H_p$.

Now $G/T' \cong \sum H'_p \oplus S_2$, where $S_1/T' = \sum H'_p$ is a direct sum of simple p-mixed groups. If we let $L = (H + T')/T'$, we get

$$G/T' \cong S_1/T' \oplus (S_2 \cap L) \oplus M,$$

where $M = \sum_{i \neq i(u)} G_i$, $1 \leqslant u \leqslant s$. Now $L = S_1/T' \oplus (S_2 \cap L)$ so that by Proposition 1 $S_2 \cap L$ is a direct sum of simple p-mixed groups. But $S_2 \cong (S_2 \cap L) \oplus M$; hence S_2 also has this property.

References

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