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# A NOTE ON INJECTIVE MODULES OVER A d.g. NEAR-RING

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In [3] an attempt was made at proving the following result:

(A) An $N$-module $M$ over a d.g. near-ring is injective if and only if for each right ideal $u$ of $N$ and each $N$-homomorphism $f:u \to M$ there exists an element $m \in M$ with $f(a) = ma$ for all $a \in u$.

In this note we present two examples. The first is a counterexample to (A) and the second illustrates one point at which the attempt made in [3] fails. As in [3] our near-rings are left near-rings (i.e. $a(b+c) = ab + ac$) with multiplicative identities. We recall that a near-ring $N$ is distributively generated (d.g.) if there is a multiplicative subsemigroup $S$ of $N$ with $(n_1 + n_2)s = n_1 s + n_2 s$ for all $n_1, n_2 \in N$, $s \in S$, and $N$ is generated additively by $\{\sigma : \sigma \in S \text{ or } -\sigma \in S\}$. In this way we can regard the ring $Z$ of integers as a near-ring generated by $\{1\}$. To indicate that we are doing this we write $(Z, 1)$ for the d.g. near-ring of integers. All groups can now be considered as $(Z, 1)$-modules. (See [1] for further details).

Clearly (A) is equivalent to

(B) An $N$-module $M$ over a d.g. near-ring $N$ is injective if and only if for each right ideal $u$ of $N$ and each $N$-homomorphism $f:u \to M$ there is an $N$-homomorphism $\phi : N \to M$ such that $\phi$ restricted to $u$ is $f$.

We write all our groups additively (without implying commutativity) and then recall that a group $G$ is divisible if and only if for each $a \in G$ and each positive integer $n$ there is an element $b \in G$ with $nb = a$.

LEMMA. *If* (B) *is true then a group $G$ is divisible if and only if it is an injective* $(Z, 1)$-*module.*

**Proof.** This is a straightforward generalisation of the corresponding result for rings (See e.g. [2; p. 51]).

In particular, if (B) is true then $Q$, the group of rational numbers, is an injective $(Z, 1)$-module.

If $M$ is any set we let Sym($M$) be the symmetric group on $M$. For $f \in \mathrm{Sym}(M)$ let $S(f) = \{m \in M : mf \neq m\}$. If $A$ is any infinite cardinal then $\mathrm{Sym}(M, A) = \{f \in \mathrm{Sym}(M) : |S(f)| < A\}$. We also denote by $A^+$ the next larger cardinal after

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$A$. Let $B$ be any infinite cardinal with $B > |Q|^+$ and $M$ be any set with $|M| = B$. Then by [4; 11.5.4.] $Q$ can be imbedded in the simple group $S = \mathrm{Sym}(M, B^+)/\mathrm{Sym}(M, B)$ and by [4, 11.5.9.], $[\mathrm{Sym}(M, B^+):\mathrm{Sym}(M, B)] > B > |Q|$. But $Q$ is $(Z, 1)$-injective so the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & Q & \xrightarrow{\alpha} & S \\ & & {\scriptstyle i}\downarrow & & \\ & & Q & & \end{array}$$

where $\alpha$ is the imbedding and $i$ the identity on $Q$ can be completed to the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & Q & \xrightarrow{\alpha} & S \\ & & {\scriptstyle i}\downarrow & \swarrow{\scriptstyle \beta} & \\ & & Q & & \end{array}$$

However, since $\ker(\beta)$ is a normal subgroup of $S$ we get $\ker(\beta) = 0$ from which $\beta$ is $1-1$ which contradicts the cardinalities or $\ker(\beta) = S$ which contradicts $\beta$ being an extension of $i$. It follows that (B) is false.

Of course (B) is only false in the "if" part. This is also true of (A). In [3] consideration is given to the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \quad \text{(exact)} \\ & & {\scriptstyle g}\downarrow & & \\ & & M & & \end{array}$$

where $A$, $B$, $M$ are $N$-modules, and we write $C = f(A) \subset B$. The submodule $\bar{C}$ $B$ generated by $C$ is

$$\bar{C} = \left\{ \sum_{i=1}^{n} (-b_i + f(a_i) + b_i) : b_i \in B,\ a_i \in A \right\}.$$

A mapping $h: \bar{C} \to M$ is introduced and defined by

(i) $$h\left(\sum (-b_i + f(a_i) + b_i)\right) = \sum h(f(a_i))$$

where $h: C \to M$ is defined by $h(f(a)) = g(a)$. It is then asserted that $h$ restricted to $C$ is $h$ but this is not necessarily true.

Again take $N = (Z, 1)$ and $A = B = M$ to be non-abelian groups regarded as $(Z, 1)$-modules. Let $f$, $g$ be the identity. For $a, a_1 \in A$

$$h(f(-a_1) + f(a) + f(a_1)) = h(f(a)) = a \quad \text{(by (i), since } f(a_1) \in B)$$

However,

$$h(f(-a_1) + f(a) + f(a_1)) = h(f(-a_1 + a + a_1)) = h(f(-a_1 + a + a_1)) = -a_1 + a + a_1$$

and we deduce that $a_1+a=a+a_1$ for all $a$, $a_1 \in A$ which is false. It is now impossible to establish that $h$ is well defined.

## Bibliography

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