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STRONG EXTENSIONS VS. WEAK EXTENSIONS OF C^*-ALGEBRAS

BY
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Let $\mathcal{H}$ be a separable complex infinite dimensional Hilbert space, $\mathcal{L}(\mathcal{H})$ the algebra of bounded operators in $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the ideal of compact operators, $\mathcal{A}(\mathcal{H})=\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and $\pi::\mathcal{L}(\mathcal{H})\to\mathcal{A}(\mathcal{H})$ the quotient map. Throughout this paper A denotes a separable nuclear C^*-algebra with unit. An extension of A is a unital *-monomorphism of A into $\mathcal{A}(\mathcal{H})$. Two extensions τ_1 and τ_2 are strongly (weakly) equivalent if there exists a unitary (Fredholm partial isometry) U in $\mathcal{L}(\mathcal{H})$ such that

$$\tau_1(a)=\pi(U^*)\tau_2(a)\pi(U)$$

for all a in A. We denote the family of strong equivalence classes of extensions of A by $\operatorname{Ext}^s A$. Recent results of Voiculescu [7] and Choi-Effros [4] show that $\operatorname{Ext}^s A$ is always an abelian group. For more information about Ext^s of commutative C^*-algebras see [1, 2, 3]. We denote the strong equivalence class of an extension τ by $[\tau]$. Since if τ has a lifting (i.e. there exists a *-monomorphism σ of A into $\mathcal{L}(\mathcal{H})$ such that $\pi\sigma=\tau$) then every $\tau'\in[\tau]$ has a lifting, we can say without ambiguity that $[\tau]$ has a lifting.

Let

$$T^+=\{[\tau]\in \operatorname{Ext}^s A \mid [\tau] \text{ has a lifting}\}.$$

Let T be the subgroup of $\operatorname{Ext}^s A$ generated by T^+. We denote the quotient group $\operatorname{Ext}^s A/T$ by $\operatorname{Exy}^w A$.

Remark 1. If A has a one-dimensional representation, then for each $[\tau]\in T^+$ by adding an appropriate multiple of that one-dimensional representation to the lifting of $[\tau]$ we can make the corresponding lifting a unital one. Hence $T=0$ and $\operatorname{Ext}^w A=\operatorname{Ext}^s A$. This will be the case if A has a non-zero commutative quotient, and in particular $\operatorname{Ext}^s(A\oplus\mathbb{C})=\operatorname{Ext}^w(A\oplus\mathbb{C})$ for any nuclear algebra A.

Remark 2. For any finite dimensional C^*-algebra A, $\operatorname{Ext}^w A=0$.

Remark 3. The subgroup T is a homomorphic image of $\mathbb{Z}$. To see this, for each $n>0$ we let $\alpha(n)$ be an element in T^+ which has a lifting σ of

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codimension n, i.e. $\dim(1_H-\sigma(1))=n$. Suppose that two extensions τ_1 and τ_2 have liftings σ_1 and σ_2, respectively, of the same codimension. Since σ_i is a faithful representation on $\sigma_i(1)\mathcal{H}$, by (Theorem 1.5, [7]), there is a unitary U in $\mathcal{L}(\sigma_1(1)\mathcal{H}, \sigma_2(1)\mathcal{H})$ such that

$$\sigma_1(x)=U^*\sigma_2(x)U\in\mathcal{K}(\sigma_1(1)\mathcal{H})$$

for all x in A. Since $\operatorname{ind} U=0$ in $\mathcal{L}(\mathcal{H})$, we can make U a unitary in $\mathcal{L}(\mathcal{H})$. Hence $[\tau_1]=[\tau_2]$. Thus α is a well-defined monoid morphism of non-negative integers onto T^+. Hence we can extend α to a group homomorphism of $\mathbb{Z}$ onto T.

The following proposition justifies the notation for Ext^w.

Proposition 1. *An extension τ belongs to the weak equivalence class of the trivial extension if and only if $[\tau]\in T$.*

Proof. Suppose τ belongs to the weak equivalence class of the trivial extension. Then there exists a Fredholm partial isometry W such that $\pi(W^*)\tau(\cdot)\pi(W)$ has a unital lifting σ. We may assume that $W^*W=1$ or $WW^*=1$. If $W^*W=1_H$ then $W\sigma(\cdot)W^*$ is a lifting of τ. And if $WW^*=1_{\mathcal{H}}$, then $W^*\sigma(\cdot)W$ is a *-homomorphism of A into $\mathcal{L}(\mathcal{H})$. Let $\tau_1=\pi(W^*\sigma(\cdot)W)$. Consider $\tau+\tau_1$.

$$\begin{aligned}(\tau+\tau_1)(a)&=\tau(a)\oplus\tau_1(a)=\pi(W)\sigma(a)\pi(W^*)\oplus\pi(W^*)\sigma(a)\pi(W)\\&=\pi(W\oplus W^*)(\sigma(a)\oplus\sigma(a))\pi(W^*\oplus W)\end{aligned}$$

for all a in A. Since $\operatorname{ind}(W\oplus W^*)=0$ in $\mathcal{L}(\mathcal{H}\oplus\mathcal{H})$, the above relations show that $[\tau_1]=[\tau]^{-1}$. Hence $[\tau]\in T$.

For the other half of the proof, it is easy to see that if an extension τ is weakly equivalent to a trivial extension, then so is $[\tau]^{-1}$ (here we mean that $\tau'\in[\tau]^{-1}$ is weakly equivalent to a trivial one). And also it is obvious that the sum of two weakly trivial extensions is weakly equivalent to a trivial one. Thus any $[\tau]$ in T is weakly equivalent to a trivial extension.

Corollary. *Two extensions τ_1 and τ_2 are weakly equivalent iff τ_1 and τ_2 determine the element in* $\operatorname{Ext}^w A$.

Proof. τ_1 and τ_2 determine the same element in $\operatorname{Ext}^w A$ iff $[\tau_1]+[\tau_2]^{-1}\in T$ iff $[\tau_1]+[\tau_2]^{-1}$ is weakly trivial iff τ_1 and τ_2 are weakly equivalent.

Remark 4. Since $\operatorname{Ext}^w(A\oplus B)=\operatorname{Ext}^w A\oplus\operatorname{Ext}^w B$ (see Proposition 3.21, [3]), $\operatorname{Ext}^w A=\operatorname{Ext}^s(A\oplus\mathbb{C})$. This way of looking at $\operatorname{Ext}^w A$ arose in conversation with M. D. Choi.

For certain classes of C^*-algebras one can compute Ext. We begin with matrix algebras.

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LEMMA 1. (Proposition 1, [6]). *Let τ_i be two extensions of a matrix algebra M_n of rank n and let σ_i be lifting of τ_i for $i=1,2$. Then τ_1 and τ_2 are strongly equivalent iff*

$$\text{codimension } \sigma_1(1) \equiv \text{codimension } \sigma_2(1) \ (\text{mod } n).$$

This result was also obtained independently by Brown-Duglas-Fillmore, Bunce-Deddens, and Pearcy-Salinas.

Now suppose $A = \overline{\bigcup_{n=1}^{\infty} A_n}$, where the A_n's have the same unit and A_n is contained in A_{n+1}, and suppose that all A_n's and A are nuclear. Then $\text{Ext}^s A_n$ with connecting homomorphisms $i_n^*: \text{Ext}^s A_{n+1} \to \text{Ext}^s A_n$, where i_n are inclusions, form an inverse system of groups. It is easy to see that

$$\Phi: \text{Ext}^s A \to \varprojlim \text{Ext}^s A_n$$

by $\Phi([\tau]) = \{[\tau \mid A_n]\}$ is always surjective. (See Theorem 2.5, [3]) We will show that Φ is an isomorphism for UHF algebras.

DEFINITION. *A* C^*-algebra A with unit is approximately finite (AF) if there is an increasing sequence of finite dimensional algebras A_n with the same unit such that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$. If there is an increasing sequence of full matrix algebras with the same unit, the algebra is said to be uniformly hyperfinite (UHF). Suppose that M_{n_1} is contained in M_{n_2}. Then n_1 divides n_2 and the homomorphism induced by the inclusion is the obvious map of $\mathbb{Z}/n_2\mathbb{Z}$ onto $\mathbb{Z}/n_1\mathbb{Z}$. If $A = \overline{\bigcup_{k=1}^{\infty} M_{n_k}}$ is UHF, then for $[\tau] \in \text{Ext}^s A$, $\Phi([\tau])$ can be regarded as a sequence $\{a_k\}$, where a_n is the minimum of dimension $(1-\sigma_k(1))$ where $\pi\sigma_k = \tau \mid M_{n_k}$.

LEMMA 2. (Lemma 2, [6]). *Suppose $M_{n_1} \subset M_{n_2}$. If $[\tau_0]$ is the identity element of* $\text{Ext}^s M_{n_2}$ *i.e. τ_0 has a unital lifting, then every unital lifting σ of $\tau_0 \mid M_{n_1}$ can be extended to a unital lifting of τ_0.*

PROPOSITION 2. *For any* UHF *algebra $A = \overline{\bigcup_{k=1}^{\infty} M_{n_k}}$, $\Phi: \text{Ext}^s A \to \varprojlim \mathbb{Z}/n_k\mathbb{Z}$ is an isomorphism.*

Proof. Since any UHF algebra is nuclear, $\text{Ext}^s A$ is a group. It suffices to prove that Φ is one-to-one. For this purpose, suppose $\Phi([\tau]) = 0$. By applying Lemma 2 to each $\tau \mid M_{n_1}$, we get a unital *-monomorphism σ of $\bigcup_{k=1}^{\infty} M_{n_k}$ into $\mathcal{L}(\mathcal{H})$ such that $\tau \mid \bigcup_{k=1}^{\infty} M_{n_m} = \pi\sigma$. By continuity we get a unital lifting of $[\tau]$.

This result was obtained independently by Primsner and Popa [5].

PROPOSITION 3. *If $A = \overline{\bigcup_{k=1}^{\infty} M_{n_k}}$ is* UHF, *then* $\text{Ext}^w A \cong (\varprojlim \mathbb{Z}/n_k\mathbb{Z})/\mathbb{Z}'$, *where $\mathbb{Z}'$ is the subgroup generated by* $(1, 1, \ldots, 1, \ldots)$.

Proof. It is obvious that if τ has a lifting then the corresponding sequence described prior to Lemma 2 is constant eventually. Conversely, if the corresponding sequence is constant eventually, then by Lemma 2 τ has a lifting. Hence the subgroup T is isomorphic to the subgroup generated by $(1, 1, \ldots)$.

For AF algebras, Φ is not always an isomorphism. For if A is UHF, then $A \oplus \mathbb{C}$ is AF, and by Remark 1 $\operatorname{Ext}^s(A \oplus \mathbb{C}) = \operatorname{Ext}^w(A \oplus \mathbb{C}) = \operatorname{Ext}^w A$, and the latter is nonzero by Proposition 3. But $\operatorname{Ext}^s(M_{n_k} \oplus \mathbb{C}) = \operatorname{Ext}^w(M_{n_k} \oplus \mathbb{C}) = 0$ by Remark 2. In a private communication, L. G. Brown has indicated that $\lim^{(1)}$ sequence of [3] holds for AF algebras (this gives an expression for ker Φ).

We have used the fact that $\beta : \operatorname{Ext}^w A \oplus \operatorname{Ext}^w B \to \operatorname{Ext}^w(A \oplus B)$, defined by

$$\beta(\tau_1, \tau_2)(a \oplus b) = \tau_1(a) \oplus \tau_2(b)$$

for a in A and b in B, is an isomorphism. The same map defines a homomorphism β^s of $\operatorname{Ext}^s A \oplus \operatorname{Ext}^s B$ onto $\operatorname{Ext}^s(A \oplus B)$. For two UHF algebras β^s is never one-to-one. The following generalization of the original statement for UHF algebras was pointed out by J. Phillips and the referee.

Proposition 4. *For two nuclear C^*-algebras A and B, $\beta^s([\tau_1], [\tau_2]) = 0$ if and only if either $[\tau_1]$ and $[\tau_2]^{-1}$ have lifting of the same codimension or $[\tau_1]^{-1}$ and $[\tau_2]$ have lifting of the same codimension.*

Proof. ($\Leftarrow$) Without loss of generality we assume τ_1 and τ_2^{-1} have lifting of codimension k_0, say $\tau_1 = \pi\sigma_1$ and $\tau_2^{-1} = \pi\sigma_2$, where codimension of $\sigma_i(1)$ is k_0. Let $\sigma_1 = \sigma|_{A \oplus 0}$, $\sigma_2 = \sigma|_{0 \oplus B}$, $\sigma(1)(H_1 \oplus H_2) = K_1$ and $\sigma_2(1)(H_1 \oplus H_2) = K_2$. Since where U is a unitary of H_2 onto $\sigma_2(1)H_2$. Then $[\tau_B] = [\tau_2]$ and $\tau_1 + \tau_B$ has a unital lifting. ($\Rightarrow$)

Suppose $\beta^s([\tau_1], [\tau_2]) = 0$. Then there exists a unital lifting σ of $\beta^s([\tau_1], [\tau_2])$. Let $\sigma_1 = \sigma|_{A \oplus 0}$, $\sigma_2 = \sigma|_{0 \oplus B}$, $\sigma(1)(H_1 \oplus H_2) = K_1$ and $\sigma_2(1)(H_1 \oplus H_2) = K_2$. Since $\pi\sigma_i(1) = \tau_i(1)$, there exists partial isometries W_i such that $\pi(W_i) = \tau_i(1)$, $W_i W_i^* \le \sigma_i(1)$ and $W_i^* W_i \le P_i$, where P_i are projections onto H_i for $i = 1, 2$. Since $\pi(W_1 \oplus W_2) = 1$, $\operatorname{ind}(W_1 \oplus W_2) = 0$, which implies that ind W_1(in $L(H_1, K_1) = -$ind W_2(in $L(H_2, K_2)$). Therefore we get a unitary extension $U_1 \oplus U_2$ of $W_1 \oplus W_2$ such that either

$$U_1 : H_1 \oplus \mathbb{C}^{k_0} \to K_1 \quad \text{and} \quad U_2 : H_2 \ominus \mathbb{C}^{k_0} \to K_2$$

or

$$U_1 : H_1 \ominus \mathbb{C}^{k_0} \to K_1 \quad \text{and} \quad U_2 : H_2 \oplus \mathbb{C}^{k_0} \to K_2$$

where $K_0 = |\operatorname{ind} W_1|$. Again we may assume the latter occurs. Since $\pi(U_1 \oplus U_2) = 1$, $\pi(U_i) = \tau_i(1)$, and since

$$(U_1 \oplus U_2)^*(\sigma_{1(.)} \oplus \sigma_{2(.)})(U_1 \oplus U_2) = U_1^* \sigma_{1(.)} U_1 \oplus U_2^* \sigma_{2(.)} U_2,$$

we can assume that $K_1 = H_1 \ominus \mathbb{C}$ and $K_2 = H_2 \oplus \mathbb{C}$. Therefore τ_1 has a lifting $U_1^* \sigma_{1(\cdot)} U_1$ of codimension K_0 and τ_2^{-1} has a lifting of codimension K_0.

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