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LINEAR INEQUALITIES OVER COMPLEX CONES

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Introduction. The basic solvability theorems of Farkas [2] and Levinson [4] were recently extended in different directions by Ben-Israel [1] and Kaul [3].

The theorem stated in this note generalizes both results of Ben-Israel and Kaul and is applicable to nonlinear programming over complex cones.

Notation and preliminaries.

$C^n[R^n]$ the n dimensional complex [real] space.

R^n_+ the nonnegative orthant in R^n.

For $\alpha=(\alpha_i) \in R^n$ satisfying $0 \leq \alpha_i \leq \pi/2$:

$$T_\alpha = \{z \in C^n; |\arg z_i| \leq \alpha_i\}.$$

$C^{m\times n}[R^{m\times n}]$ the $m\times n$ complex [real] matrices.

For $A \in C^{m\times n}$:

A^* the conjugate transpose of A.

$N(A)$ the null space of A.

A nonempty set S in C^n is a *convex cone* if $0 \leq \lambda \Rightarrow \lambda S \subset S$ and $S+S \subset S$. S is a *polyhedral* (convex) cone if for some positive integer k there is a $B \subset C^{n\times k}$ such that $S=BR^k_+$. The *polar* of a nonempty set S, denoted by S^*, is the closed convex cone ([1, Theorem 1.3.a.])

$$S^* = \{y \in C^n; \operatorname{Re}(y, S) \geq 0\}.$$

Let $A \in C^{m\times n}$ and let S be a polyhedral cone in C^n. Then $N(A)+S$ is closed (or equivalently AS is closed), e.g. [1, Theorem 3.5].

For more properties and examples of cones consult [1] and the references there.

We mention here that R^n_+ and T_α are polyhedral convex cones, R^n_+ is self polar and

$$T_\alpha^* = \left\{\omega \in C^n, |\arg \omega_i| \leq \frac{\pi}{2} - \alpha_i\right\}.$$

THEOREM. *Let $A \in C^{m\times n}$, $b \in C^m$ and let C be an Hermitian positive semi definite matrix of order m and S a closed convex cone in C^n such that $N(A)+S$ is closed.*

Then the following are equivalent:

(a) *The system*

(1) $Ax-Cy=b$

(2) $x \in S$

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(3) $A^*y \in S^*$

(4) $y^*Cy \leq 1$

is consistent.

(b) $A^*z \in S^* \Rightarrow \text{Re}(b, z) + (z^*Cz)^{1/2} \geq 0.$

REMARKS. (1) Choosing C to be the zero matrix the theorem reduces to the solvability theorem of Ben-Israel ([1, Theorem 3.5]).

(2) Choosing $S=T_\alpha$, the theorem reduces after a slight change of notation, to the solvability theorem of Kaul [3], since $N(A)+T_\alpha$ is closed by the polyhedrality of T_α.

(3) The proof is a modification of the one in [3], where the theorem of Ben-Israel replaces the theorem of Levinson, and is sketched below.

Proof.

$\underline{(a) \Rightarrow (b)}$.

(5) $\text{Re}(Ax, z) = \text{Re}(b, z) + \text{Re}(Cy, z)$ by (1)

(6) $\text{Re}(Cy, z) \leq (z^*Cz)^{1/2}(y^*Cy)^{1/2}$ by the Cauchy-Schwartz inequality.

$\leq (z^*Cz)^{1/2}$ by (4)

$(5)+(6) \Rightarrow$

$\text{Re}(b, z) + (z^*Cz)^{1/2} \geq \text{Re}(Ax, z)$

$= \text{Re}(x, A^*z) \geq 0$ if $A^*z \in S^*$ by (2)

$\Rightarrow (b)$.

$\underline{(b) \Rightarrow (a)}$.

Let W denote the set

$$W = \{Ax - Cy, x \in S, A^*y \in S^*, y^*Cy \leq 1\}$$

For the second part of the proof it is crucial to show that W is closed.

Let u be in the closure of W. Then there exist sequences $\{x_k\}$, $\{y_k\}$ and $\{u_k\}$ such that x_k satisfies (2), y_k satisfies (3) and (4) and $u_k = Ax_k - Cy_k \rightarrow u$. $\{y_k\}$ can be chosen so that it has a limit point e.g. [3]. Let y be a limit point of $\{y_k\}$. Then $A^*y \in S^*$, $y^*Cy \leq 1$ and one has to show that there exists $x \in S$ such that $u = Ax - Cy$. Since $x_k \in S$, $A^*z \in S^* \Rightarrow \text{Re}(A^*z, x_k) \geq 0$,

$$\Rightarrow \text{Re}(Ax_k, z) \geq 0 \Rightarrow \text{Re}(u_k + Cy_k, z) \geq 0 \Rightarrow \text{Re}(u + Cy, z) \Rightarrow 0.$$

By theorem 3.5 of [1](1), this is equivalent to the consistency of $Ax = u + Cy$, $x \in S$ and thus $u \in W$.

Suppose now that (a) is false. Then b is separated from W; and since W is closed, there exist a vector $y_0 \neq 0$ and a scalar k such that

(7) $\text{Re}(Ax - Cy, y_0) \geq k > \text{Re}(b, y_0)$

for every x, y satisfying (2), (3), and (4). Substituting $y=0$ in (5) gives

(1) The assumption that $N(A)+S$ is closed is needed here. Example 2.5 in [1] shows that it is essential.

$\mathrm{Re}(x, A^*y_0) \geq k$ for every $x \in S$ and since S is a cone this implies that y_0 satisfies (3). Thus by (b):

(8) $k > \mathrm{Re}(b, y_0) \geq -(y_0^* C y_0)^{1/2}$. Substituting $x=y=0$ in (7) implies $k \leq 0$ and thus $(y_0^* C y_0)^{1/2} > 0$.

Now (7)+(8)$\Rightarrow$

(9) $\mathrm{Re}(Ax - Cy, y_0) > -(y_0^* C y_0)^{1/2}$

for every x, y satisfying (2), (3), and (4). Let $y_1 = (y_0^* C y_0)^{-1/2} y_0$. Then $y_1^* C y_1 = 1$ and so y_1 satisfies (3) and (4), $x=0$ and $y=y_1$ satisfy (2), (3), and (4) and substituting them in (9) gives:

$$-y_0^* C (y_0^* C y_0)^{-1/2} y_0 > -(y_0^* C y_0)^{1/2}.$$

Contradiction.

References

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