BULL. AUSTRAL. MATH. SOC. 35B65
VOL. 22 (1980), 219-230. (35K15)

ON SOLUTIONS OF PARABOLIC EQUATIONS IN REGIONS WITH EDGES

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In this paper, smoothness properties of solutions of the initial-Dirichlet problem for parabolic equations in regions with edges are considered. We obtain bounds for solutions and derivatives, and prove the Hölder continuity of the first derivatives and of the second derivatives multiplied by a suitable power of the distance from the edges.

1. Introduction

This investigation is concerned with the behaviour of solutions of linear parabolic equations in regions with edges, and it may be motivated as follows. For *elliptic equations*, the behaviour of solutions near singularities of the boundary of the domain has been studied under various assumptions by means of complex analysis, Hölder and Sobolev space methods (*cf.* for example, Dziuk [*1*], Kondrat'ev [*4*], Wigley [*10*], and the review by Grisvard [3]) and, recently, by methods of geometric measure theory (*cf.* Simon [*8*]). The earliest interest in these problems arose from conformal mapping, in connection with the behaviour of mapping functions near the boundary (*cf.* Warschawski [9]). Later work also had important applications in elasticity theory, fluid flow, and numerical analysis ("subtraction of singularities", estimation of truncation errors in difference methods; *cf.* Laasonen [5]).

The methods used in the above and related papers do not extend to

Received 17 March 1980.

https://doi.org/10.1017/S0004972700006511 Published online by Cambridge University Press

parabolic equations; indeed, in this case, comparatively little is known about general smoothness properties of solutions if the boundary of the domain has singularities. We shall be concerned with the initial-Dirichlet problem for parabolic equations

$$(1)\quad Lu(x, t) = a_{ij}(x)u_{x_i x_j} + a_i(x, t)u_{x_i} + a(x, t)u - u_t = f(x, t) ,$$

$x = (x_1, x_2)$, with $C^\alpha(\bar\Omega)$-coefficients, $0 < \alpha < 1$, in a cylindrical region $\Omega = G \times J_1$, where $J_1 = \{t \mid 0 < t \le t_1\}$, $t_1 > 0$, $G \subset R^2$ is a domain with corners to be specified below, and in (1) we use the summation convention. We shall introduce a method of obtaining smoothness statements for solutions of (1) satisfying initial-boundary conditions of Dirichlet type.

2. Bounds for solutions in a cylindrical sector

The first step of the method consists of deriving bounds for solutions of (1) in the special region $\Omega = G \times J_1$, where G is a circular sector. We use the notations

$$(2)\quad \begin{aligned} G_\sigma &= \{(r, \theta, 0) \mid 0 < r < \sigma, \theta \in I\} , \\ B_\sigma &= \{(r, \theta) \mid 0 < r < \sigma, \theta = \beta \text{ or } \beta+\omega\} , \\ J_2 &= \{t \mid 0 < t \le t_2\} , \quad t_2 \le t_1 , \end{aligned}$$

where $I = (\beta, \beta+\omega)$, the angle $\omega \in (0, \pi)$ is given, $\beta > 0$ is such that $\pi/2 < \omega + 2\beta < \pi$, and r, θ are defined by $x_1 = r \cos\theta$, $x_2 = r \sin\theta$.

THEOREM 1. *Let u be a bounded solution of the problem*

$$(3)\quad Lu = f \quad in \quad \Omega_1 = G_\sigma \times J_2 ,$$

$$(4)\quad u = 0 \quad on \quad \bar G_\sigma \quad and \quad B_\sigma \times J_2 ,$$

where $a_{ij}(0) = \delta_{ij}$, $a_{ij} \in C^0(\bar\Omega_1)$, and a_i, a, f are bounded in $\bar\Omega_1$. Then there exist $r_0 \in (0, \sigma/2)$ and $\nu \in (0, 1)$ such that in $\bar\Omega_2$, where $\Omega_2 = G_{2r_0} \times J_2$,

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(5) $$|u(x,\ t)| \le Mr^{1+\nu}$$

with $M > 0$ *independent of* $(x,\ t)$.

Proof. We write (1) in the form

$$Lu = \Delta u + \left(a_{ij}-\delta_{ij}\right)u_{x_ix_j} + a_iu_{x_i} + au - u_t = f$$

and introduce the function

$$v(x) = \tilde{v}(r,\ \theta) = -Mr^{\mu}\sin\lambda\theta \quad (M > 0)\ ,$$

assuming that

$$1 < \mu < \lambda = \frac{\pi}{2\beta+\omega} < 2\ .$$

Then

$$Lv(x) = \left(\lambda^2-\mu^2\right)Mr^{\mu-2}\sin\lambda\theta + Mh_{ij}(x)\left(a_{ij}(x)-\delta_{ij}\right)r^{\mu-2} + Mh_1(x,\ t)r^{\mu-1} + Mh_2(x,\ t)r^{\mu}\ ,$$

where h_{ij}, h_1, h_2 denote functions which are bounded in $\overline{\Omega}_1$, say,

$$|h_{ij}(x)| \le R_0\ ,\quad |h_i(x,\ t)| \le R_0\ ,\quad i,\ j = 1,\ 2\ .$$

Since $a_{ij} - \delta_{ij}$ is continuous and vanishes at $x = 0$, given $\varepsilon > 0$, there is an $r_0 \in (0,\ \sigma/2)$ such that for $r \le 2r_0$,

$$|a_{ij}(x)-\delta_{ij}| < \varepsilon/4R_0\ .$$

Since $\sin\lambda\theta \ge \sin\lambda\beta$ if $\theta \in \overline{I}$, we thus obtain in Ω_2 ,

$$Lv(x) \ge M\left[\left(\lambda^2-\mu^2\right)\sin\lambda\beta - \varepsilon\right]r^{\mu-2} - MR_0r^{\mu-1} - MR_0r^{\mu}\ .$$

For a positive $\varepsilon < \left(\lambda^2-\mu^2\right)\sin\lambda\beta$ the right-hand side tends to infinity as $r \to 0$. Since f is bounded in $\overline{\Omega}_1$, by choosing $r_0 > 0$ sufficiently small, we thus can make

$$Lv(x) \ge f(x,\ t)\ .$$

Hence, in Ω_2 ,

(6) $$L\left(u(x,\ t)-v(x)\right) \le 0 \ .$$

We prove next that $u - v$ is nonnegative on $S = \partial\Omega_2 \backslash \hat{G}_{2r_0}$ for sufficiently large M ; here

$$\hat{G}_\sigma = \left\{\left(r,\ \theta,\ t_2\right) \mid 0 < r < \sigma,\ \theta \in I\right\} ,$$

u is zero on $\overline{G}_{2r_0}$ and $B_{2r_0} \times J_2$, so that on this part of S ,

(7) $$u(x,\ t) - v(x) = -v(x) \ge 0 \ .$$

On the other part of S ,

$$u(x,\ t) - v(x) \ge -\|u\|_0 + M\left(2r_0\right)^\mu \sin \lambda\beta \ ,$$

where $\|\cdot\|_0$ is the sup norm in $\overline{\Omega}_2$. This shows that the left-hand side is positive for sufficiently large M . From this, (6), (7), and the maximum principle (*cf*. [6], pp. 174-175) it follows that in $\overline{\Omega}_2$,

$$u(x,\ t) \ge v(x) = -Mr^\mu \sin \lambda\theta \ge -Mr^\mu \ .$$

Similarly, for sufficiently large M and small $r_0 > 0$ the maximum principle also yields

$$u(x,\ t) \le Mr^\mu$$

in $\overline{\Omega}_2$. Since $1 < \mu < 2$, setting $\nu = \mu - 1$, we have (5). Theorem 1 is proved.

3. Bounds for derivatives of solutions in a cylindrical sector

In the second step of our method, using Theorem 1 and the notations of Section 2, we estimate the first and second partial derivatives of solutions in a cylindrical sector.

THEOREM 2. *Let* u *be a bounded solution of the problem* (3), (4) *in* $\Omega_2 = G_{2r_0} \times J_2$ *with* r_0 *as in Theorem* 1, *and assume the coefficients of* L *and the function* f *to be of class* $C^\alpha\left(\overline{\Omega}_2\right)$, $0 < \alpha < 1$. *Then in* $\overline{\Omega}_3$, *where* $\Omega_3 = G_{r_0} \times J_2$,

(8) $$\left|D_x^k u(x,\ t)\right| \le M_k r^{1+\nu-k}\ ,\quad k = 1,\ 2\ ,$$

with ν *as in Theorem* 1; *here,* $D_x^k u$ *denotes any* k*th partial derivative of* u *with respect to* $x_1,\ x_2$.

Proof. For $n = -1,\ 0,\ 1,\ \ldots$ let

$$H_n = \left\{(r,\ \theta,\ t)\ \middle|\ 2^{-n-2}r_0 \le r \le 2^{-n-1}r_0,\ \theta \in \bar{I},\ 0 < t < t_2\right\}\ ,$$

and $\tilde{H}_n = H_{n-1} \cup H_n \cup H_{n+1}$, $n = 0,\ 1,\ \ldots$. Then $H_n,\ \tilde{H}_n \subset \bar{\Omega}_2$. The transformation

(9) $$x = 2^{-n}y\ ,\quad y = \left(y_1,\ y_2\right)$$

maps H_n onto H_0 and $\tilde{H}_n$ onto $\tilde{H}_0$. Also $w(y,\ t) = u\left(2^{-n}y,\ t\right)$ satisfies in $\tilde{H}_0$ the parabolic equation

(10) $$b_{ij}w_{y_iy_j} + 2^{-n}b_iw_{y_i} + 2^{-2n}bw - 2^{-2n}w_t = 2^{-2n}g\ ,$$

where $b_{ij}(y) = a_{ij}\left(2^{-n}y\right)$, and so on, and $g(y,\ t) = f\left(2^{-n}y,\ t\right)$; *cf.* (1). We now use a Schauder type estimate (*cf.* [2]) in H_0 and $\tilde{H}_0$:

(11) $$\|w\|_{2+\alpha}^{H_0} \le \eta\left(\|w\|_0^{\tilde{H}_0} + 2^{-2n}\|g\|_\alpha^{\tilde{H}_0}\right)\ .$$

Here the constant η does not depend on w , and the norms are defined as usual, that is,

$$\|w\|_{2+\alpha}^{H_0} = \sum_{i=0}^{2}\left\|D_y^i w\right\|_0^{H_0} + h_\alpha^{H_0}\left(D_y^2 w\right)\ ,$$

where $\|\cdot\|_0$ denotes the sup norm and $h_\alpha^{H_0}$ the Hölder coefficient, and so on. In (11), by Theorem 1,

$$\|w\|_0^{\tilde{H}_0} \le M_0\left(2^{-n}\right)^\mu \quad (\mu = 1+\nu)\ .$$

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Since $\|g\|_\alpha^{\tilde{H}_0} < \infty$, from (11) we thus have

(12) $$\|w\|_{2+\alpha}^{H_0} \le \eta_1\left(2^{-\mu n}+2^{-2n}\right) \le 2\eta_1 2^{-\mu n} .$$

Now, by (9), for corresponding derivatives in the x and y systems,

$$D_y^k w = \partial^k w/\partial y_1^{k_1}\partial y_2^{k-k_1} = 2^{-nk}D_x^k u = 2^{-nk}\left\{\partial^k u/\partial x_1^{k_1}\partial x_2^{k-k_1}\right\} ,$$

where $k_1 = 0, \ldots, k$, and $k = 1, 2$. In H_0 ,

$$\left|D_y^k w(y, t)\right| \le \|w\|_{2+\alpha}^{H_0} , \quad k = 1, 2 ,$$

so that (12) yields in H_n ,

(13) $$2^{-nk}\left|D_x^k u(x, t)\right| \le \eta_2 2^{-\mu n} , \quad k = 1, 2 .$$

Now $2^{-n-2}r_0 \le r \le 2^{-n-1}r_0$ in H_n . Hence by (13), in H_n ,

$$\left|D_x^k u(x, t)\right| \le \eta_2\left(2^{-n}\right)^{\mu-k} \le M_k r^{\mu-k} , \quad k = 1, 2 .$$

Since $\mu = 1 + \nu$, Theorem 2 follows.

4. Smoothness of solutions in a cylindrical sector

We now show that a bounded solution of the problem (3), (4) is of class $C^{1+\nu}(\bar{\Omega}_3)$ in x , where $\Omega_3 = G_{r_0} \times J_2$, as in Theorem 2. Note that u is of class $C^{1+\alpha/2}(\bar{\Omega}_3)$ in t , so that the corner of G_{r_0} does not affect the t-smoothness of the solution.

THEOREM 3. *Let u be a bounded solution of the problem* (3), (4) *in Ω_2 . Assume that the conditions in Theorem* 2 *are satisfied. Then for u , considered as a function of x , we have*

$$u \in C^{1+\nu}(\bar{\Omega}_3) \quad (0 < \nu < 1)$$

with r_0 and ν as in Theorem 1. *Furthermore, for any* $\kappa \in (1-\nu, 1)$,

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(14) $$r^{\kappa}D_x^2 u \in C^{\chi}(\bar{\Omega}_3) ,$$

where $\chi = \min(\kappa+\nu-1, \alpha)$.

Proof. We use the metric defined by

$$d(P, \tilde{P})^2 = |x-\tilde{x}|^2 + |t-\tilde{t}| ,$$

where $P : (x, t)$, $\tilde{P} : (\tilde{x}, \tilde{t})$. In cylindrical coordinates we also write $P : (r, \theta, t)$, and $u(x, t)$ will be written $u(P)$, for simplicity. Consider $P : (r_1, \theta_1, t)$ and $Q : (r_2, \theta_2, t)$ in $\bar{\Omega}_3$, corresponding to the same t , but otherwise arbitrary; let $0 \le r_2 \le r_1 \le r_0$, without restriction.

Case 1. If $r_2 \le r_1/2$, then $d(P, Q) \ge r_1/2$, so that (8) yields

$$|D_x u(P)-D_x u(Q)|/d(P, Q)^{\nu} \le M_1\left(r_1^{\nu}+r_2^{\nu}\right)/\left(r_1/2\right)^{\nu} \le K = \text{const.}$$

In a similar fashion it follows that $r^{\kappa}D_x^2 u$ satisfies an analogous inequality.

Case 2. Let $r_2 > r_1/2$. We consider the transformation

(15) $$x = \xi y , \quad \xi = 2r_1/r_0 ,$$

where $y = (y_1, y_2)$. We set $\rho = r/\xi$. Now P has the image $P^* : (\rho_1, \theta_1, t)$, $\rho_1 = r_0/2$, and Q has the image $Q^* : (\rho_2, \theta_2, t)$, $\rho_2 = r_2/\xi > r_0/4$. The region

$$R = \{(r, \theta, t) \mid r_1/2 \le r \le r_1, \theta \in \bar{I}, 0 \le t \le t_2\}$$

is mapped onto

$$R^* = \{(\rho, \theta, t) \mid r_0/4 \le \rho \le r_0/2, \theta \in \bar{I}, 0 \le t \le t_2\} ,$$

and the region

$$S = \{(r, \theta, t) \mid r_1/4 \le r \le 2r_1, \theta \in \bar{I}, 0 \le t \le t_2\}$$

onto

$$S^* = \{(\rho, \theta, t) \mid r_0/8 \le \rho \le r_0, \theta \in \bar{I}, 0 \le t \le t_2\} .$$

We now consider the function w defined by $w(y, t) = u(x, t)$. This function is zero on the bottom part $(t = 0)$ and the two plane lateral parts of the boundary of S^*, and in S^* it satisfies the parabolic equation

$$c_{ij}w_{y_iy_j} + \xi c_i w_{y_i} + \xi^2 cw - \xi^2 w_t = \xi^2 g ,$$

where $c_{ij}(y) = a_{ij}(\xi y)$, and so on, and $g(y, t) = f(\xi y, t)$. In R^* and S^* we again apply a Schauder type estimate for parabolic equations:

$$\|w\|_{2+\alpha}^{R^*} \le \eta\left\{\|w\|_0^{S^*}+\xi^2\|g\|_\alpha^{S^*}\right\} .$$

As in the proof of Theorem 2, from this it follows that

$$\|w\|_{2+\alpha}^{R^*} \le \eta_1 r_1^\mu \quad (\mu = 1+\nu) .$$

As before, for corresponding derivatives in the x and y systems we have

$$D_y^k w = \xi^k D_x^k u , \quad k = 1, 2 ,$$

and, furthermore,

$$h_\nu^{R^*}\left(D_y w\right) = \xi^\mu h_\nu^R\left(D_x u\right)$$

as well as, for any $\chi \in (0, \alpha]$,

$$h_\chi^{R^*}\left(D_y^2 w\right) = \xi^{2+\chi} h_\chi^R\left(D_x^2 u\right) .$$

Consequently,

$$h_\nu^R\left(D_x u\right) \le K_1 ,$$

that is, $D_x u \in C^\nu(R)$. It implies that $u \in C^{1+\nu}(R)$, considered as a function of x. Similarly,

$$h_\chi^R\left(D_x^2 u\right) \le K_2 r_1^{\mu-2-\chi} .$$

Hence by Theorem 2 and formula (14),

$$\left|r_1^\kappa D_x^2 u(P) - r_2^\kappa D_x^2 u(Q)\right| / d(P, Q)^\chi \le K_3 .$$

This proves the second statement of the theorem in Case 2 and completes the proof.

5. Smoothness of solutions in general cylindrical regions

In this last step of our method, using Theorem 3 we obtain the main result, which concerns the smoothness of solutions of the initial-Dirichlet problem

(16) $$Lu = f \quad \text{in} \quad \dot{\Omega} = G \times J_2 \ ,$$

(17a) $$u|_{\bar{G}\times\{0\}} = 0 \ ,$$

(17b) $$u|_{\partial G\times J_2} = 0$$

in a general region $\Omega = G \times J_2$ with L as in (1) and J_2 as in (2), where G has corners. Obviously, it suffices to consider the case of a single corner, from which the case of finitely many corners (each satisfying the conditions of Theorem 4, below) results in a trivial way. Accordingly, we assume the following.

(A1) $G \subset \mathbb{R}^2$ is a simply connected bounded domain. Its boundary ∂G is of class $C^{2+\alpha}$, except at a point P, at which ∂G has a corner with interior angle $\gamma > 0$. Let ω denote the angle into which γ is transformed under the transformation of

(18) $$a_{ij}(P)u_{x_i x_j} = 0$$

into canonical form.

(A2) The coefficients of L and the function f are of class $C^\alpha(\bar{\Omega})$.

Note that ω does not depend on the special choice of that transformation. Furthermore, if $G^* \subset G$ is a compact region having positive distance from the corner P, then a bounded solution u of (16), (17) in Ω, considered as a function of x, is of class $C^{2+\alpha}(\bar{\Omega}^*) \cap C^0(\bar{\Omega})$, where $\Omega^* = G^* \times J_2$; *cf.* [2], Chapter 3.

THEOREM 4. *Let* u *be a bounded solution of the problem* (16), (17) *in* Ω *with* L *given in* (1), *and assume that* (A1) *and* (A2) *are satisfied. If* $\omega < \pi$, *then for* u , *considered as a function of* x , *we have*

$$u \in C^{1+\nu}(\bar{\Omega})$$

with a suitable $\nu \in (0, 1)$.

Proof. Let the corner P be at $x = (0, 0) = 0$, without restriction. Let

$$x_1 = \varphi_1(x_2) \quad \text{and} \quad x_2 = \varphi_2(x_1)$$

represent the two arcs of ∂G emanating from P , in a neighborhood of P . Then $\varphi_1(0) = \varphi_2(0) = 0$. Moreover, we assume that

$$\varphi_1'(0) = \cot \gamma \quad \text{and} \quad \varphi_2'(0) = 0 .$$

We consider the subregion $\Omega^1 = N^\lambda \times J_2 \subset \Omega$, where

$$N^\lambda = \{x \in G \mid |x| \le \lambda\}$$

with $\lambda > 0$ sufficiently small. Since in $\bar{\Omega}\backslash\overline{\Omega^1}$ the solution u is of class $C^{2+\alpha}$ in x , it is sufficient to prove the theorem in Ω^1 (instead of Ω). We first apply a mapping $(x_1, x_2) \mapsto (y_1, y_2)$ such that at the corner the coefficients of the transformed principal part of L have the values δ_{ij} and $N^{\tilde{\lambda}} = \{x \in G \mid |x| \le \tilde{\lambda}\}$ is mapped onto a region $\hat{N}$ in the y_1y_2-plane whose boundary consists of two straight segments in the direction $y_2 = 0$ and $y_2 = y_1 \tan \omega$, respectively, and a simple arc joining the other endpoints of those segments. Such a mapping is

$$(19) \qquad \begin{aligned} y_1 &= \left[k_{12}\left(x_1-\varphi_1(x_2)\right)+k_{11}\left(x_2-\varphi_2(x_1)\right)\right]/\delta\sqrt{k_{11}} , \\ y_2 &= \left(x_1-\varphi_1(x_2)\right)/\sqrt{k_{11}} , \end{aligned}$$

where

$$k_{11} = a_{11}(0) - 2\varphi_1'(0)a_{12}(0) + \varphi_1'^2(0)a_{22}(0) ,$$

$$k_{12} = a_{22}(0)\varphi_1'(0) - a_{12}(0) ,$$

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$$\delta = \left[a_{11}(0)a_{22}(0)-a_{12}(0)^2\right]^{\frac{1}{2}} .$$

Here $\tan\omega = \delta/k_{12}$, $0 < \omega < \pi$. Note that ω depends only on γ and on the values of the coefficients of the principal part of L at the origin. The function $u_1(y, t) = u(x, t)$ satisfies in $\hat{\Omega} = \hat{N} \times J_2$ a parabolic equation which is obtained from (16) and (19). From (17) and (19) it follows that u_1 vanishes for $t = 0$ as well as on the plane parts of $\partial\hat{\Omega}$, which correspond to those two straight segments. Since $\omega < \pi$, we can rotate $\hat{\Omega}$ about the t-axis through an angle β such that $\pi/2 < \omega+2\beta < \pi$. The composite of the two mappings is a mapping $(x_1, x_2) \mapsto (z_1, z_2)$ such that $u_2(z, t) = u(x, t)$ satisfies the conditions in Theorems 1 to 3 in the transformed region. This implies that $u_2 \in C^{1+\nu}(\bar{\Omega}^*)$, where $\Omega^* = G^* \times J_2$ and G^* is a circular sector in the z_1z_2-plane whose radius r_0 is sufficiently small. Since the value of the Jacobi determinant of (19) at $(0, t)$ is $1/\delta \neq 0$, it follows that $u \in C^{1+\nu}(\overline{\Omega^1})$. This completes the proof.

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