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A GENERATION PROCEDURE FOR THE SIMPLE 3-POLYTOPES WITH CYCLICALLY 5-CONNECTED GRAPHS

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In this paper we derive a generation procedure for the simple (3-valent) 3-polytopes with cyclically 5-connected graphs. (A graph is called *cyclically n-connected* if it cannot be broken into two components, each containing a cycle, by the removal of fewer than n edges.) We define three new types of face splitting and we show, in Theorems 16 and 17, that the simple 3-polytopes with cyclically 5-connected graphs are exactly the polytopes obtained from the dodecahedron by these face splittings.

We clarify our terminology with a definition. The polytope G' will be said to be obtained from the polytope G by a *simple face splitting* if G' is obtained from G by adding a new vertex on each of two distinct edges of some face of G and a new edge connecting these vertices across the face, as illustrated in Figure 0.

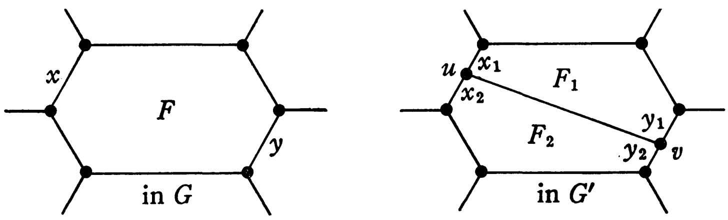

FIGURE 0

Procedures for the generation of all 3-polytopes from the tetrahedron have been given by Eberhard [**3**], Brückner [**2**], Steinitz [**13**], Steinitz and Rademacher [**14**]. See Klee [**10**] and Grünbaum [**8**; **9**] for a summary of these results. From one of the several proofs of Steinitz's theorem, given in Steinitz and Rademacher [**14**], one sees that the simple 3-polytopes whose graphs are cyclically 3-connected, that is all simple 3-polytopes, are exactly the polytopes generated from the tetrahedron by simple face splittings. Kotzig [**11**], Faulkner [**6**] and Faulkner and Younger [**7**] have shown that the simple 3-polytopes with cyclically 4-connected graphs are exactly those polytopes obtained from the

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cube by successive simple face splittings performed on non-adjacent edges of a face, that is, such that neither of the two new faces is a triangle.

Following the notation of Kotzig [**12**], a graph will be called a *Z-graph* if it is cyclically 5-connected, planar and 3-valent. A set of edges, X, of a graph G is a *cut* if removing the edges separates G into two components and no proper subset of X has this property. The components are called the *banks* of the cut. A cut will be called *non-trivial* if each bank contains a circuit, *trivial* otherwise. If the cardinality of the cut is n, it will be called an *n-cut*. We note that if X is a 3-cut in a Z-graph, it must be a trivial cut, and therefore one bank must be a vertex. Similarly, if X is a 4-cut in a Z-graph, it is a trivial cut and one bank must consist exactly of one edge with its two vertices. A set of n distinct faces, $s_1, \ldots, s_n$ of a graph, G, is called an *n-ring* if there exist distinct edges $a_1, \ldots, a_n$ in G such that

$$s_i \text{ Adj } s_{i+1} \text{ on } a_i,\ 1 \leqq i < n, \text{ and } s_n \text{ Adj } s_1 \text{ on } a_n$$

and $\{a_1, \ldots, a_n\}$ is an n-cut in G. It is called a *non-trivial n-ring* if the cut is a non-trivial n-cut. We will use the notation s Adj t (s adjacent to t), to indicate that the two faces, s and t, have a common edge, and the notation s Adj $(e)t$ or s Adj t on e, to indicate that s and t have the common edge, e.

We now define three new types of face splitting and the corresponding reductions. Note that these reductions, as all inverse operations, are not always performable in the class of Z-graphs. In fact, whether or not these reductions can be performed plays an essential rule in the proof of our main result, Theorem 16.

Face splittings.

Type 1 is any simple face split, as defined above, which does not create a face with fewer than five sides.

Type 2 is the split of two adjacent pentagonal faces into four pentagonal faces, as illustrated in Figure 1, by introducing four more vertices and six more edges.

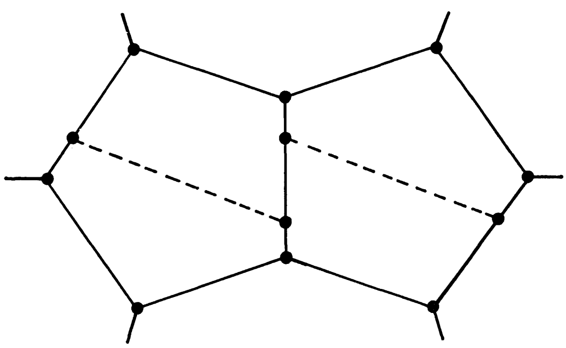

FIGURE 1

Type 3 is the split of a pentagonal face into six pentagonal faces, as illustrated in Figure 2, by the introduction of ten more vertices and fifteen more edges.

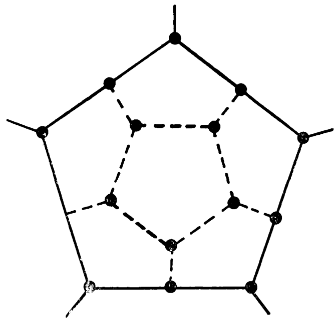

FIGURE 2

Reductions.

Type 1 is the merging of any two adjacent faces by removing the common edge and suppressing the two associated vertices.

Type 2 is the reduction of four pentagons, adjacent as illustrated in Figure 3, to two pentagons by removing the two edges indicated by dark lines, and suppressing the associated vertices.

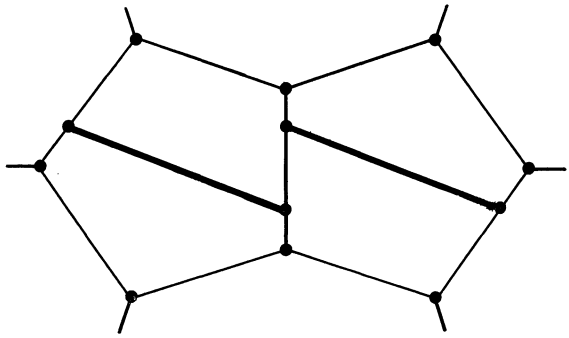

FIGURE 3

Type 3 is the reduction of six pentagons, adjacent as in Figure 4, to one pentagon by removing the ten edges indicated by dark lines and suppressing

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the associated vertices.

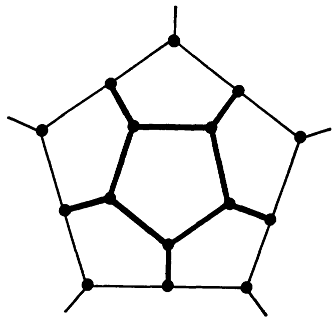

FIGURE 4

LEMMA 1. *If G is a Z-graph and G' is obtained from G by any face split of Type* 1, 2, *or* 3 *then G' is a Z-graph.*

Proof. Clearly, these three face splits preserve 3-valency, planarity and cyclically 5-connectedness.

LEMMA 2. *If G is cyclically n-connected, $n \geqq 4$, 3-valent and planar, and G' is obtained from G by any reduction of Type* 1, 2 *or* 3, *then G' is 3-valent, planar and 3-edge connected.*

Proof. Assume G is cyclically n-connected, $n \geqq 4$, 3-valent and planar. It follows that G is 3-edge connected and cannot have a non-trivial k-cut, $k < 4$. Clearly, by the nature of our reductions, G' is 3-valent and planar. If G' is not 3-edge connected then there is an m-cut, X, $m < 3$, in G'. Since G' is planar and 3-valent, each bank of the cut contains a cycle. Therefore X is a non-trivial m-cut in G'.

If the reduction is of Type 1 on e, as in Figure 5, with a and b the two edges in G' which are joined by e in G, we cannot have a and b in the same bank of X, nor $\{a, b\} = X$, because then there would be a non-trivial m-cut in G. If a and b are in opposite banks of X then $\{e\} \cup X$ is a non-trivial 3-cut in G, which is impossible. If $a \in X$

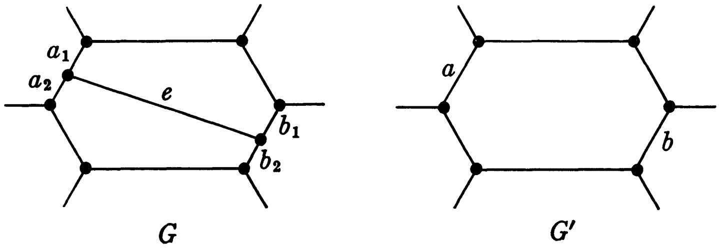

FIGURE 5

and $b \notin X$, again, we can construct a non-trivial m-cut in G by including e in the same bank as b.

If the reduction is of Type 2 on the edges e and f, as illustrated in Figure 6, then clearly $X \not\subseteq \{a, b, c\}$, since the third edge would connect the two banks.

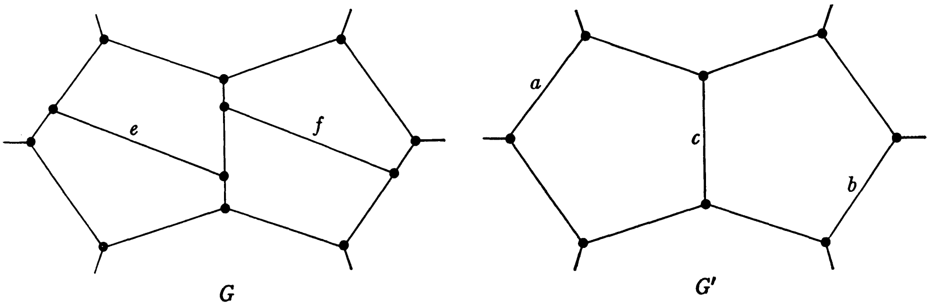

FIGURE 6

If a only, or b only, is in X we could construct a non-trivial m-cut in G. We cannot have c only in X. If a, b and c are all in the same bank, then X is a non-trivial m-cut in G. Finally, if a and c (or b and c) are in different banks we can construct a non-trivial 3-cut in G.

If the reduction is of Type 3, as illustrated in Figure 7, and $X = \{a, b\}$ then

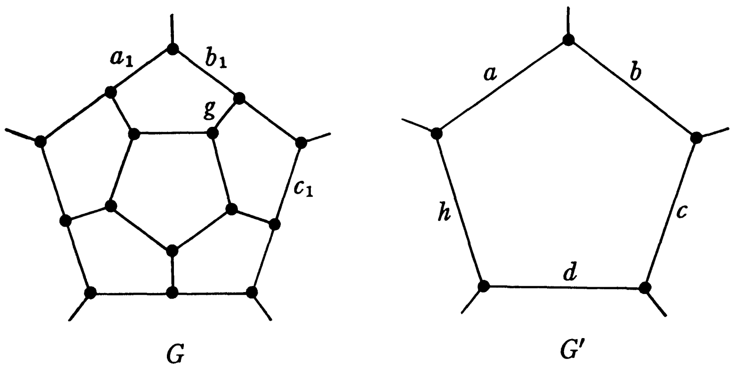

FIGURE 7

$\{a_1, b_1\}$ is a non-trivial 2-cut in G. If $X = \{a, c\}$ then $\{a_1, g, c_1\}$ is a non-trivial 3-cut in G. We cannot have exactly one of the edges a, b, c, d or h in X. If none of the edges a, b, c, d, h are in X, they must all be in the same bank and X is a non-trivial m-cut in G. This establishes the lemma.

LEMMA 3. *If G is a Z-graph and G' is obtained from G by one of the three reductions, then either G' is a Z-graph or G' has a non-trivial n-cut, $n = 3$ or 4.*

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Proof. By Lemma 2, G' is 3-valent, planar and 3-edge connected, hence cyclically 3-connected. Therefore, if G' is not a Z-graph, G' must have a non-trivial n-cut, $3 \leqq n < 5$.

The following two lemmas are special cases of [**6**, Lemma 3.2].

LEMMA 4. *If G is a Z-graph, e an edge of G such that removing it creates a graph G' which is not a Z-graph, then the edge e belongs to a non-trivial 5-cut in G.*

Proof. Removing e is a reduction of Type 1. By Lemma 3, G' has a non-trivial n-cut, X, $3 \leqq n < 5$. Now e must join the two banks of the cut, otherwise we would have a non-trivial n-cut, $n < 5$, in G. Hence $\{e\} \cup X$ is a non-trivial $(n + 1)$-cut, in G. Since G is a Z-graph; $n + 1 = 5$, and G has a non-trivial 5-cut containing the edge e.

LEMMA 5. *If G is a Z-graph for which no Type 1 reduction is possible then every edge of G belongs to a non-trivial 5-cut in G.*

Proof. Let e be any edge of G. Removing e creates a non-Z graph, since no Type 1 reduction is possible. By Lemma 4, e belongs to a non-trivial 5-cut in G.

The next lemma is a special case of [**6**, Lemma 2.7].

LEMMA 6. *If G is 3-valent, planar, cyclically n-connected, $n \geqq 4$, and X is a non-trivial n-cut, then no two edges in X are adjacent in G.*

Proof. It two edges in X were adjacent in G, since G is 3-valent there is a third edge at the common vertex. This edge is in the same bank as the common vertex and therefore could replace the two edges in X, giving an $(n - 1)$-cut in G. This is impossible since G is cyclically n-connected. This is a generalization of [**11**, Theorem 8].

LEMMA 7. *If G is a Z-graph, X a 5-cut in G, then any face of G contains exactly* 0 *or* 2 *members of X.*

Proof. By [**11**], any circuit in G contains an even number of edges from any cut in G. Therefore, the perimeter of any face has 0, 2 or 4 members from X. But if the perimeter of a face, s, contained 4 members of X, there would be four faces, each adjacent to s on a member of X, and each of these four faces would have to have another edge belonging to the set X. But this is impossible because X has only five elements and since G is 3-edge connected no two faces of G can be adjacent on two distinct edges.

LEMMA 8. *If G is a Z-graph, X a non-trivial 5-cut in G, and $a \in X$, then there*

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exist 5 *distinct faces*, s_0, s_1, s_2, s_3, s_4 *in G with*

$$s_0 \text{ Adj } s_1 \text{ Adj } s_2 \text{ Adj } s_3 \text{ Adj } s_4,\ s_4 \text{ Adj } s_0 \textit{ on } a,$$

and no other adjacencies among these 5 *faces.*

Proof. Let s_0 and s_4 be the two faces adjacent to the edge a, s_0 Adj s_4 on a. By Lemma 7, s_0 has two edges in X. Let b be the other edge, and let s_1 be the unique face such that s_0 Adj s_1 on b. Then $s_1 \neq s_4$ or we would have a non-trivial 3-cut in G. Since G is cyclically 5-connected we can continue in this manner until we get five distinct faces, s_0, s_1, s_2, s_3, s_4 with

$$s_4 \text{ Adj}(a)\ s_0 \text{ Adj}(b)\ s_1 \text{Adj}(c)\ s_2 \text{ Adj}(d)\ s_3 \text{Adj}(e)\ s_4 \text{ and } X = \{a, b, c, d, e\}.$$

Furthermore, there can be no other adjacencies among these five faces.

LEMMA 9. *If G is a Z-graph containing the configuration of Figure* 8 *and the edges a, b and c belong to a* 5-*cut, X, in G, then the face D is a pentagon.*

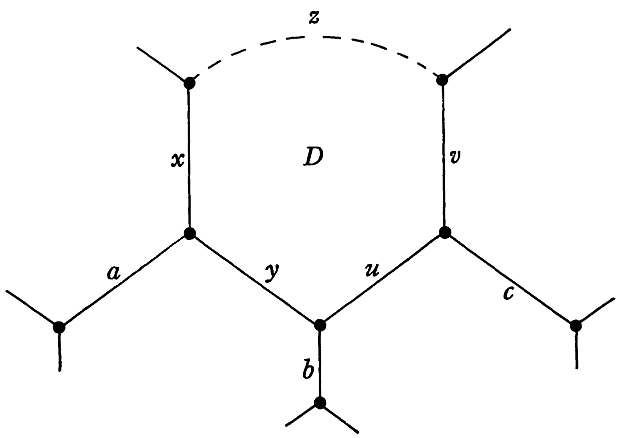

FIGURE 8

Proof. Assume D has 6 or more sides. The perimeter of D cannot be the only cycle in the bank of X in which it is contained, since that would require at least 6 edges in the cut X. Hence if the cycle uses the edges x, y, u, v they can be replaced by the arc z, and hence the edges a, b, c in the cut can be replaced by the edges x and v. This creates a non-trivial 4-cut in G. Since G is a Z-graph, this is impossible. Therefore D is a pentagon.

LEMMA 10. *If G is a Z-graph in which no reductions of Types* 1 *or* 2 *are possible and G contains the configuration of Figure* 9 *then one of the two faces A or B is a pentagon.*

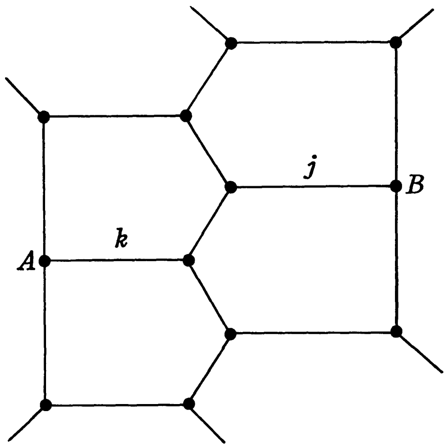

FIGURE 9

Proof. Removing the edges k and j produces a graph G' containing the configuration of Figure 10. By Lemma 3, G' has a non-trivial n-cut, S,

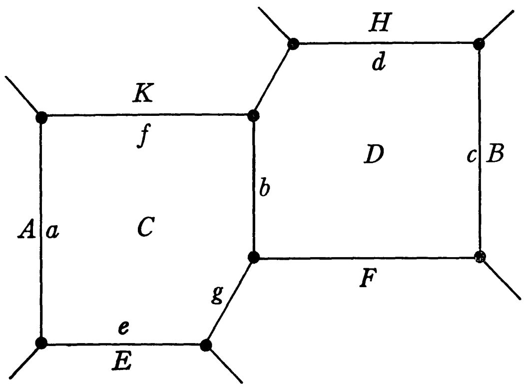

FIGURE 10

$3 \leqq n < 5$. The faces C and D cannot both be included in one bank of S or S would be a non-trivial n-cut, $n < 5$, in G. Also,

$$\{a, b, c\} \nsubseteq S, \quad \{a, b, d\} \nsubseteq S, \quad \{e, b, c\} \nsubseteq S,$$

since each of these three cases would give a non-trivial n-cut, $n = 3$ or 4, in G. Using Lemma 7, and disregarding symmetric cases, either $\{e, f\} \subseteq S$ or $\{e, b, d\} \subseteq S$ or $\{f, g\} \subseteq S$.

Case (a). If $\{e, f\} \subseteq S$ then, by Lemma 9, A is a pentagon.

Case (b). Assume $\{e, b, d\} \subseteq S$: If $\{e, b, d\} = S$ then $E = H$ and there is a non-trivial 3-cut in G which is impossible. Assume $S \neq \{e, b, d\}$. Therefore S

is a non-trivial 4-cut in G'. We have E Adj H in G'; hence E Adj H in G and $\{u, v, q, p\}$ is a 4-cut in G, as illustrated in Figure 11. But G is cyclically 5-connected, so this must be a trivial 4-cut. Therefore one bank consists of a single edge, and either F or B must be a quadrilateral. Hence this case is impossible.

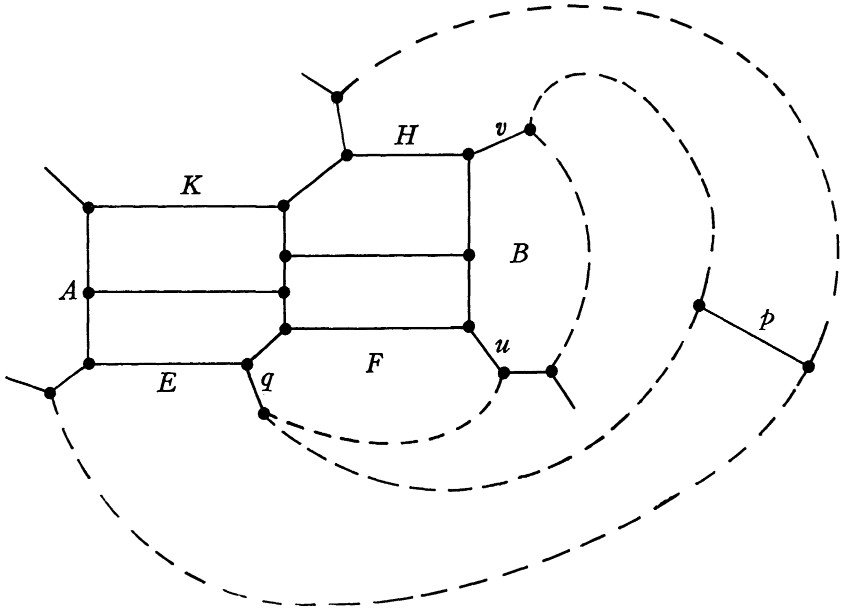

FIGURE 11

Case (c). $\{f, g\} \subseteq S$: S cannot be a non-trivial 3-cut in G', since, if it were, we would have K Adj F and the ring K, C', C'', F would give a non-trivial 4-cut in G.

Assume S is a non-trivial 4-cut in G'. We will show that B must be a pentagon. There must be a face J such that K Adj J Adj F in G, as in Figure 12. Note that $B \neq J$ or the ring K, D', B would give a non-trivial 3-cut in G. Now consider the edge d. By Lemma 4, there is a non-trivial 5-cut, T, in G, with $d \in T$.

Let us assume that B is not a pentagon. Then by Lemmas 7 and 9, one of $\{d, j, m\} \subseteq T$, or $\{d, h, k\} \subseteq T$, or $\{d, h, i\} \subseteq T$ must hold. We consider each of these three cases separately, and show that none of them are possible.

Case (c)-1. $\{d, j, m\} \subseteq T$: T and $S \cup \{k\}$ are non-trivial 5-cuts in G. As indicated in Figure 13 we rename the faces, C', F, J, K respectively s_0, s_2, s_3, s_4; and the faces D'', D', H respectively t_0, t_4, t_3. C'' will be labelled both s_1 and t_1. Since T is a 5-cut, there is a face, t_2, such that t_1 Adj t_2 Adj t_3. Now t_2 cannot be s_0 or s_2, since both are Adj t_0. Therefore t_2 and t_0 are in opposite banks of the 5-cut $S \cup \{k\}$, with t_0 Adj s_0 and t_2 Adj s_1. Therefore either t_3 or t_4 is one of the s_i.

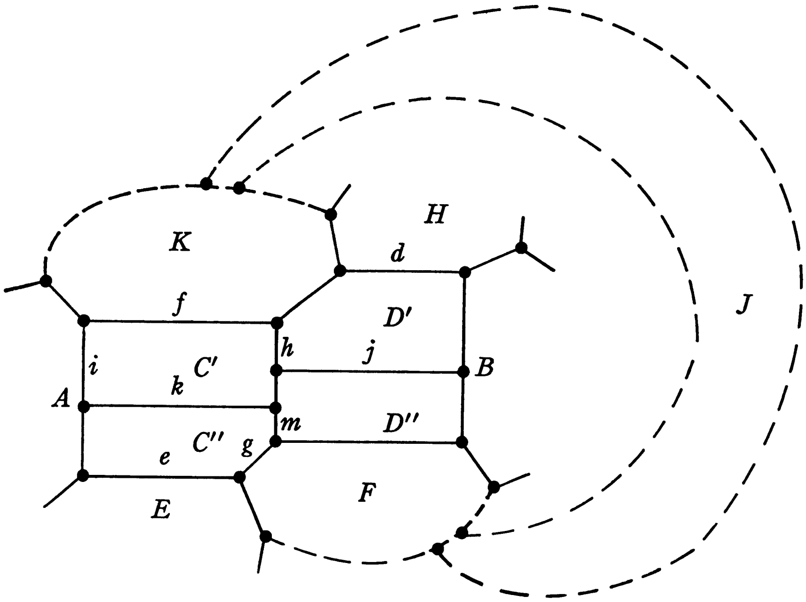

FIGURE 12

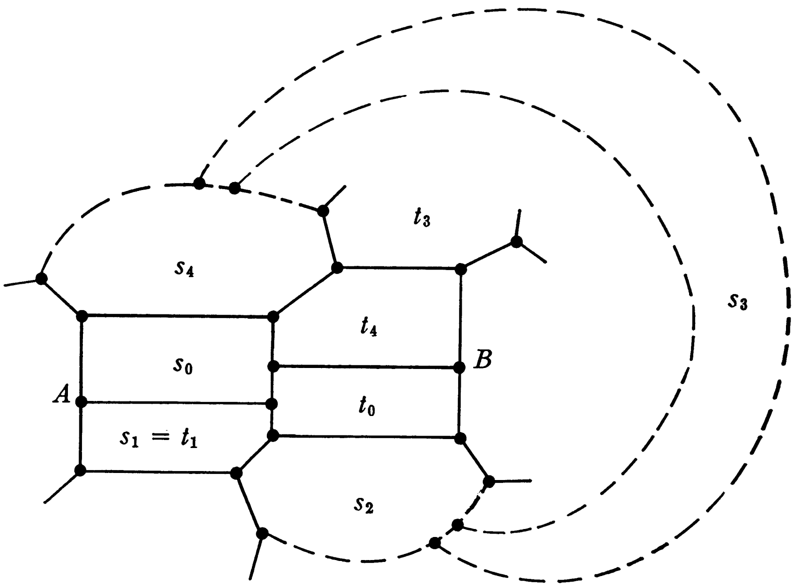

FIGURE 13

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But t_3, $t_4 \neq s_0$, s_1, s_2 or s_4. Also $t_3 \neq s_3$ or t_3, t_4, t_0, s_2 would give a non-trivial 4-cut in G. Similarly, $t_4 \neq s_3$ or t_4, t_0, s_2 would give a non-trivial 3-cut in G. Therefore case (c)-1 is impossible.

Case (c)-2. $\{d, h, k\} \subseteq T$: Rename the s_i's as in Case (c)-1. Rename D', H respectively t_4, t_3. Rename C' both s_0 and t_0, C'' both s_1 and t_1, as indicated in Figure 14. As before, there is a face, t_2, such that t_1 Adj t_2 Adj t_3. Now t_0 and t_1

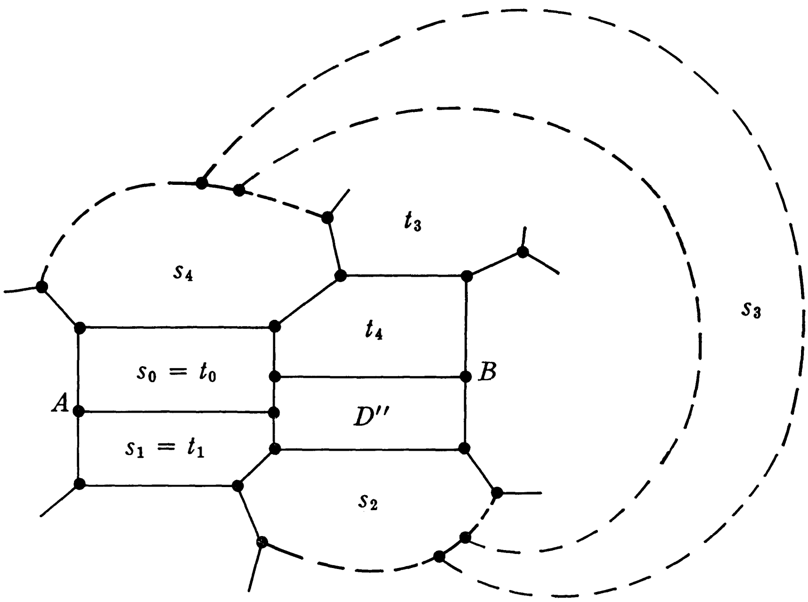

FIGURE 14

are in the s-ring. Clearly $t_2 \neq s_4$ since s_4 Adj t_4 and $t_2 \neq D''$ since s_4 Adj D''. Also $t_2 \neq s_2$, since if $t_2 = s_2$ the ring t_4, t_3, t_2, D'' would give a non-trivial 4-cut in G. Therefore t_2 and t_4 are in opposite banks of the cut, S, with t_2 Adj s_1 and t_4 Adj s_0. Therefore t_3 must be a member of the s-ring. Since t_3 cannot be adjacent to either t_0 or t_1 and s_0 Adj t_1, s_1 Adj t_0, s_4 Adj t_0, s_2 Adj t_1 it follows that $t_3 \neq s_0$, s_1, s_4 or s_2. Finally $t_3 \neq s_3$, since if $t_3 = s_3$, the ring t_3, t_4, D'', s_2 would give a non-trivial 4-cut in G. Therefore t_3 cannot be in the s-ring and case (c)-2 is impossible.

Case (c)-3. $\{d, h, i\} \subseteq T$: If $\{d, h, i\} \subseteq T$, by Lemma 9, K is a pentagon. We consider the face J which is adjacent to K. Since J Adj F certainly $J \neq C'$. Also $J \neq D'$ or D', D'', F would be a non-trivial 3-ring in G. Similarly $J \neq H$ or H, D', D'', F would be a non-trivial 4-ring in G, and $J \neq A$ or A, C'', F would be a non-trivial 3-ring in G. But J cannot be the fifth face adjacent to K, since if it were then J, A, E, F would be a non-trivial 4-ring in G. Thus case (c)-3 is also impossible, which establishes the lemma.

LEMMA 11. *If G is a Z-graph in which no Type* 1 *reductions are possible and G contains the configuration of Figure* 15, *then one of the three faces A, B or C is a pentagon.*

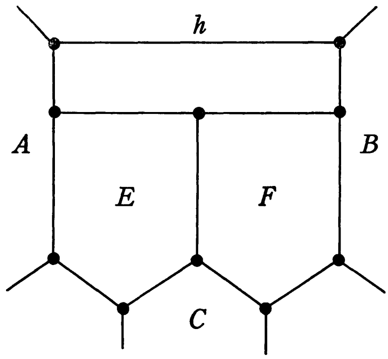

FIGURE 15

Proof. By Lemma 5, h belongs to a 5-cut in G. Let s_0, s_1, s_2, s_3, s_4 be the associated 5-ring, with s_0 Adj s_1 on h, $s_2 = E$, as indicated in Figure 16. (If $s_2 = F$ the argument is similar.) Now, either $s_3 = D$ or $s_3 = C$. If $s_3 = D$, by

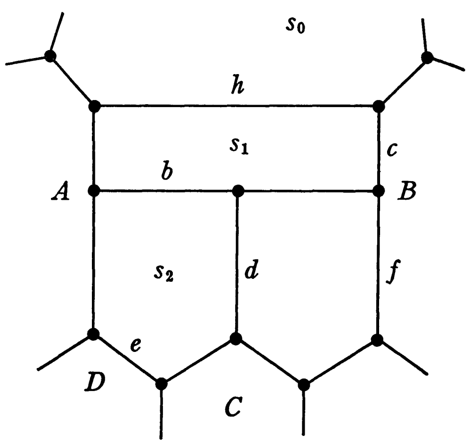

FIGURE 16

Lemma 9, A is a pentagon, so assume $s_3 = C$. Consider the edge e. Again, by Lemma 5, e belongs to a 5-cut, T, in G. Let t_0, t_1, t_2, t_3, t_4 be the associated 5-ring.

If none of the faces A, B or C is a pentagon, then by Lemma 9 either $\{e, b, c\} \subseteq T$ or $\{e, d, f\} \subseteq T$. We treat the two cases separately.

Case (a). $\{e, b, c\} \subseteq T$. We have the configuration of Figure 17, with t_3 Adj t_4. Now, t_1 and t_2 are among the s_i, t_0 and t_3 are in different banks of the cut S, with t_0 Adj s_1 and t_3 Adj s_2. Therefore t_4 must be one of the s_i. But $t_4 \neq s_0, s_1, s_2$ or s_3 since all are adjacent to t_1 or t_2. Also $t_4 \neq s_4$ or the ring t_0, F, s_3, s_4 would yield a non-trivial 4-cut. So case (a) is impossible.

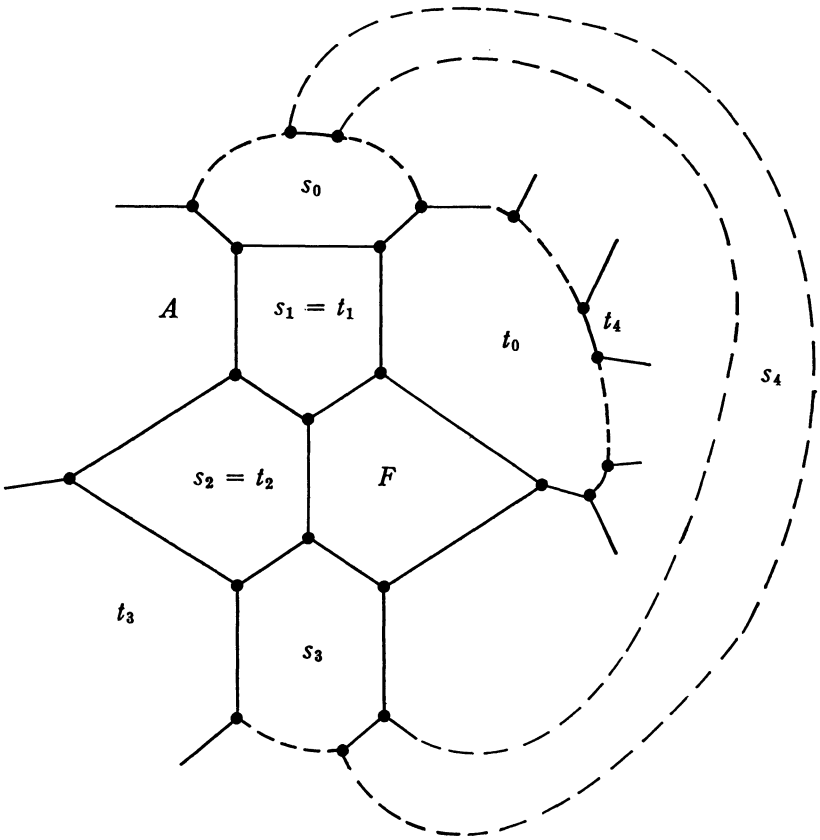

FIGURE 17

Case (b). $\{e, d, f\} \subseteq T$: We have the configuration of Figure 18, with t_0 Adj t_4. Again, t_2 is in the s-ring, and t_1 and t_3 are in opposite banks of the cut S, with t_1 Adj s_2 and t_3 Adj s_2. Hence either t_0 or t_4 is one of the s_i. But $t_0 \neq s_1$, s_2 or s_3 since each is adjacent to t_3, and $t_4 \neq s_0$, s_1, s_2 or s_3, since each is adjacent to t_4 or t_1. Also $t_0 \neq s_0$, since then the ring s_0, s_1, s_2, t_1 would give a non-trivial 4-cut. Similarly $t_4 \neq s_4$, or the ring t_4, t_3, s_3 would give a non-trivial 3-cut. Finally $t_0 \neq s_4$, or s_4, s_3, t_3, t_4 would be a 4-ring, so t_0 Adj t_3 which is impossible. This proves the lemma.

LEMMA 12. *If G is a Z-graph for which none of the three reductions are possible and G contains the configuration of Figure* 19, *then one of the faces A or B is a pentagon.*

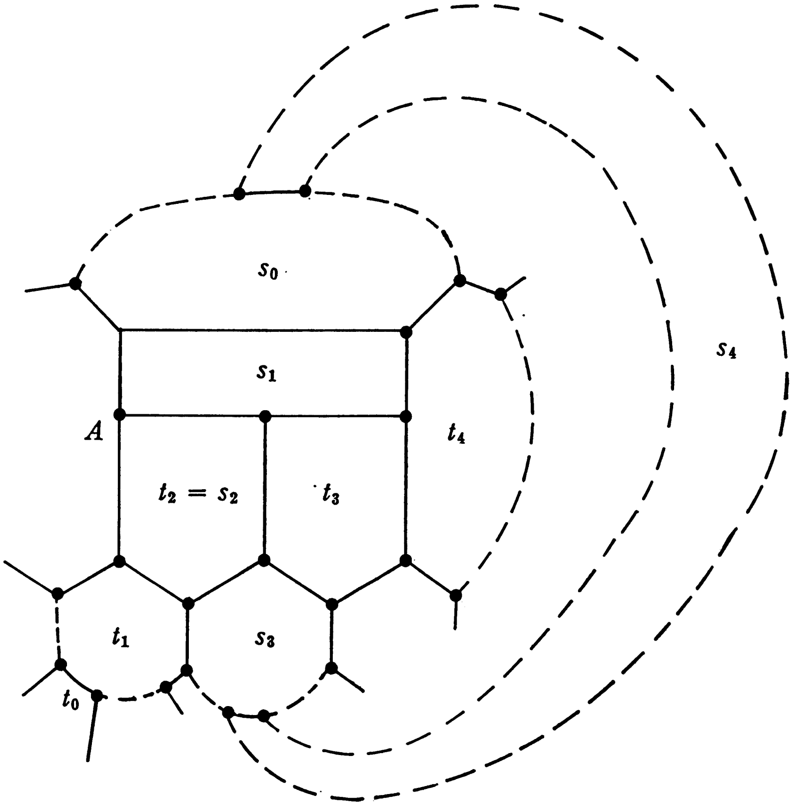

FIGURE 18

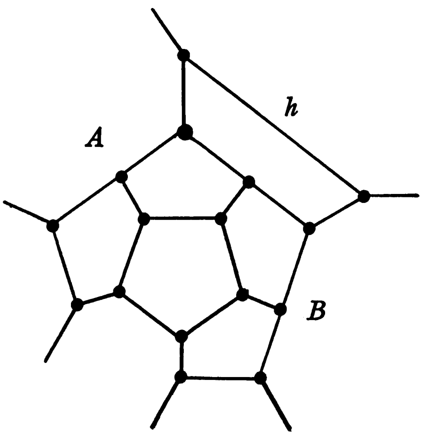

FIGURE 19

Proof. By Lemma 5, h belongs to a 5-cut, s_0, s_1, s_2, s_3, s_4, Assume it is as illustrated in Figure 20. (The case $s_1 = D$ is similar.) Now either $s_2 = E$ or $s_2 = F$.

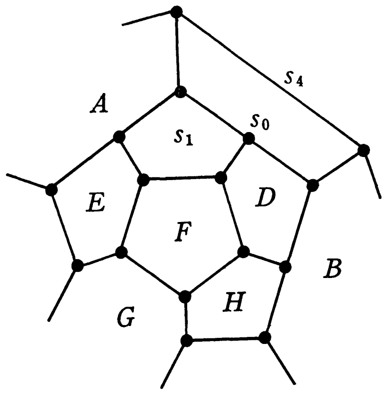

FIGURE 20

Case (a). $s_2 = E$: By Lemma 9, A is a pentagon.

Case (b). $s_2 = F$: Either $s_3 = H$ or $s_3 = G$. If $s_3 = H$, then the ring s_4, s_0, D, s_3 gives a non-trivial 4-cut in G which is impossible. If $s_3 = G$, then either the ring s_4, G, H, B gives a non-trivial 4-cut in G or B is a quadrilateral, either of which is impossible.

LEMMA 13. *If G is a Z-graph for which none of the reductions, are possible and G contains the configuration of Figure* 21, *then one of the faces A, B, C, D or E is a pentagon.*

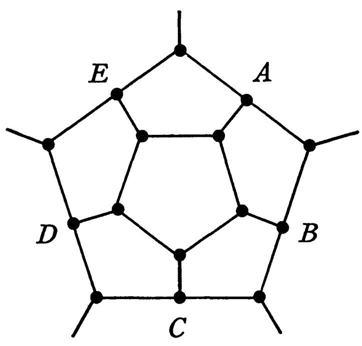

FIGURE 21

Proof. Form G' by a Type 3 reduction, as illustrated in Figure 22. By Lemma 3, G' has a non-trivial n-cut, X, $3 \leqq n < 5$. The cut X must not contain all of the pentagon, F, in one bank, or X would be an n-cut in G. Therefore X must contain two non-adjacent edges of F, say a and b. But then $X \cup \{a_1, e, b_1\} \sim \{a, b\}$ is 5-cut in G, and by Lemma 9, A is a pentagon.

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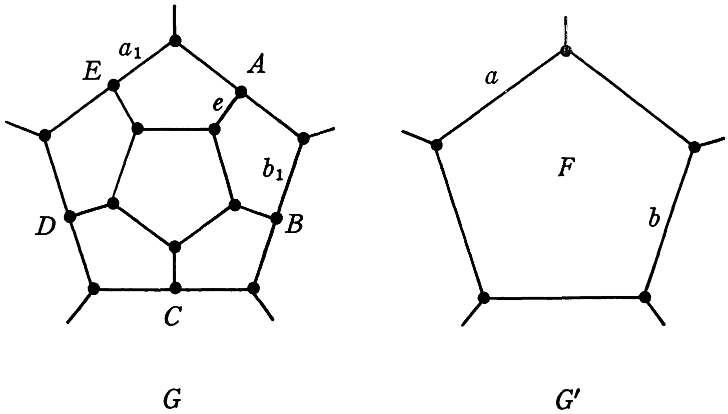

FIGURE 22

LEMMA 14. *If G is a Z-graph in which no reductions can be made then G has two adjacent pentagons.*

Proof. Since G is a Z-graph, G has no quadrilateral or triangular faces and hence, by Euler's Theorem [**4**; **5**], G has at least 12 pentagons. Choose one pentagon, call it s_0. Choose an edge, a, of s_0. Since no reductions of Type 1 are possible, by Lemma 5 the edge, a, belongs to a non-trivial 5-cut S in G. Let s_0, s_1, s_2, s_3, s_4 be the associated 5-ring with s_0 Adj s_4 on a. Let b be the edge such that s_0 Adj s_1 on b. Let c be the edge of s_0 adjacent to both a and b. By Lemma 5, c belongs to a non-trivial 5-cut T in G. Let t_0, t_1, t_2, t_3, t_4 be the associated 5-ring, as indicated in Figure 23. We have $s_0 = t_0$, t_1 Adj s_0, t_4 Adj s_0, with t_1 and t_4 in opposite banks of the 5-cut S. Therefore, t_2 or t_3 is an s_i. But $t_2 \neq s_0, s_1, s_4$, since they are all adjacent to t_4. Also $t_3 \neq s_0, s_1, s_4$ since they are adjacent to either t_0 or t_1.

If $t_2 = s_2$ then t_2, s_1, t_4, t_3 is a 4-ring. Since t_2 cannot be adjacent to t_4 we must have t_3 Adj s_1 as indicated in Figure 24. But now t_2, s_1, s_0, t_1 is a non-trivial 4-ring, which is impossible, so $t_2 \neq s_2$. Again referring to Figure 23, if $t_2 = s_3$ then t_2, s_4, t_1 is a 3-ring and so t_2, s_4 and t_1 must meet at a common vertex. But then t_4, t_3, t_2, s_4 is a 4-ring and since t_4 cannot be adjacent to t_2 we must have t_3 Adj t_4 and thus s_4 is a pentagon.

Similarly if $t_3 = s_2$, then t_3, s_1, t_4 is a 3-ring, and so t_3, s_1 and t_4 meet at a common vertex. But then t_3, t_4, s_4, s_3 is a 4-ring and since we cannot have s_2 adjacent to s_4 we must have s_3 Adj t_4, and thus t_4 is a pentagon. If $t_3 = s_3$ by a similar argument t_4 is again a pentagon. Hence G contains two adjacent pentagons.

LEMMA 15. *If G is a Z-graph in which no reductions of Type* 1 *are possible and G has two adjacent pentagonal faces A and B, then there is a third pentagonal face adjacent to both A and B.*

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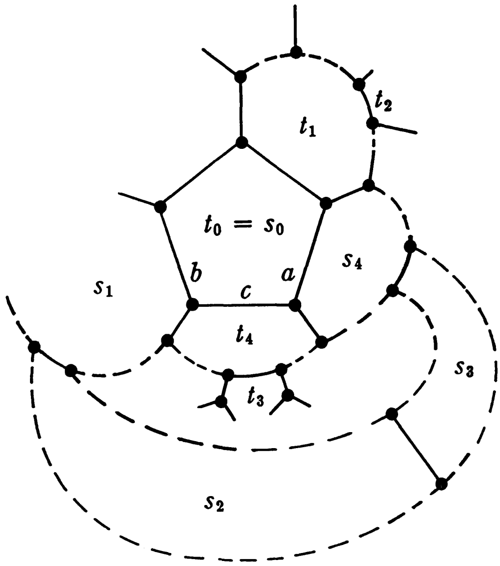

FIGURE 23

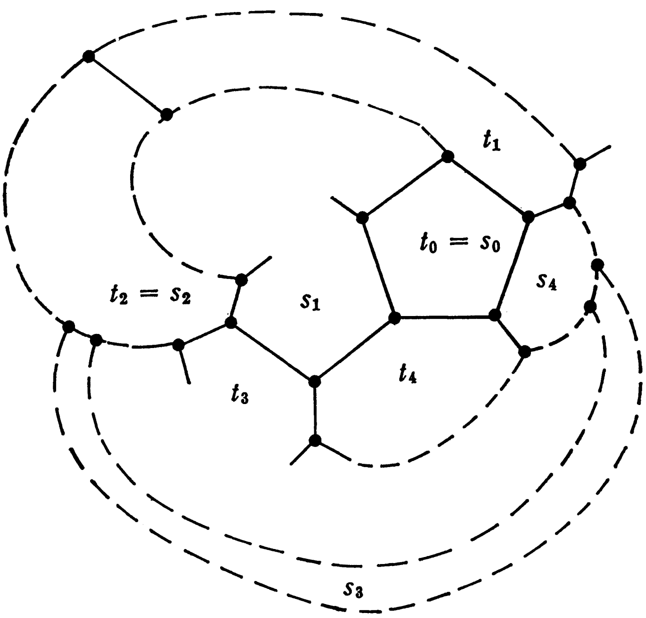

FIGURE 24

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Proof. Let c be the edge common to the faces A and B, as indicated in Figure 25. By Lemma 4, there is a 5-cut, S, in G, with $c \in S$. Let s_0, s_1, s_2, s_3, s_4 be the associated 5-ring, $s_1 = A$, $s_0 = B$. By Lemmas 6 and 7 either e or $b \in S$, and

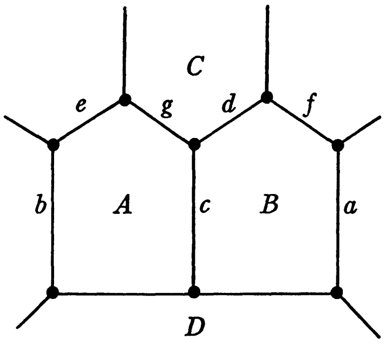

FIGURE 25

either f or $a \in S$. If $\{e, c, f\} \subseteq S$ by Lemma 9, C is a pentagon. Similarly, if $\{b, c, a\} \subseteq S$, D is a pentagon. There are only two other cases to consider, $\{e, c, a\} \subseteq S$ and $\{b, c, f\} \subseteq S$. These are symmetric cases. Therefore, we assume $\{e, c, a\} \subseteq S$. Consider the edge b. By Lemma 4, b belongs to a 5-cut, T, in G. If $\{b, c, a\} \subseteq T$, D is a pentagon. There are two other possibilities: $\{b, c, f\} \subseteq T$ or $\{b, g\} \subseteq T$. We consider each case separately.

Case (a). Assume $\{b, c, f\} \subseteq T$: Let t_0, t_1, t_2, t_3, t_4 be the associated 5-ring with t_2 Adj s_2 as indicated in Figure 26. The faces t_0, t_1 are among the s_i, and t_2 and t_4 are in opposite banks of the cut S, with t_2 Adj s_1, t_4 Adj s_0. Therefore t_3 must be an s_i. But $t_3 \neq s_1, s_0, s_2, s_4$. If $t_3 = s_3$, then t_4, t_3, s_4 is a 3-ring. Therefore t_4, s_3, s_4 meet at a common vertex, and s_2, s_3, t_4, C is a 4-ring. Hence C must be a pentagon.

Case (b). $\{b, g\} \subseteq T$: Let t_0, t_1, t_2, t_3, t_4 be the associated 5-ring, with t_0 Adj t_1 on b, t_1 Adj t_2 on g, as indicated in Figure 27, with s_2 Adj s_3 and t_4 Adj t_3. As before $s_1 = t_1$, and t_0 and t_2 are in opposite banks of S, with t_2 Adj s_1, t_0 Adj s_1. Therefore either t_3 or t_4 is an s_i. Now $t_3 \neq s_0, s_1, s_2$ since they are adjacent to t_0 or t_1, and $t_4 \neq s_0, s_1, s_2$ since they are adjacent to t_2. Also $t_3 \neq s_3$ since if $t_3 = s_3$ then s_3, t_2, s_0, s_4 is a trivial 4-ring, and s_3 Adj s_0, which is impossible. And $t_4 \neq s_4$ or t_4, t_0, t_1, s_0 would give a non-trivial 4-cut which, again, is impossible. Also $t_3 \neq s_4$, or t_2, s_0, s_4 would give a non-trivial 3-cut in G which is impossible. Finally, if $t_4 = s_3$, then t_4, t_0, D, s_4 gives a non-trivial 4-cut in G, see Figure 28, and hence D must be a pentagon, which establishes the lemma.

THEOREM 16. *If G is a Z-graph in which no reduction of Type* 1, 2 *or* 3 *can be made then G is the dodecahedron.*

Proof. By Lemma 14, G has two adjacent pentagons, hence by Lemma 15 three pentagons adjacent at a common vertex, as illustrated in Figure 29.

By Lemma 11, one of the faces A, B, or C is a pentagon, so we have the configuration of Figure 30. By Lemma 10, one of the faces X or Y is a pentagon and we have the configuration of Figure 31. Now by Lemma 10 one of the faces U or V is a pentagon. If U is a pentagon we have the configuration of Figure 32. If V is a pentagon we have the configuration of Figure 33, and by Lemma 12 one of the faces W or T is a pentagon, giving, in any case,

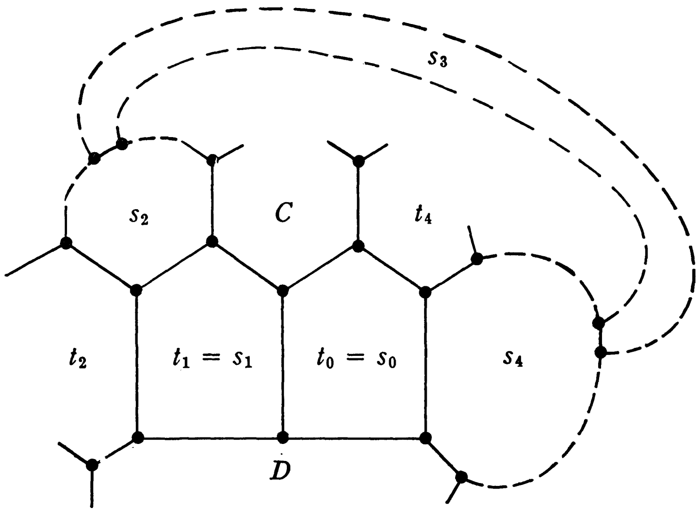

FIGURE 26

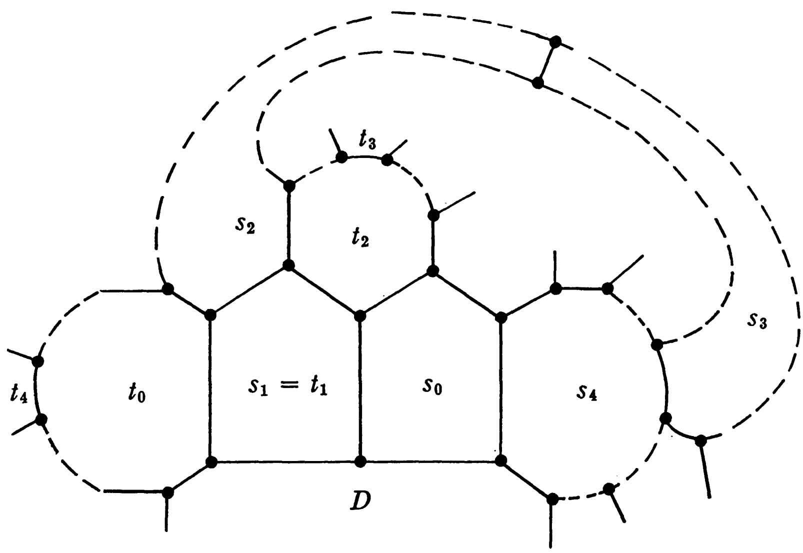

FIGURE 27

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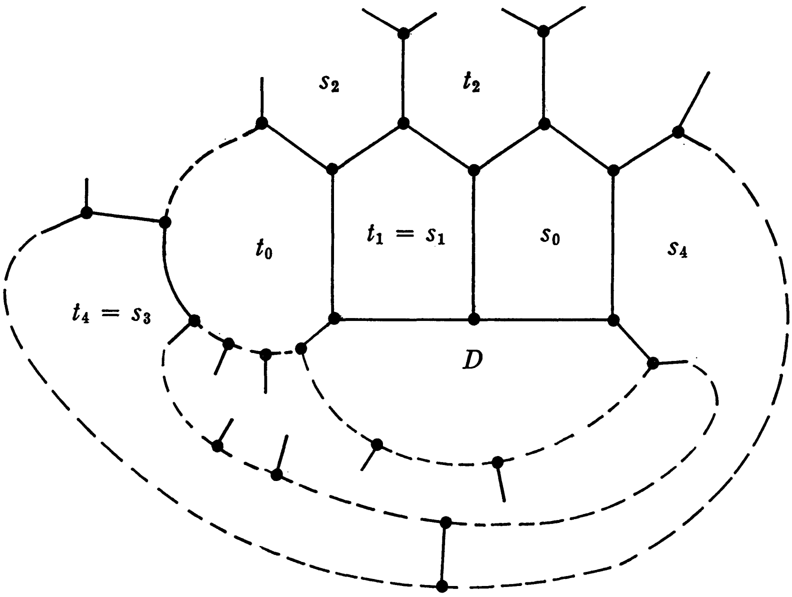

FIGURE 28

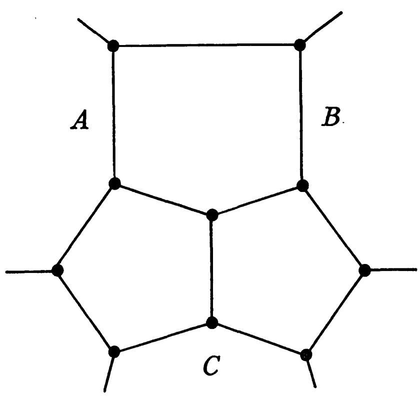

FIGURE 29

the configuration of Figure 32. By Lemma 13 one of the faces A, B, C, D, or E is a pentagon, giving the configuration of Figure 34, and, by Lemma 12 again, A or B is a pentagon and we have the configuration of Figure 35. But, since G is a Z-graph, G cannot have a non-trivial 4-cut. Therefore G is the dodecahedron.

THEOREM 17. *The class of* 3*-valent, convex* 3*-polytopes whose graphs are cyclically* 5*-connected is the smallest class which contains the dodecahedron and is closed under splits of Types* 1, 2 *and* 3. *Therefore, any such polytope can be*

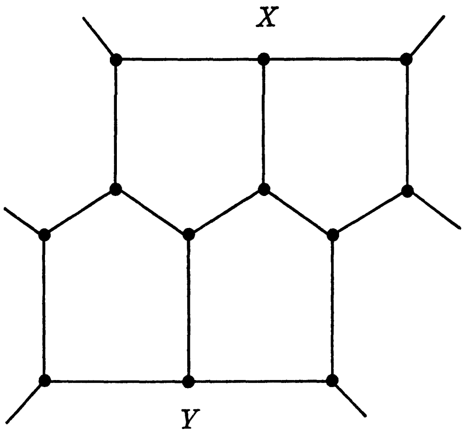

FIGURE 30

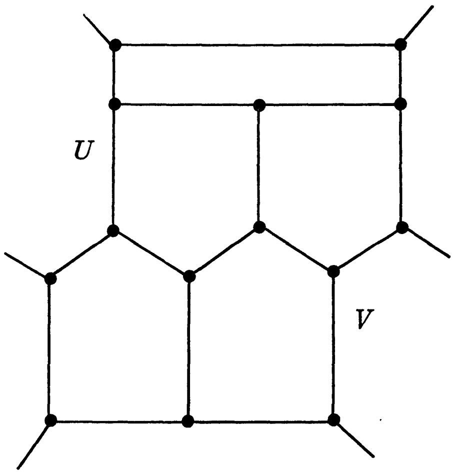

FIGURE 31

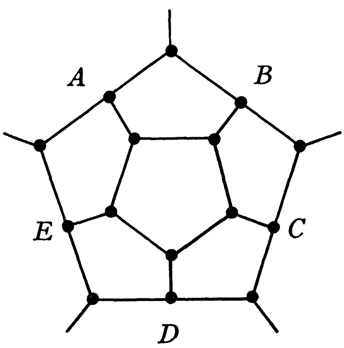

FIGURE 32

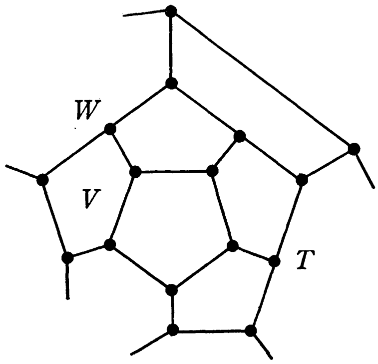

FIGURE 33

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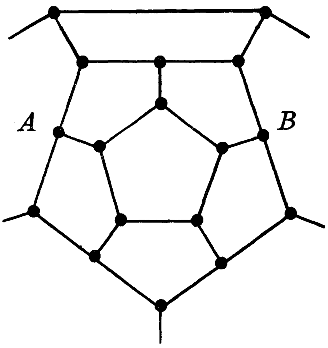

FIGURE 34

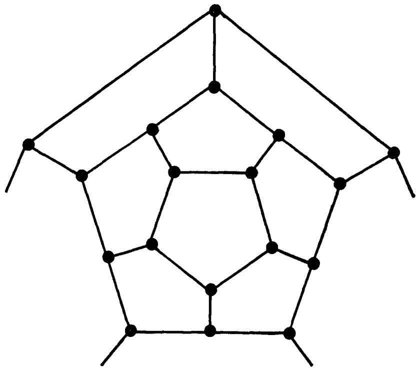

FIGURE 35

obtained from the dodecahedron by the successive application of finitely many (zero or more) of these operations.

Proof. Let Q be any class of 3-valent, convex 3-polytopes whose graphs are cyclically 5-connected, which contains the dodecahedron and is closed under the three types of face splitting. Let Z be the class of Z-graphs. Clearly the dodecahedron is in Z, and these splittings all preserve cyclically 5-connectedness, planarity and 3-valency. Thus $Q \subseteq Z$. To show that $Z \subseteq Q$, we note that the dodecahedron is in Q, and that if G is in Z and is not the dodecahedron, then, by Theorem 16, a reduction of Type 1, 2 or 3 can be made, producing a Z-graph with fewer vertices. Eventually the dodecahedron will be reached. Hence, by reversing the procedure G can be obtained from the dodecahedron by finitely many of these face splittings and is therefore in Q.

It is also interesting to note that these three face splittings are all essential. Since a face split of Type 1 requires a face with at least six sides, the first split must be of Type 2 or 3. Since Type 3 introduces ten new vertices we easily see that the 24 vertex polytope obtained by one Type 2 face split cannot be obtained using Types 1 and 3, and any 26 vertex polytope obtained by a Type 2 followed by a Type 1 cannot be obtained from 2 and 3 alone. By Kotzig [**12**] we see that a Type 3 face split cannot be produced by any combinations of splits of Types 1 and 2.

Remark. The referee has informed us that the results presented in the preceding paper have also been obtained by D. Barnette. His article will appear in Discrete Mathematics.

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