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ENDOMORPHISMS OF FIBRED GROUPS

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A collection $\mathscr{F}=\{G_\alpha \mid \alpha\in A\}$ of proper subgroups G_α of a group G is a *fibration* of G if

$$G=\bigcup_{\alpha\in A} G_\alpha,\quad G_\alpha\cap G_\beta=\{1\}\quad\text{for}\quad \alpha\neq\beta.$$

It is of geometric interest to associate two semigroups to a group G with fibration $\mathscr{F}$:

$$E:=E(G,\mathscr{F}):=\{h\in\operatorname{End} G \mid h(G_\alpha)\subseteq G_\alpha \text{ for all } \alpha\in A\}$$

$$S:=S(G,\mathscr{F}):=\{h\in\operatorname{End} G \mid \text{for each } \alpha\in A \text{ there is some } \beta\in A \text{ with } h(G_\alpha)\subseteq G_\beta\}.$$

The elements of E are dilatations of the associated translation plane, while the elements of S are endomorphisms of G which are at the same time operators for this translation plane (for more details on this, see e.g. [**6**]).

All groups are finite. Clearly, $E\subseteq S$ always holds.

Theorem. *For each finite fibred group G, $E\neq S$.*

Proof. The proof requires several steps. We assume $E=S$.

(1) Suppose G has a non-trivial centre $Z(G)$, By [**3**, p. 199], either G has a prime exponent p or $Z(G)$ is contained in a single fibre G_0 (say). In the first case, G is nilpotent and hence has a maximal normal subgroup of index p. In the second case $Z(G)$ has prime exponent p [**1**, Bemerkung, 2.4] and all elements of G of order $\neq p$ are contained in G_0 [**1**, Lemma 2.1]. If N is the subgroup generated by $Z(G)$ and all elements of order $\neq p$ then N is a subgroup of G_0. Since conjugation preserves order, it is routine to check that N is a normal subgroup of G, contained in G_0. Hence G/N is a p-group and by the homomorphism theorem we again get a maximal normal subgroup of index p in G.

(2) If N is a normal subgroup of G of prime index, we have $G/N\cong \mathbf{Z}_p$. Let $g\in G\backslash N$ be of order p. Then $G/N\cong\langle g\rangle$; so we get an endomorphism $h\neq id$ of G mapping all of G into the single fibre containing g. Hence $h\in S$, but $h\notin E$, a contradiction.

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(3) Hence we can assume that $Z(G)=\{1\}$. If $E\neq\{0,id\}$, Theorem II.3 of [7] implies that G must have a non-trivial centre (0 denotes the trivial endomorphism). Hence we are down to the case $E=S=\{0,id\}$, $Z(G)=\{1\}$.

(4) Suppose that the Fitting subgroup FG of G is trivial. By [5] or [9] G must fall into one of the following classes:

(i) $G\cong PGL(2,p^n)$, $p^n\geqq 4$

(ii) $G\cong PSL(2,p^n)$, $p^n\geqq 4$

(iii) G is a simple Suzuki group.

Recall that by [9, Lemma 1], any fibration can be refined into a normal (=kinematic in [3]) one. Since a normal fibration has $\{id\}\neq \operatorname{Inn} G\subseteq S$, we can exclude these ones. In all cases (i)–(iii), an examination of the proofs of the results of [5] shows that $\mathscr{F}$ arises from such a normal fibration $\mathscr{N}$ by taking the normalizer N of a suitable Sylow subgroup of G and all fibres of $\mathscr{N}$ not contained in N. Take some $x\in N$, $x\neq 1$. Then x determines an inner automorphism $\phi_x\neq id$ of G. Since $\phi_x\in \operatorname{Inn} G$ and $\mathscr{N}$ is normal, each $\phi_x(N_\alpha)\in\mathscr{N}$ for $N_\alpha\in\mathscr{N}$. Trivially $\phi_x(N)=N$ since $x\in N$. Hence $id\neq\phi_x\in S$, and $FG=\{1\}$ cannot happen.

(5) Finally, we study the case $FG\neq\{1\}$. From [2] and [4], either

(i) G is a p-group, or

(ii) G is a Frobenius group, or

(iii) $G\cong S_4$, or

(iv) $Z(G)\neq\{1\}$.

Now (i) and (iv) are excluded by (1) and (2).

In case (ii), we study a normal refinement $\mathscr{F}^*$ (see [8]) of $\mathscr{F}$. By [1, Satz 4.1], $\mathscr{F}^*$ consists of subgroups G_i $(i\in I)$ of FG and of subgroups G_j $(j\in J)$ which are self-normalizing. By Satz 4.7 of [1], $\mathscr{F}^*$ consists of some (possibly different) subgroups $G_k (k\in K)$ of FG and the same subgroups G_j as above. Since $\mathscr{F}$ is normal, for each inner automorphism ϕ_x induced by $x\in G$, $\phi_x(G_j)$ is some $G_{j'} (j,j'\in J)$.

FG is nilpotent and hence has a non-trivial centre. Take $z\neq 1$ in the centre. Then $\phi_z=id$ on FG, but $\phi_z(G_j)=G_j$ is impossible for $j\in J$, since each G_j coincides with its normalizer. Hence ϕ_z is in S, but not in E.

Finally, let $G\cong S_4$. In this case, A_4 is normal of prime index. We can proceed as in (2) to get some $h\in S\backslash E$, and we are done.

Corollary 1. *A finite group cannot have a fibration of fully invariant subgroups.*

Now we write G additively (this does not imply commutativity). It is also of geometric interest (see [7]) to consider the collection of all possible sums $dg\,E$ of elements of $E=E(G,\mathscr{F})$. E is a distributively generated near-ring (see e.g. [8]). The same applies to $dg\,S$. Clearly, $dg\,E$ is a subnear-ring of $dg\,S$. If $dg\,E=dg\,S$, each $s\in S\subseteq dg\,S$

must map each cell into itself, because every sum in $dg\,E$ behaves that way. Hence we have the following.

Corollary 2. *If G is a finite fibred group then* $dg\ E \neq dg\ S$.

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