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A functional calculus for continuous affine operators

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In the Appendix to a recent paper by J.J. Koliha and A.P. Leung (*Math. Ann.* 216 (1975), 273-284), a functional calculus for continuous affine operators was constructed on the basis of the Taylor-Dunford calculus. This calculus applied only to functions defined and analytic in an open set containing the spectrum of an operator and the point $\lambda = 1$. In the present paper I examine the affine resolvent, and develop independently a more general calculus applicable to functions which are analytic in any open neighbourhood of the spectrum of an affine operator.

Let X be a complex Banach space. An operator $A : X \to X$ is *affine* if $A(\alpha x+(1-\alpha)y) = \alpha Ax + (1-\alpha)Ay$ for all $x, y \in X$ and all complex α. The *trace* of A is the linear operator $A^{\#}$ on X defined by

$$A^{\#}x = Ax - A0 , \quad x \in X .$$

PROPOSITION 1. *Let A, B be affine operators on X, and let λ, μ be complex numbers. Then:*

(i) A is continuous iff $A^{\#}$ is continuous;

(ii) $(\lambda A+\mu B)^{\#} = \lambda A^{\#} + \mu B^{\#}$, $(AB)^{\#} = A^{\#}B^{\#}$;

(iii) if A is bijective, then the inverse A^{-1} is affine, and $(A^{-1})^{\#} = (A^{\#})^{-1}$;

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(iv) A *is bijective iff* $A^{\#}$ *is bijective;*

(v) *if* A *is continuous and bijective, its inverse* A^{-1} *is continuous.*

The proof is omitted.

PROPOSITION 2. *The set* $\mathsf{A}(X)$ *of all continuous affine operators on* X *is a Banach space under the norm*

$$\|A\| = \|A0\| + \|A^{\#}\| .$$

The norm topology of $\mathsf{A}(X)$ *coincides with the topology of uniform convergence on bounded subsets of* X *.*

The proof is omitted.

We note that $\mathsf{A}(X)$ is a near algebra with the unit I, satisfying the laws

$$(A+B)C = AC + BC , \quad (\alpha A)B = \alpha(AB) .$$

Furthermore,

$$C(A+B)x = (CA+CB)x - C0 ,$$

$$A(\alpha B)x = \alpha(AB)x + (1-\alpha)A0 .$$

For any operator $A \in \mathsf{A}(X)$, we define the *resolvent set* $\rho(A)$ of A as the set of all complex λ such that the operator $\lambda I - A$ is bijective; the *spectrum* $\sigma(A)$ is the complement of $\rho(A)$ in the complex plane. (This definition differs from the one given in [3], where the point $\lambda = 1$ was adjoined to $\sigma(A)$ when A was non-linear.) In view of Proposition 1,

$$\rho(A) = \rho(A^{\#}) , \quad \sigma(A) = \sigma(A^{\#}) .$$

It follows from [2, pp. 123-125] that the resolvent set is open, and that the spectrum is non-empty and compact. The *spectral radius* $r(A)$ of $A \in \mathsf{A}(X)$ is the number $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$.

For $A \in \mathsf{A}(X)$, the function $R(\lambda; A) = (\lambda I - A)^{-1}$ defined for $\lambda \in \rho(A)$ is the *resolvent* of A. We note that $R(\lambda; A)^{\#} = R(\lambda; A^{\#})$.

THEOREM 1. *For any* $A \in \mathsf{A}(X)$ *the function* $\lambda \mapsto R(\lambda; A)$ *on* $\rho(A)$ *to* $\mathsf{A}(X)$ *is analytic in the norm topology of* $\mathsf{A}(X)$.

Proof. First we show that

(1) $$R(\lambda;\, A)x = R(\lambda;\, A^{\#})(x+AO)\ ,\quad \lambda \in \rho(A)\ .$$

Indeed, applying $\lambda I - A$ to the vector on the right in (1), we get $(\lambda I-A^{\#})R(\lambda;\, A^{\#})(x+AO) + (\lambda I-A)0 = x$, and (1) follows.

Choose $\lambda_0 \in \rho(A)$. For all λ in the disc $|\lambda-\lambda_0| < \left\|R\left(\lambda_0;\, A^{\#}\right)\right\|^{-1}$ the series $\sum_{n=0}^{\infty} (\lambda_0-\lambda)^n R\left(\lambda_0;\, A^{\#}\right)^{n+1}$ converges to $R(\lambda;\, A^{\#})$ in norm by Theorem 4.7.1 in [2, p. 123]. Consequently,

(2) $$R(\lambda;\, A)x = \sum_{n=0}^{\infty} (\lambda_0-\lambda)^n R\left(\lambda_0;\, A^{\#}\right)^{n+1}(x+AO)$$

uniformly on bounded subsets of X . □

Let K be a compact subset of an open set Ω in the complex plane. A cycle γ [1, p. 138] is a *Cauchy cycle with respect to the pair* (Ω, K) if γ has a representation as a sum of rectifiable loops in $\Omega\backslash K$, and if the index $n(\gamma, \lambda) = (2\pi i)^{-1}\int_\gamma (\xi-\lambda)^{-1}d\xi$ equals 0 for all $\lambda \in \mathrm{C}\backslash\Omega$, and 1 for all $\lambda \in K$. The existence of such cycle is demonstrated as follows. Let $\varepsilon > 0$ be such that $|\mu-\lambda| \geq \varepsilon$ if $\mu \in \mathrm{C}\backslash\Omega$ and $\lambda \in K$. Cover the complex plane with a mesh of squares, each of diameter less than ε , and let $\partial S_1, \ldots, \partial S_n$ be the positively oriented boundary loops of those closed squares $S_1, \ldots, S_n$ that meet K . Then $\gamma = \partial S_1 + \ldots + \partial S_n$ is a desired cycle.

With each operator $A \in \mathrm{A}(X)$ we associate the class $\mathrm{F}(A)$ of complex valued functions f defined and analytic in an open neighbourhood $\Delta(f)$ of the spectrum $\sigma(A)$. For $f \in \mathrm{F}(A)$, the *germ* $[f]$ is the set of all $g \in \mathrm{F}(A)$ such that $g(\lambda) = f(\lambda)$ for all λ in some open neighbourhood of $\sigma(A)$.

Let $f \in \mathrm{F}(A)$ for some $A \in \mathrm{A}(X)$. We put $\Omega(f) = \Delta(f)\backslash\{1\}$ if $\lambda = 1$ is in the resolvent set of A , and $\Omega(f) = \Delta(f)$ otherwise. We define $f_{\#}$ as the unique function analytic in $\Omega(f)$ satisfying

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$$f(\lambda) = \tau + (\lambda-1)f_{\#}(\lambda) \ , \quad \lambda \in \Omega(f) \ ,$$

where $\tau = \tau_{f,A}$ equals $f(1)$ if $1 \in \sigma(A)$, and 0 if $1 \in \rho(A)$. Finally, define f_* on $\Omega(f)$ by

$$f_* = f_{\#} - f \ .$$

If $A \in \mathsf{A}(X)$ and $f \in \mathsf{F}(A)$, we define $f(A)x$ for each $x \in X$ by the formula

$$(3) \qquad f(A)x = \frac{1}{2\pi i}\int_{\gamma} f(\lambda)R(\lambda;\, A)x\,d\lambda + \frac{1}{2\pi i}\int_{\gamma} f_*(\lambda)R(\lambda;\, A)0\,d\lambda \ ,$$

where γ is any Cauchy cycle with respect to the pair $(\Omega(f),\, \sigma(A))$.

THEOREM 2. *For any* $A \in \mathsf{A}(X)$ *and any* $f \in \mathsf{F}(A)$, $f(A)$ *is a continuous affine operator on* X *dependent only on the germ* $[f]$.

Proof. The map $x \mapsto R(\lambda;\, A)x$ is affine, and the correspondence $h \mapsto \int_{\gamma} h$ is linear; so $f(A)$ is affine. Let $\gamma = \sigma_1 + \ldots + \sigma_n$ be a representation of γ by loops in $\Omega(f)$, and let

$$M = \frac{1}{2\pi}\sum_{j=1}^{n} \sup_{\lambda\in|\sigma_j|} |f(\lambda)|\,\|R(\lambda;\, A^{\#})\|\,V(\sigma_j) \ .$$

Noting that $R(\lambda;\, A)x_1 - R(\lambda;\, A)x_2 = R(\lambda;\, A^{\#})(x_1-x_2)$ for all $x_1, x_2 \in X$, we deduce that $\|f(A)x_1-f(A)x_2\| \le M\|x_1-x_2\|$, which proves the (Lipschitz) continuity of $f(A)$.

Let f_1, f_2 be members of $\mathsf{F}(A)$ belonging to the germ $[f]$. Let γ_k be a Cauchy cycle with respect to $(\Omega(f_k),\, \sigma(A))$, $k = 1, 2$. By assumption, there is an open neighbourhood Ω of $\sigma(A)$ such that $f_1(\lambda) = f_2(\lambda)$ for all $\lambda \in \Omega$. Choose a Cauchy cycle γ with respect to $(\Omega,\, \sigma(A))$. For $k \in \{1, 2\}$, γ is also a Cauchy cycle with respect to $(\Omega(f_k),\, \sigma(A))$, and $n(\gamma-\gamma_k,\, \lambda) = 0$ if $\lambda \notin \Omega(f_k)\backslash\sigma(A)$. Hence $\gamma - \gamma_k$ is a cycle homologous to zero in $\Omega(f_k)\backslash\sigma(A)$. The homology form of Cauchy's Theorem [1, p. 145] implies that $\int_{\gamma_k} h_k = \int_{\gamma} h_k$ for any analytic function h_k on $\Omega(f_k)\backslash\sigma(A)$ to X. If, in addition, h_1 and h_2 are equal on Ω, then

$$\int_{\gamma_1} h_1 = \int_{\gamma} h_1 = \int_{\gamma} h_2 = \int_{\gamma_2} h_2 .$$

The conclusion now follows as $\lambda \longmapsto R(\lambda;\, A)x$ is analytic in $\rho(A)$ for each fixed $x \in X$ by Theorem 1. □

If A is linear, the second integral in (3) vanishes, and we have

$$f(A)x = \frac{1}{2\pi i}\int_{\gamma} f(\lambda)R(\lambda;\, A)x\,d\lambda ,$$

in agreement with the Taylor-Dunford calculus.

THEOREM 3. *For any* $A \in \mathrm{A}(X)$ *and any* $f \in \mathrm{F}(A)$,

$$(4) \qquad f(A)x = f\left(A^{\#}\right)x + f_{\#}\left(A^{\#}\right)A0 ,$$

where

$$f\left(A^{\#}\right) = f(A)^{\#} , \quad f_{\#}\left(A^{\#}\right)A0 = f(A)0 .$$

Proof. Let γ be a Cauchy cycle with respect to the pair $\left(\Omega(f),\, \sigma(A)\right)$. The defining formula (3) implies that $f(A)x - f(A)0$ is equal to the integral

$$\frac{1}{2\pi i}\int_{\gamma} f(\lambda)\left(R(\lambda;\, A)x - R(\lambda;\, A)0\right)d\lambda ,$$

which is seen to be $f\left(A^{\#}\right)x$. Again by (3),

$$(5) \qquad f(A)0 = \frac{1}{2\pi i}\int_{\gamma} f_{\#}(\lambda)R(\lambda;\, A)0\,d\lambda .$$

Since $R(\lambda;\, A)0 = R\left(\lambda;\, A^{\#}\right)A0$ by (1), we get $f(A)0 = f_{\#}\left(A^{\#}\right)A0$. □

A formula closely related to (4) was used in [3] to define the functional calculus for an affine operator A, admitting only functions f analytic in an open neighbourhood $\Delta(f)$ of the set $\sigma\left(A^{\#}\right) \cup \{1\}$. For any such f define $f^{\#}$ on $\Delta(f)$ by $f^{\#}(\lambda) = (\lambda-1)^{-1}\left(f(\lambda)-f(1)\right)$ if $\lambda \neq 1$, and $f^{\#}(1) = f'(1)$. The calculus presented in [3] is defined by the

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formula

$$(4)' \qquad \bar{f}(A)x = f(A^{\#})x + F^{\#}(A^{\#})A0 \ ,$$

where $f(A^{\#})$ and $f^{\#}(A^{\#})$ are interpreted in the sense of the Taylor-Dunford calculus. To prove the consistency of (4) and (4)', we show that for any member f of $\mathsf{F}(A)$ whose domain $\Delta(f)$ contains the point $\lambda = 1$ we have $f(A)0 = \bar{f}(A)0$; that is,

$$(5)' \qquad f(A)0 = \frac{1}{2\pi i}\int_{\sigma} f^{\#}(\lambda)R(\lambda;\, A)0d\lambda \ ,$$

where σ is any Cauchy cycle with respect to $(\Delta(f),\, \sigma(A))$.

If $1 \in \sigma(A)$, then $f^{\#} = f_{\#}$. Suppose that $1 \in \rho(A)$, and recall that $\Omega(f) = \Delta(f)\backslash\{1\}$. Choose a Cauchy cycle γ with respect to $(\Omega(A),\, \sigma(A))$, and a Cauchy cycle σ with respect to $(\Delta(f),\, \sigma(A))$. We note that γ is also a Cauchy cycle with respect to $(\Delta(f),\, \sigma(A))$, so that the difference

$$\frac{1}{2\pi i}\int_{\sigma} f^{\#}(\lambda)R(\lambda;\, A)0d\lambda - \frac{1}{2\pi i}\int_{\gamma} f_{\#}(\lambda)R(\lambda;\, A)0d\lambda$$

is equal to

$$\frac{1}{2\pi i}\int_{\gamma} f(1)(\lambda-1)^{-1}R(\lambda;\, A)0d\lambda \ .$$

The last integral vanishes since the integrand is analytic in $\Omega(f)$, and the cycle γ homologous to zero in $\Omega(f)$. This result combined with (5) establishes (5)'.

The foregoing argument illuminates our convention that the point $\lambda = 1$ be deleted from $\Delta(f)$ when $1 \in \rho(A)$.

To test the formula (3) as a basis for a functional calculus, we prove that for each $x \in X$,

$$f_k(A)x = A^k x \quad \text{if} \quad f_k(\lambda) = \lambda^k \ , \quad k = 0,\, 1,\, \ldots \ .$$

According to the formula (4), this is equivalent to

$$(6) \qquad f_k(A^{\#})x = A^{\#k}x \quad \text{and} \quad f_k(A)0 = A^k 0 \ .$$

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The first equation in (6) follows from the well known power series expansion for the linear resolvent $R(\lambda;\, A^{\#})$ (Theorem 4.7.2 in [2, p. 124]). In view of (5)', the second equation in (6) is equivalent to

$$\frac{1}{2\pi i}\int_{\sigma}\left(\sum_{j=0}^{k-1}\lambda^{j}\right)R(\lambda;\, A)0d\lambda = A^{k}0 \ ,$$

where σ is any Cauchy cycle with respect to $(\mathbb{C},\, \sigma(A))$, and where $\sum_{j=0}^{-1} = 0$. Proceeding by induction, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma}\left(\sum_{j=0}^{k}\lambda^{j}\right)R(\lambda;\, A)0d\lambda &= A^{k}0 + \frac{1}{2\pi i}\int_{\sigma}\lambda^{k}R(\lambda;\, A^{\#})A0d\lambda \\ &= A^{k}0 + A^{\#k}A0 \\ &= A^{k+1}0 \ .\end{aligned}$$

THEOREM 4. *Let* $A \in \mathsf{A}(X)$, *let* $f, g \in \mathsf{F}(A)$, *and let* α, β *be complex numbers. Then:*

(i) $\alpha f + \beta g \in \mathsf{F}(A)$, *and* $(\alpha f+\beta g)(A) = \alpha f(A) + \beta g(A)$;

(ii) $f\cdot g \in \mathsf{F}(A)$, *and*

$$f(A)g(A)x = (f\cdot g)(A)x + (1-\tau)f(A)0 \ ,$$

where $\tau = \tau_{g,A}$ *equals* $g(1)$ *if* $1 \in \sigma(A)$, *and* 0 *if* $1 \in \rho(A)$;

(iii) if f *has the power series expansion* $f(\lambda) = \sum_{k=0}^{\infty}\alpha_{k}\lambda^{k}$ *valid in an open neighbourhood of* $\sigma(A)$, *then*

$$f(A) = \sum_{k=0}^{\infty}\alpha_{k}A^{k} \ \text{ in the norm of } \mathsf{A}(X) \ ;$$

(iv) $\sigma\big(f(A)\big) = f\big(\sigma(A)\big)$.

Proof. *(i)* This follows from the defining formula (3) and the identity $(\alpha f+\beta g)_{*} = \alpha f_{*} + \beta g_{*}$.

(ii) If A is linear, we apply the argument given in (5.2.7) [2, p. 169] with Γ and Γ' chosen as follows: let $\Omega = \Omega(f) \cap \Omega(g)$, and let D be a bounded open neighbourhood of $\sigma(A)$ whose closure $\overline{D}$ is

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contained in Ω. Then select Γ as a Cauchy cycle with respect to $(D, \sigma(A))$, and Γ' as a Cauchy cycle with respect to $(\Omega, \overline{D})$. We conclude that

$$f(A)g(A) = (f \cdot g)(A) .$$

Let A be affine. In view of Theorem 3 and the preceding result for linear operators, (7) will be established when we show that

$$f(A)g(A)0 = (f \cdot g)(A)0 + (1-\tau)f(A)0 . \tag{8}$$

Applying (4), the preceding result for linear operators, and part *(i)* of the present theorem, we reduce (8) to

$$\left(f \cdot g_{\#} + f_{\#}\right)\left(A^{\#}\right)A0 = \left((f \cdot g)_{\#} + (1-\tau)f_{\#}\right)\left(A^{\#}\right)A0 ;$$

this equation holds as $(f \cdot g)_{\#} = f \cdot g_{\#} + \tau f_{\#}$.

(iii) Using the first equation in (6) and the limit passage under the integral sign, we obtain the series expansion

$$f\left(A^{\#}\right) = \sum_{k=0}^{\infty} \alpha_k A^{\# k} \quad \text{(in the operator norm).} \tag{9}$$

Let $1 \in \sigma(A)$. Then $f_{\#} = f^{\#}$, and

$$f^{\#}(\lambda) = \sum_{k=0}^{\infty} \alpha_k \left(\sum_{j=0}^{k-1} \lambda^j\right)$$

uniformly on compact subsets of $\Delta(f)$ by (A9) in [3]. According to the formula (5) and the second equation in (6), $f(A)0$ is given by

$$\sum_{k=0}^{\infty} \alpha_k \left(\frac{1}{2\pi i} \int_{\gamma} \left(\sum_{j=0}^{k-1} \lambda^j\right) R(\lambda; A)0 d\lambda\right) = \sum_{k=0}^{\infty} \alpha_k A^k 0$$

Let $1 \in \rho(A)$. Then $f_{\#}(\lambda) = (\lambda-1)^{-1} f(\lambda)$ for all $\lambda \in \Delta(f) \setminus \{1\}$, and $f(A)0$ is equal to

$$\sum_{k=0}^{\infty} \alpha_k \left(\frac{1}{2\pi i} \int_{\gamma} (\lambda-1)^{-1} \lambda^k R(\lambda; A)0 d\lambda\right)$$

for any Cauchy cycle γ with respect to $(\Delta(f) \setminus \{1\}, \sigma(A))$. The integral under the summation sign is equal to

$$\frac{1}{2\pi i}\int_\gamma\left(\sum_{j=0}^{k-1}\lambda^j\right)R(\lambda;\, A)0d\lambda + \frac{1}{2\pi i}\int_\gamma (\lambda-1)^{-1}R(\lambda;\, A)0d\lambda \;;$$

the second integral vanishes, and we have again

$$f(A)0 = \sum_{k=0}^{\infty}\alpha_k A^k 0 \;. \tag{10}$$

The result follows from (9) and (10).

(iv) Since $\sigma(f(A)) = \sigma(f(A)^{\#}) = \sigma(f(A^{\#}))$, we can apply the spectral mapping theorem for bounded linear operators [2, p. 171]. □

Theorem 4 *(i)*, *(ii)*, *(iii)* extend the correspondingly numbered parts of Theorem A1 in [3] to arbitrary members f, g of $\mathrm{F}(A)$. The best result on composite functions seems to be Theorem A1 (iv) of [3] which states that

$$h(f(A)) = (h\circ f)(A)$$

if $f \in \mathrm{F}(A)$ is such that $f(1) = 1$, and if $h \in \mathrm{F}(f(A))$. When we relinquish the requirement $f(1) = 1$, we can only conclude that $h(f(A)) - (h\circ f)(A)$ is a constant operator.

We observe that the operators $f(A), g(A)$ do not commute in general; however, the commutator $[f(A), g(A)] = f(A)g(A) - g(A)f(A)$ is a constant operator, namely

$$[f(A), g(A)]x = [f(A), g(A)]0 \;,\quad x \in X \;.$$

We conclude the paper with an application.

EXAMPLE. Let T be a bounded linear operator on X , and let $y, z \in X$ be given. We show that the differential equation

$$\frac{dy(t)}{dt} = Ty(t) + e^t z \;,\quad y(0) = y \;,$$

in the real variable t has a unique solution given by

$$y(t) = e^{tA}y \;,$$

where A is the affine operator defined by $Ax = Tx + z$.

Clearly, it is enough to prove that

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$$\frac{d}{dt} e^{tA}y = Ae^{tA}y + \left(e^{t}-1\right)z .$$

Put $G(t, \lambda) = e^{t\lambda}$, and define $G(t, A)$ in accordance with (3). Differentiating under the integral sign, and observing that $\partial G_*/\partial t = (\partial G/\partial t)_*$, we obtain that

$$\frac{d}{dt} e^{tA} = \frac{\partial G}{\partial t} (t, A) .$$

The result then follows when we find that

$$\frac{\partial G}{\partial t} (t, A)y = Ae^{tA}y + \left(e^{t}-1\right)z$$

by Theorem 4 *(ii)* with $f(\lambda) = \lambda$ and $g(\lambda) = e^{t\lambda}$.

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