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UNIQUENESS THEOREMS FOR A SINGULAR PARTIAL DIFFERENTIAL EQUATION

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0. Introduction and summary. A singular partial differential equation which occurs frequently in mathematical physics is given by

$$\Delta u + \frac{s}{x_1}\frac{\partial u}{\partial x_1} + ku = 0$$

where $\Delta \equiv \sum_{i=1}^{n} \partial^2/\partial x_i{}^2$ is the Laplacian operator on $\mathbf{R}^n$ of which the generic point is denoted by $x = (x_1, \ldots, x_n)$ and s and k are real numbers. The study of solutions of this equation for the case $k = 0$ was initiated by A. Weinstein [**5**], who named it 'Generalized Axially Symmetric Potential Theory'. Numerous references to the literature on this equation can be found in [**1**; **3**; **6**]. The analytic theory of equations of the type mentioned above has extensively been treated in [**2**].

In this paper uniqueness theorems for more general second order linear partial differential equations whose coefficients (of the first order derivatives) may become unbounded on the co-ordinate hyperplanes are obtained. These equations are assumed to be 'quasi-elliptic' in a sense to be defined.

In § 1 certain notations are explained and the notion of 'quasi-ellipticity' of a linear second order partial differential operator $L_{n,m}$ in $\mathbf{R}^n$ with unbounded coefficients is introduced.

In § 2 a uniqueness theorem for the boundary-value problem associated with the equation $L_{n,1}[u] = f$ is established; the case of the bounded domain is proved in full and modifications for the case of the unbounded domain are indicated. Consideration is restricted to solutions u satisfying an "evenness condition" (hypothesis (iv)) and, more crucially, also a restriction (hypothesis (v)) on the nature of $\partial u/\partial x_1$ near the region of singularity $x_1 = 0$. In the case of the unbounded domain only solutions whose growth-rate at infinity is constant are considered. The principal tool used in establishing these results is the 'Strong maximum Principle' due to E. Hopf [**4**].

In § 3 the results of § 2 are extended to the case of the operator $L_{n,m}$ $(m < n)$ which has singularities on m of the n co-ordinate hyperplanes.

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1. Notations and definitions. 1. For $\tilde{x} = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$, $(\tilde{x}|_k y)$ will denote the point $(x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_{n-1}) \in \mathbf{R}^n$ for

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$k = 2, 3, \ldots, n-2$ while $(\tilde{x}|_1 y)$ will denote the point $(y, x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^n$, and $(\tilde{x}|_n y)$ will denote the point $(x_1, x_2, \ldots, x_{n-1}, y) \in \mathbf{R}^n$.

2. For $D \subset \mathbf{R}^n$, and $k \in \{1, 2, \ldots, n\}$

$$D_k^+ = \{(x_1, \ldots, x_n) \in D : x_k > 0\},$$
$$D_k^- = \{(x_1, \ldots, x_n) \in D : x_k < 0\},$$
$$D_k^0 = \{(x_1, \ldots, x_n) \in D : x_k = 0\}.$$

Obviously, D_k^0 may be identified with $\{\tilde{x} \in \mathbf{R}^{n-1} : (\tilde{x}|_k 0) \in D\}$ and this will be done whenever needed.

3. $L_{n,m}$ will denote the linear second order partial differential operator on $C^2(D)$ defined by

$$L_{n,m}[u] = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m} \frac{1}{x_k^{\gamma_k}} \sum_{i=1}^{n} b_{ik}(x) \frac{\partial u}{\partial x_i} + a(x)u$$

where $\{\gamma_k\}_{k=1}^m$ are positive integers and a_{ij}, b_{ik}, a are real-valued functions defined on $D \subset \mathbf{R}^n$. These functions a_{ij}, b_{ik}, a will be called the coefficients of $L_{n,m}$. It will be assumed that $n \geqq 2$ and $m < n$. For convenience, in the case $m = 1$, we will write D_+, D_-, D_0 instead of D_1^+, D_1^-, D_1^0; $(y, \tilde{x})$ instead of $(\tilde{x}|_1 y)$, γ instead of γ_1, L instead of $L_{n,1}$ and a_i instead of b_{i1}.

Definition. $L_{n,m}$ is said to be *quasi-elliptic* in D if and only if

(i) $\sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j$ is positive definite for each $x \in D_+ = \bigcap_{k=1}^{m} D_k^+$ and

(ii) $\sum_{i,j=1;\, i,j\neq k}^{n} a_{ij}(x)\lambda_i\lambda_j$ is positive definite for each $x \in D_k^0$ for $k = 1, \ldots, m$.

2. An application of the maximum principle. Before proceeding to the uniqueness theorems, it is desirable to record here an "obvious" result regarding the usual topology of the n-dimensional Euclidean space $\mathbf{R}^n$.

LEMMA 2.1. *If $D \subset \mathbf{R}^n$ is such that $D_+ = \{x \in D : x_1 > 0\}$ is a non-empty proper subset of $\mathbf{R}^n_+$ and if Δ is a component of D_+, then $\partial\Delta \cap \partial D \neq \emptyset$.*

Proof. Set $H = \{x \in \mathbf{R}^n : x_1 = 0\}$. Since $\partial\Delta \subset \partial D \cup H$, if $\partial\Delta \cap \partial D = \emptyset$ then $\partial\Delta \subset H$ so that $\partial\Delta \cap \mathbf{R}^n_+ = \emptyset$. Hence Δ is closed and open in $\mathbf{R}^n_+$ so that $\Delta = \mathbf{R}^n_+$ a contradiction to $D_+ \subsetneq \mathbf{R}^n_+$.

THEOREM 2.1. *If L is quasi-elliptic in a non-empty open subset D of $\mathbf{R}^n$, if the "coefficients of L" are continuous in D, and if for each $i = 2, 3, \ldots, n$, $\alpha_i : D_0 \to \mathbf{R}$ defined by:*

$$\alpha_i(x) = \lim_{x_1 \to 0} \frac{a_i((x_1, x))}{x_1^{\gamma}}$$

exists and is continuous and $a(D_+) \subset (-\infty, 0]$, then the boundary-value problem

(i) $L[u] = 0$ *in* D,

(ii) $u = 0$ *on* ∂D,

(iii) $u \in C^2(D) \cap C(\bar{D})$,

(iv) $u(D_-) \subset u(D_+)$,

(v) $\lim_{x_1 \to 0} \frac{1}{x_1{}^\gamma} \frac{\partial u}{\partial x_1} = 0$ *everywhere in* D_0

with either D *bounded, or* D *unbounded and*

(vi) $\lim_{||x|| \to \infty} u(x) = 0$ *has only the trivial solution* $u \equiv 0$ *in* D.

Proof. First consider the case of a bounded D. Suppose u is a non-trivial solution. Then there is a point $x \in D$ such that $u(x) \neq 0$. Since $-u$ is a solution whenever u is, it can be assumed without loss of generality that $u(x) > 0$. Hence u attains a positive maximum K on $\bar{D}$. Since $u = 0$ on ∂D it follows that the non-empty subset Ω of $\bar{D}$ defined by $\Omega = \{\xi \in \bar{D} : u(\xi) = K\}$ is contained in D. Before continuing we shall prove the following lemma.

Lemma (a). *The set* Ω *is a subset of* D_0.

Proof. Suppose $\Omega \not\subset D_0$. Then, either $\Omega \cap D_+ \neq \emptyset$ or $\Omega \cap D_- \neq \emptyset$. Since $u(D_-) \subset u(D_+)$, it follows that $\Omega \cap D_- \neq \emptyset \Rightarrow \Omega \cap D_+ \neq \emptyset$. Thus $\Omega \not\subset D_0 \Rightarrow \Omega \cap D_+ \neq \emptyset$. Let $z \in \Omega \cap D_+$ and Δ be the component of D_+ containing z; let $\xi \in \Delta \cap \Omega$. Since D_+ is open and hence locally connected, Δ is an open subset of $\mathbf{R}^n$. Therefore, from the continuity of u and the fact that $u(\xi) = K > 0$, it follows that there exists a closed ball B around ξ such that $B \subset \Delta$ and $u > 0$ on B. Also, $a(x) \leqq 0$ on B and $x_1 > 0$ on B so that $x_1{}^\gamma a(x)u(x) \leqq 0$ on B. This shows that if the operator M is defined by

$$M[w] = x_1{}^\gamma \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial w}{\partial x_i},$$

then

$$M[u] = -x_1{}^\gamma a(x)u(x) \geqq 0 \text{ on } B.$$

Since $x_1 > 0$ on B, the quasi-ellipticity of L in D implies the ellipticity of M in B; also the coefficients of M are continuous on B. Further, u attains its maximum value K on B at the point $\xi \in \operatorname{Int} B$. Hence, by Hopf's strong maximum principle [**4**], it follows that $u \equiv$ constant in B. Thus, for each $x \in B$, $u(x) = u(\xi) = K$. This shows that $B \subset \Omega$ and hence that $B \subset \Delta \cap \Omega$. Thus, each point $\xi \in \Delta \cap \Omega$ has a closed ball B around it such that $B \subset \Delta \cap \Omega$. Hence $\Delta \cap \Omega$ is open (in fact, in $\mathbf{R}^n$). But since u is continuous, $\Delta \cap \Omega = \Delta \cap u^{-1}(K)$ is closed in Δ. Since Δ is connected and $\Delta \cap \Omega \neq \emptyset$, it follows that $\Delta \cap \Omega = \Delta$.

Therefore $u(x) = K$ for each $x \in \Delta$ and hence by continuity of u,

(1) $$u(x) = K \text{ for each } x \in \bar{\Delta}.$$

Since Δ is a component of D_+, by Lemma 2.1, $\partial\Delta \cap \partial D \neq \emptyset$. Hence there is an $x \in \partial\Delta$ such that $u(x) = 0$. This, however, contradicts (1) and proves Lemma (a).

Let $v : D_0 \to \mathbf{R}$ be defined by $v(\tilde{x}) = u((0, \tilde{x}))$. The hypothesis $\lim_{x_1\to 0}(1/x_1^\gamma)(\partial u/\partial x_1) = 0$ implies that $\lim_{x_1\to 0}\partial u/\partial x_1 = 0$ since $\gamma \geqq 1$. Hence

$$\left(\frac{\partial^2 u}{\partial x_1^2}\right)_{x_1=0} = \lim_{x_1\to 0}\frac{1}{x_1}\left(\frac{\partial u}{\partial x_1} - \left(\frac{\partial u}{\partial x_1}\right)_{x_1=0}\right) = \lim_{x_1\to 0}\frac{1}{x_1}\frac{\partial u}{\partial x_1}$$

$$= \lim_{x_1\to 0} x_1^{\gamma-1}\frac{1}{x_1^\gamma}\frac{\partial u}{\partial x_1} = 0 \text{ again because } \gamma \geqslant 1.$$

Also for $i \neq 1$, we have

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_1}\right)_{x_1=0} = \lim_{x_1\to 0}\frac{\partial^2 u}{\partial x_i\partial x_1} = \lim_{x_1\to 0}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_i}\left(\lim_{x_1\to 0}\frac{\partial u}{\partial x_1}\right) = 0.$$

Therefore, rewriting $L[u] = 0$ at $(x_1, \tilde{x})$ where $\tilde{x} \in D_0$ and taking limits as $x_1 \to 0$, we have

$$\sum_{i,j=2}^{n} \alpha_{ij}(\tilde{x})\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=2}^{n}\alpha_i(\tilde{x})\frac{\partial v}{\partial x_i} + \alpha(\tilde{x})v = 0, \tag{2}$$

where $\alpha_{ij}(\tilde{x}) = a_{ij}((0, \tilde{x}))$ and $\alpha(\tilde{x}) = a((0, \tilde{x}))$. Clearly, the operator $\tilde{L}$ defined by

$$\tilde{L}[w] = \sum_{i,j=2}^{n}\alpha_{ij}(\tilde{x})\frac{\partial^2 w}{\partial x_i\partial x_j} + \sum_{i=2}^{n}\alpha_i(\tilde{x})\frac{\partial w}{\partial x_i} + \alpha(\tilde{x})w$$

is elliptic in D_0 (by the condition (ii) in the definition of quasi-ellipticity of L) and has continuous coefficients in D_0. Also by the continuity of a in D and the hypothesis $a(D_+) \subset (-\infty, 0]$, we have $\alpha(D_0) \subset (-\infty, 0]$. Moreover, in terms of the function v, the Lemma (a) established above shows that $\Omega = v^{-1}(K)$.

Now in $\mathbf{R}^{n-1}$, let $\tilde{x}_0 \in \Omega$ and Σ be the component of D_0 containing $\tilde{x}_0$; let $\xi \in \Sigma \cap v^{-1}(K)$. Again, Σ is open by the local connectedness of the open set D_0, $\xi \in \Sigma$ and $v(\xi) = K > 0$. Therefore, by the continuity of v, it follows that ξ has a closed ball N surrounding it such that $N \subset \Sigma$ and $v > 0$ on N. Hence, if the operator $\tilde{M}$ is defined by

$$\tilde{M}[w] \equiv \sum_{i,j=2}^{n}\alpha_{ij}(\tilde{x})\frac{\partial^2 w}{\partial x_i\partial x_j} + \sum_{i=2}^{n}\alpha_i(\tilde{x})\frac{\partial w}{\partial x_i},$$

then

$$\tilde{M}[v] = -\alpha(\tilde{x})v \geqq 0 \text{ on } N.$$

Since $\tilde{M}$ is elliptic with continuous coefficients in N and v attains its maximum K on N at the point $\xi \in \operatorname{Int} N$, it follows again by Hopf [**4**] that $v \equiv$ constant on N. Thus, for each $\tilde{x} \in N$, $v(\tilde{x}) = v(\xi) = K$. This shows that $N \subset v^{-1}(K)$ and hence that $N \subset \Sigma \cap v^{-1}(K)$. Thus, $\Sigma \cap v^{-1}(K)$ is open in $\mathbf{R}^{n-1}$ and hence both closed and open in Σ. Also $\Sigma \cap v^{-1}(K) \neq \emptyset$ because $\tilde{x}_0 \in \Sigma \cap v^{-1}(K)$. Hence it follows from the connectedness of Σ that $\Sigma \cap v^{-1}(K) = \Sigma$. Therefore $v(\tilde{x}) = K$ for each $\tilde{x} \in \Sigma$ and, again, by the continuity of v,

$$(3) \qquad v(\tilde{x}) = K \text{ for each } \tilde{x} \in \bar{\Sigma}.$$

Since $\partial D_0 \subset \partial D$ on which $u = 0$, it follows that $v = 0$ on ∂D_0. But since Σ is a component of D_0, $\partial\Sigma \subset \partial D_0$. Hence we have $v(\tilde{x}) = 0$ for each $\tilde{x} \in \partial\Sigma$. This contradicts (3) since $K > 0$, and completes the proof of the theorem in the case of a bounded D.

For the case of an unbounded D, the foregoing reasoning can be modified as follows.

Let $x_0 \in D$ be such that $u(x_0) > 0$ and for $r > 0$ define

$$B_r(0) = \{x \in \mathbf{R}^n : ||x|| < r\}.$$

Since $u(x) \to 0$ as $||x|| \to \infty$, a positive r can be found such that $|u(x)| < u(x_0)$ in $D \backslash B_r(0)$. If $E = D \cap B_r(0)$, then $x_0 \in E$ and hence u attains a maximum K on $\bar{E}$ such that $K \geqq u(x_0) > 0$. Also, clearly, $\partial E \subset \partial D \cup \partial B_r(0)$ and, by hypothesis, $u = 0$ on ∂D while, by the choice of r, $|u(x)| < u(x_0)$ for each $x \in D \cap \partial B_r(0)$. Hence

$$(4) \qquad |u(x)| < u(x_0) \text{ for each } x \in \partial E,$$

so that

$$\Omega = \{x \in \bar{D} : u(x) = K\} = \{x \in \bar{E} : u(x) = K\}$$

is a non-empty subset of E. We now establish

LEMMA (b). *The set Ω is a subset of E_0.*

Proof. Suppose $\Omega \not\subset E_0$. Then $\Omega \subset E \Rightarrow$ either $\Omega \cap E_+ \neq \emptyset$ or $\Omega \cap E_- \neq \emptyset$. But from $u(D_-) \subset u(D_+)$, we have, *a fortiori*, $u(E_-) \subset u(D_+)$. Therefore, $\Omega \cap E_- \neq \emptyset \Rightarrow \Omega \cap D_+ \neq \emptyset$. However, in $D_+ \backslash E_+$ we have, by the choice of r, $|u(x)| < u(x_0) \leqq K$. Therefore $\Omega \cap D_+ \neq \emptyset \Rightarrow \Omega \cap E_+ \neq \emptyset$. Thus $\Omega \not\subset E_0 \Rightarrow \Omega \cap E_+ \neq \emptyset$. Let then, $z \in \Omega \cap E_+$ and Δ be the component of E_+ containing z. Then as in the proof of the Lemma (a) we have

$$(5) \qquad u(x) = K \text{ for each } x \in \bar{\Delta}.$$

But, again by Lemma 2.1, $\partial\Delta \cap \partial E \neq \emptyset$ and if $\bar{x} \in \partial\Delta \cap \partial E$, then by (4), $|u(\bar{x})| < u(x_0) \leqq K$ while by (5), $u(\bar{x}) = K$. This contradiction proves Lemma (b).

Now the succeeding arguments in the proof of the case of bounded D may be repeated with D replaced by E to show that

$$(6) \qquad v(\tilde{x}) = K \text{ for each } \tilde{x} \in \bar{\Sigma},$$

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where $\tilde{x}_0 \in \Omega \subset E_0 \subset \mathbf{R}^{n-1}$ and Σ is the component of $E_0 \subset \mathbf{R}^{n-1}$ containing $\tilde{x}_0$. Since $\partial E_0 \subset \partial E$, and by (4), $|u(x)| < u(x_0) \leqq K$ for each $x \in \partial E$, we have $|v(\tilde{x})| < K$ for each $\tilde{x} \in \partial E_0$. But Σ a component of $E_0 \Rightarrow \partial\Sigma \subset \partial E_0$. Hence we have, *a fortiori*, $|v(\tilde{x})| < K$ for each $\tilde{x} \in \partial\Sigma$. This, however, contradicts (6) and the proof in the case of unbounded D is complete.

Note 1. It may be observed from the proof of the Theorem that the hypothesis $u(D_-) \subset u(D_+)$ on u can be replaced by the weaker hypothesis "there exists an $\tilde{x} \in \bar{D}_+$ such that $u(\tilde{x}) = \max\{u(x):x \in \bar{D}\}$". If this be done, then in the case of bounded D, Lemma (a) is replaced by the weaker assertion $\Omega \cap D_0 \neq \emptyset$. The proof of this assertion may be constructed the same way as that of Lemma (a) because denial of the assertion implies, by the new hypothesis, that $\Omega \cap D_+ \neq \emptyset$. Once the result $\Omega \cap D_0 \neq \emptyset$ is proved, it can be interpreted in terms of v as "there exists $\tilde{x}_0 \in D_0$ such that $v(\tilde{x}_0) = K$". The rest of the proof follows without change by taking Σ to be the component of D_0 containing $\tilde{x}_0$, and so on. The case of unbounded D can also be dealt with in like manner.

Note 2. It is obvious that the double hypothesis "D symmetric about the hyperplane $x_1 = 0$ and for each $(x_1, \ldots, x_n) \in D$, $u((x_1, \ldots, x_n)) = u((-x_1, x_2, \ldots, x_n))$" implies the hypothesis $u(D_-) \subset u(D_+)$.

Note 3. The preceding theorem does not imply uniqueness of the solution to $L[u] = f$ where f is a given continuous function because the hypothesis (iv) is non-linear in the sense that if u and v satisfy (iv) it does not follow that $u - v$ does. For this reason it is desirable to replace (iv) by some linear hypothesis that implies (iv). One such linear hypothesis is the "double hypothesis" mentioned in Note 2 above.

3. Extension to several singularities. In this section, the result of § 2 is extended to the case $m > 1$. However, instead of the hypothesis (iv) of Theorem 2.1, the "double hypothesis" mentioned in Note 2 is used for ease of formulation.

THEOREM 3.1. *If $L_{n,m}$ is quasi-elliptic in a non-empty open subset D of $\mathbf{R}^n$ which is symmetric about the hyperplanes $x_k = 0$ for $k = 1, 2, \ldots, m$, and the "coefficients of $L_{n,m}$" are continuous in D, and if for each $k \in \{1, 2, \ldots, m\}$ and for each $i \neq k$, $\beta_{ik} : D_k{}^0 \to \mathbf{R}$ defined by*

$$\beta_{ik}(\tilde{x}) = \lim_{y\to 0} \frac{b_{ik}((\tilde{x}|_k y))}{y^{\gamma_k}}$$

exists and is continuous and $a(D_+) \subset (-\infty, 0]$, then the boundary-value problem:

(i) $L_{n,m}[u] = 0$ *in* D,
(ii) $u = 0$ *on* ∂D,

(iii) $u \in C^2(D) \cap C(\bar{D})$,
(iv) $u((x_1, \ldots, x_n))$ *is even in each* x_k, $k = 1, 2, \ldots, m$,
(v) $\lim_{y \to 0}(1/y^{\gamma_k})(\partial u((\tilde{x}|_k y))/\partial x_k) = 0$ *for each* $\tilde{x} \in D_k^0$ *for* $k = 1, 2, \ldots, m$,

with either D bounded, or D unbounded, and

(vi) $\lim_{||x|| \to \infty} u(x) = 0$

has only the trivial solution $u \equiv 0$ *in* D.

Proof. We use induction on m. For $m = 1$, the result follows from Theorem 2.1. Let m be an integer $\geqq 2$ and consider the induction hypothesis that the theorem holds for all $L_{p,m-1}$ for $p > m - 1$. Let $u : D \to \mathbf{R}$ satisfy (i) through (v) and define $v : D_1^0 \to \mathbf{R}$ by $v(\tilde{x}) = u((\tilde{x}|_1 0))$. Rewriting (i) at $(\tilde{x}|_1 y)$ where $\tilde{x} \in D_1^0$ and taking limits as $y \to 0$ we have, as in the proof of Theorem 2.1

(i)′ $$\tilde{L}_{n-1,m-1}[v] = 0 \text{ in } D_1^0$$

where

$$\tilde{L}_{n-1,m-1}[v] = \sum_{i,j=2}^{n} \alpha_{ij}(\tilde{x}) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{k=2}^{m} \frac{1}{x_k^{\gamma_k}} \sum_{i=2}^{n} \beta_{ik}(\tilde{x}) \frac{\partial v}{\partial x_i} + \alpha(\tilde{x}) v,$$

α_{ij} and α being real-valued functions defined on D_1^0 by $\alpha_{ij}(\tilde{x}) = a_{ij}((\tilde{x}|_1 0))$ and $\alpha(\tilde{x}) = a((\tilde{x}|_1 0))$. Since $u = 0$ on ∂D and $\partial D_1^0 \subset \partial D$, it follows that

(ii)′ $$v = 0 \text{ on } \partial D_1^0.$$

Also, from $u \in C^2(D) \cap C(\bar{D})$ it follows that

(iii)′ $$v \in C^2(D_1^0) \cap C(\overline{D_1^0}).$$

Again, the statement $u((x_1, \ldots, x_n))$ is even in x_k for $k = 1, 2, \ldots, m$ implies that

(iv)′ $v((x_2, \ldots, x_n))$ is even in x_k for $k = 2, \ldots, m$.

Further, the condition (v) on u implies the corresponding condition (v)′ on the function v for $k = 2, 3, \ldots, m$. Moreover, hypothesis (ii) of the quasi-ellipticity of $L_{n,m}$ implies that $\tilde{L}_{n-1,m-1}$ is elliptic in D_1^0. Also the hypothesis on the coefficients b_{ik} imply the corresponding hypotheses on β_{ik}. Lastly, the continuity of a and the hypothesis $a(D_+) \subset (-\infty, 0]$ together show that $a(\bar{D}_+) \subset (-\infty, 0]$ which, in turn, implies that $\alpha((D_1^0)_+) \subset (-\infty, 0]$. From these hypotheses satisfied by $\tilde{L}_{n-1,m-1}$ and from the conditions (i)′ through (v)′ it follows by the induction hypothesis that $v \equiv 0$, so that $u = 0$ on D_1^0. In like manner, it follows that $u = 0$ on D_k^0 for $k = 1, 2, \ldots, m$. From this and the fact that $u = 0$ on ∂D we have the result: $u = 0$ on ∂D_+ because $\partial D_+ \subset \partial D \cup \bigcup_{k=1}^{m} D_k^0$.

Now let Δ be any component of D_+. Since $\partial\Delta \subset \partial D_+$, we have

$$u = 0 \text{ on } \partial\Delta. \tag{7}$$

Since D_+ is an open subset of $\mathbf{R}^n$ it follows that Δ is open and hence a domain. In this domain $L_{n,m}$ is elliptic by the condition (i) of quasi-ellipticity of $L_{n,m}$ in D and hence the same is true of the operator M defined by

$$M[w] = \left[\prod_{k=1}^{m} x_k^{\gamma_k}\right] L_{n,m}[w].$$

But M has continuous coefficients in $\bar{\Delta}$. Using again Hopf's maximum principle [**4**] we see that (7) together with the fact that $M[u] = 0$ in Δ implies that $u \equiv 0$ in Δ. From the choice of Δ it follows that $u \equiv 0$ in D_+ and hence by continuity that $u \equiv 0$ in $\bar{D}_+$. But then by the hypothesis (iv) on u it follows that $u \equiv 0$ in D. This completes the proof in the case of bounded D. The case of unbounded D is similar.

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