AUTOMORPHISMS OF THE SEMIGROUP OF ALL RELATIONS ON A SET

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An automorphism Φ of a semigroup S is said to be an inner automorphism if there exists a unit u in S such that

$$a\Phi = u^{-1}au$$

for each a in S. Let X denote a nonempty set and let $\mathcal{B}$ denote the semigroup of all binary relations on X where

$$A \circ B = \{(x,y) \in X \times X: (x,z) \in A \text{ and } (z,y) \in B \text{ for some } z \text{ in } X\}$$

for elements A and B of $\mathcal{B}$. This semigroup is discussed in [1, pp. 13-16] and in [2, pp. 33-35]. The purpose of this note is to prove the following

THEOREM. Every automorphism of $\mathcal{B}$ is inner.

Applying this theorem, we also obtain the following

COROLLARY. The automorphism group of $\mathcal{B}$ is isomorphic to the group of permutations on X.

The analogous results for the semigroup of all transformations of a set into itself are proven in [2, pp. 302-303]. It is pointed out there that I. Schreier [5], A.I. Malcev [4] and E.S. Ljapin [3] have all given proofs of the fact that every automorphism of the semigroup of all transformations on a set is inner. Schreier's paper, the first of the three, appeared in 1936.

In what follows, the empty relation will be denoted by E. For any A in $\mathcal{B}$, the domain, $\mathcal{D}(A)$, of A and the range,

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$\mathcal{R}(A)$, of A are defined by

$$\mathcal{D}(A) = \{x \in X: (x, y) \in A \text{ for some } y \text{ in } X\} \text{ and}$$

$$\mathcal{R}(A) = \{y \in X: (x, y) \in A \text{ for some } x \text{ in } X\}.$$

Finally, for x in X,

$$\{(y, x) \in X \times X: y \in X\}$$

is an element of $\mathcal{B}$ and will be denoted by $|x|$.

Proofs.

In proving the theorem, we make use of the following facts:

(1) $\mathcal{D}(A) = X$ if and only if $B \circ A = E$ implies $B = E$.

(2) $\mathcal{D}(A) \subset \mathcal{D}(B)$ if and only if $C \circ B = E$ implies $C \circ A = E$.

(3) $\mathcal{R}(A) \subset \mathcal{R}(B)$ if and only if $B \circ C = E$ implies $A \circ C = E$.

(4) $\mathcal{R}(A)$ consists of one point if and only if $A \neq E$ and there exists an element B in $\mathcal{B}$ such that

(i) $A \circ B = E$ and

(ii) if $A \circ C \neq E$ and $\mathcal{D}(B) \subset \mathcal{D}(C)$, then $\mathcal{D}(C) = X$.

The verification of each of the statements above is straightforward and will not be given here. Now let Φ be an automorphism of $\mathcal{B}$. Since E is the zero of $\mathcal{B}$, we must have

(5) $E\Phi = E$.

Statements (1) and (5) together imply

(6) $\mathcal{D}(A) = X$ if and only if $\mathcal{D}(A\Phi) = X$.

Statements (3) and (5) imply

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(7) $\mathcal{R}(A) \subset \mathcal{R}(B)$ if and only if $\mathcal{R}(A\Phi) \subset \mathcal{R}(B\Phi)$.

Statements (1), (2), (4) and (5) imply

(8) $\mathcal{R}(A)$ consists of one point if and only if $\mathcal{R}(A\Phi)$ consists of one point.

Now we are in a position to define a one-to-one transformation H from X onto X. Let any x in X be given. It follows from (6) and (8) that

$$|x|\Phi = |y| \text{ for some } y \text{ in } X .$$

We define $xH = y$. Let us observe that H is a unit of $\mathcal{B}$ and moreover

(9) $|x|\Phi = |xH|$ and $|x|\Phi^{-1} = |xH^{-1}|$

for each x in X. Now for any x in X and A in $\mathcal{B}$, we let

$$xA = \{y \in X: (x, y) \in A\} .$$

We will use the fact that for elements A and B of $\mathcal{B}$, $A = B$ if and only if $xA = xB$ for each x in X.

Using (9) (several times) we get the following string of equalities:

$$x(H\Phi) = x(|x| \circ H\Phi) = x(|xH^{-1}|\Phi \circ H\Phi) = x((|xH^{-1}| \circ H)\Phi) =$$

$$x((|x| \circ H^{-1} \circ H)\Phi) = x(|x|\Phi) = x|xH| = x(|x| \circ H) = xH .$$

Therefore

(10) $H\Phi = H$.

Now let A be an arbitrary element of $\mathcal{B}$. It follows from (7), (9) and (10) that the following statements are successively equivalent:

$y \in x(H^{-1} \circ A \circ H)$,

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$$y \in x(|xH^{-1}| \circ A \circ H),$$

$$\mathcal{R}(|y|) \subset \mathcal{R}(|xH^{-1}| \circ A \circ H),$$

$$\mathcal{R}(|y|\Phi) \subset \mathcal{R}(|xH^{-1}|\Phi \circ A\Phi \circ H\Phi),$$

$$\mathcal{R}(|yH|) \subset \mathcal{R}(|x| \circ A\Phi \circ H),$$

$$yH \in z(|x| \circ A\Phi \circ H) \text{ for some } z \text{ in } X,,$$

$$yH \in x(A\Phi \circ H),$$

$$y \in x(A\Phi).$$

Thus $x(H^{-1} \circ A \circ H) = x(A\Phi)$ for each x in X. This implies $A\Phi = H^{-1} \circ A \circ H$ and the theorem is proved.

To see how the corollary follows from the theorem, recall that the units of $\mathcal{B}$ are precisely the permutations on X. For any permutation H on X, map H onto the automorphism which carries an element A in $\mathcal{B}$ onto $H^{-1} \circ A \circ H$. This mapping is a homomorphism from the group of permutations on X into the group of automorphisms of $\mathcal{B}$. In fact, the mapping is an epimorphism onto $\mathcal{B}$ since every automorphism of $\mathcal{B}$ is inner. If H is in the kernel of the epimorphism, we must have $H^{-1} \circ A \circ H = A$ for each A in $\mathcal{B}$. But this implies that H commutes with each element of $\mathcal{B}$ which, in turn, implies that H is the identity mapping on X. Thus, the epimorphism is an isomorphism and the corollary is proved.

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