Canad. Math. Bull. Vol. **23** (2), 1980

ON THE STRUCTURE OF LOCALLY SOLID TOPOLOGIES

BY
C. D. ALIPRANTIS AND O. BURKINSHAW

1. **Introduction.** This paper considers what conditions on the order structure of a Riesz space will insure that one locally solid topology is finer than another, or when does one topology induce a finer topology than another on the order bounded subsets. The basic tool employed for the comparisons will be the carrier of a locally solid topology. We shall deal mainly with topologies whose carriers are order dense; a locally solid topology with order dense carrier will be called entire. Among the main results of this work are the following:

1) The best possible extension of Amemiya's comparison Theorem ([3], [1, Th. 12.8, p. 87]) will be obtained: *If τ is an entire topology, then τ induces a finer topology on the order bounded subsets than any Lebesgue topology on the space.* This result will allow us to compare metrizable topologies with Lebesgue topologies.
2) The relation between the topological dual L' and the order continuous dual $L_n^\sim$ is also studied. It is shown that if τ is a locally convex-solid topology on L that is entire, then L'_n is order dense in $L_n^\sim$, extending a result of Luxemburg and Zaanen [5, Th. 37.1, p. 520].

2. **Preliminaries.** For notation and basic terminology concerning Riesz spaces not explained below we refer the reader to [1].

Let L be a Riesz space. A net $\{u_\alpha\}$ of L is *order convergent* to u, in symbols $u_\alpha \xrightarrow{(0)} u$, if there exists a net $\{v_\alpha\}$ with the same indexed set such that $|u_\alpha - u| \leq v_\alpha$ for all α and $v_\alpha \downarrow \theta$. A Riesz subspace K of L is called *order dense*, if for every $\theta < u \in L$, there exists $\theta < v \in K$ with $\theta < v \leq u$.

As the title of the paper indicates, we shall deal with locally solid topologies on Riesz spaces. A *locally solid topology* τ on a Riesz space is a linear topology having a basis for zero consisting of solid sets. A locally solid topology τ on a Riesz space L may or may not satisfy one of the following properties:

(1) σ-Lebesgue; if $u_n \downarrow \theta$ in L, then $u_n \xrightarrow{\tau} \theta$.
(2) Lebesgue; if $u_\alpha \downarrow \theta$ in L, then $u_\alpha \xrightarrow{\tau} \theta$.
(3) Pre-Lebesgue; if $\theta \leq u_n \downarrow$, then $\{u_n\}$ is a τ-Cauchy sequence.
(4) σ-Fatou; if τ has a basis for zero consisting of solid and σ-order closed sets.
(5) Fatou; if τ has a basis for zero consisting of solid and order closed sets.

Received by the editors October 25, 1978.

https://doi.org/10.4153/CMB-1980-025-5 Published online by Cambridge University Press

A sequence $\{V_n\}$ of solid τ-neighborhoods of zero is called a *normal sequence* of neighborhoods if $V_{n+1}+V_{n+1}\subseteq V_n$ holds for all n. If $\{V_n\}$ is a normal sequence of τ-neighborhoods, then the ideal $N=\bigcap_{n=1}^{\infty} V_n$ is called the *null ideal* of $\{V_n\}$.

DEFINITION. Let τ be a locally solid topology on a Riesz space L, and let $\mathcal{N}$ denote the collection of all normal sequences of solid τ-neighborhoods of zero. The *carrier* C_τ of τ is defined to be:

$$C_\tau=\bigcup\left\{N^d: N=\bigcap_{n=1}^{\infty} V_n, \{V_n\}\in\mathcal{N}\right\}.$$

The carrier C_τ is always a σ-ideal of L[1, Th. 12.3, p. 84]. This ideal will be the key tool for the discussion in this paper. The carrier C_τ of a locally solid topology τ can also be introduced in terms of Riesz pseudonorms; if ρ is a Riesz pseudonorm its null ideal is defined by $N_\rho=\{u\in L:\rho(u)=0\}$. If $\mathcal{P}$ is the collection of all τ-continuous Riesz pseudonorms, then $C_\tau=\bigcup\{N_\rho^d: \rho\in\mathcal{P}\}$. It should also be noted that if two locally solid topologies on a Riesz space satisfy $\tau_1\subseteq\tau_2$, then $C_{\tau_1}\subseteq C_{\tau_2}$.

3. **Entire topologies.** In this section we shall study the properties of locally solid topologies whose carriers are order dense. We start by naming this property.

DEFINITION 3.1. An *entire topology* on a Riesz space L is a locally solid topology τ on L whose carrier C_τ is order dense in L.

From the above definition, it follows immediately that an entire topology must necessarily be Hausdorff. Some classes of entire topologies are described in the next result.

THEOREM 3.2. *Let L be a Riesz space. Then:*

(i) *Every metrizable locally solid topology on L is entire.*
(ii) *Every Hausdorff Fatou topology on L is entire.*
(iii) *Every Hausdorff Lebesgue topology on L is entire.*

Proof. Part (i) is obvious, since the carrier is exactly L. Part (ii) is proven in [1, Th. 12.3, p. 84], and (iii) follows from (ii) by observing that a Lebesgue topology is also a Fatou topology.

It should be noted that there are entire topologies which do not belong in the three classes of the previous theorem. We next describe some important Riesz spaces having all their Hausdorff locally solid topologies entire. It will follow that $C[0,1]$ has this property.

THEOREM 3.3. *Let X be a separable metric space. Then every Hausdorff locally solid topology τ on $C(X)$ satisfies $C_\tau=C(X)$, and hence τ is entire.*

https://doi.org/10.4153/CMB-1980-025-5 Published online by Cambridge University Press

Proof. Let $\{x_1, x_2, \ldots\}$ be a countable dense subset of X. For each pair of positive integers k and n, choose $f_{k,n} \in C(X)$ such that:

(1) $0 \le f_{k,n}(x) \le 1/n$ for all $x \in X$,
(2) $f_{k,n}(x) = 0$ if $d(x_k, x) \ge 1/n$ (where d is the metric),
(3) $f_{k,n}(x_k) = 1/n$.

Let $\{u_1, u_2, \ldots\}$ be an enumeration of the double sequence $\{f_{k,n}\}$.

Now if τ is a Hausdorff locally solid topology on $C(X)$, use induction to select a normal sequence $\{V_n\}$ of solid τ-neighborhoods of zero such that $u_n \notin V_n$ for all n. We claim that $N = \bigcap_{n=1}^{\infty} V_n = \{\theta\}$. Indeed, if $\theta \le f \in N$ satisfies $f(x_i) > 0$ for some i, then there exists some k with $\theta < u_k \le f$. So, $u_k \in N \subseteq V_k$, contrary to $u_k \notin V_k$. Thus $N = \{\theta\}$, and hence $N^d = C(X)$ so that $C_\tau = C(X)$.

Recall that $C_c(X)$ denotes the Riesz space of all real valued continuous functions with compact support defined on the topological space X.

Corollary 3.4. *Let X be a metric space. Then every Hausdorff locally solid topology on $C_c(X)$ is entire.*

The next result characterizes the entire topologies in terms of Fatou topologies and has a number of important consequences.

Theorem 3.5. *Assume that a Riesz space L admits a Hausdorff Lebesgue topology. Then for a locally solid topology τ on L the following statements are equivalent:*

(i) *τ is entire,* i.e., *C_τ is order dense in L.*
(ii) *L admits a Hausdorff Fatou topology coarser than τ.*

Proof. (i)$\Rightarrow$(ii). Let τ^* be a Hausdorff Lebesgue topology on L. Choose a basis $\{V\}$ of zero for τ consisting of solid τ-neighborhoods. For each $V \in \{V\}$, let V^* denote the τ^*-closure of V; clearly each V^* is solid and order closed. It should also be clear that $\{V^*\}$ defines a Fatou topology on L which is coarser than τ. To complete the proof we shall show that this topology is Hausdorff.

By way of contradiction assume that $\theta < u \in V^*$ for all V^*. Since C_τ is order dense, there exists $\theta < v \in C_\tau$ with $\theta < v \le u$. Similarly since C_{τ^*} is order dense, there exists $\theta < w \in C_{\tau^*}$ with $\theta < w \le v$. Replacing u by w we can suppose that $u \in C_\tau \cap C_{\tau^*}$. Pick a normal sequence $\{V_n\}$ of solid τ-neighborhoods of zero with $u \in N^d$, where $N = \bigcap_{n=1}^{\infty} V_n$. Similarly, pick another normal sequence $\{W_n\}$ of solid τ^*-neighborhoods of zero with $u \in M^d$, where $M = \bigcap_{n=1}^{\infty} W_n$. Clearly $u \in V_n^*$ for all n. Now for each n, choose $u_n \in V_n$ with $\theta \le u_n \le u$ and $u - u_n \in W_n$. Then $\{u_n\} \subseteq M^d$ and $u_n \xrightarrow{\tau_1} u$ in M^d, where τ_1 is the metrizable Lebesgue topology on M^d generated by $\{W_n \cap M^d\}$. Now by [1, Th. 15.9, p. 107], there exists a subsequence $\{v_n\}$ of $\{u_n\}$ and a net $\{w_\alpha\}$ satisfying $\theta \le w_\alpha \uparrow u$ in M^d, and such that given α, there exists n_α with $w_\alpha \le v_n$ for all $n \ge n_\alpha$. It

https://doi.org/10.4153/CMB-1980-025-5 Published online by Cambridge University Press

follows that $\{w_\alpha\} \subseteq N$. But since $\{w_\alpha\} \subseteq N^d$, we get $w_\alpha = \theta$ for all α. Hence $u = \theta$, a contradiction. Statement (ii) is now immediate.

(ii)$\Rightarrow$(i). If τ_1 is a Hausdorff Fatou topology, then by Theorem 3.2(ii), C_{τ_1} is order dense in L. Now if $\tau_1 \subseteq \tau$, then (by virtue of $C_{\tau_1} \subseteq C_\tau$) C_τ is order dense in L, regardless of whether or not L admits a Hausdorff Lebesgue topology.

Recall that the order dual $L^\sim$ of a Riesz space L is the Riesz space consisting of all order bounded linear functionals on L. Also, $L_n^\sim$ denotes the order continuous linear functionals on L, and $L_c^\sim$ the σ-order continuous linear functionals. If $\varphi \in L^\sim$, we remind the reader that $C_\varphi = N_\varphi^d$, where $N_\varphi = \{u \in L : |\varphi|(|u|) = 0\}$.

Lemma 3.6. *Let τ be a Hausdorff σ-Fatou topology on a Riesz space L. Then $C_\varphi \subseteq C_\tau$ holds for each $\varphi \in L^\sim$.*

Proof. Let $\theta < u \in C_\varphi$. Since $C_\tau \oplus C_\tau^d$ is order dense in L and C_φ has the countable sup property, there exists a sequence $\{u_n + v_n\} \subseteq C_\tau \oplus C_\tau^d$ with $\theta \leq u_n + v_n \uparrow u$ in L. We shall show that $v_n = \theta$ for all n. It will then follow from the σ-order closedness of C_τ that $u \in C_\tau$.

Assume that $v_k \neq \theta$ for some k (note that $v_k \in C_\varphi \cap C_\tau^d$). Choose a normal sequence $\{V_n\}$ of σ-order closed solid τ-neighborhoods of zero with $v_k \notin V_1$. Since $N \oplus N^d$ is order dense in $L(N = \bigcap_{n=1}^\infty V_n)$, there exists a sequence $\{x_n + y_n\} \subseteq N \oplus N^d$ with $\theta \leq x_n + y_n \uparrow v_k$. It follows that $y_n = \theta$ for all n, and so $v_k \in V_1$, which is impossible. The proof of the Lemma is now complete.

Theorem 3.7. *If $L_n^\sim$ separates the points of L, then every Hausdorff σ-Fatou topology on L is entire.*

Proof. Let τ be a Hausdorff σ-Fatou topology on L, and let $\theta < u \in L$. Choose $\theta < \varphi \in L_n^\sim$ with $\varphi(u) > 0$, and then pick a net $\{u_\alpha + v_\alpha\} \subseteq N_\varphi \oplus C_\varphi$ with $\theta \leq u_\alpha + v_\alpha \uparrow u$. Since N_φ is a band of L, there exists an index α with $\theta < v_\alpha \leq u$. By Lemma 3.6, $v_\alpha \in C_\varphi \subseteq C_\tau$, so that C_τ is order dense in L.

For a locally solid Riesz space (L, τ) with topological dual L', we put $L'_c = L' \cap L_c^\sim$ and $L'_n = L' \cap L_n^\sim$. In [5, Th. 37.1, p. 520] it was shown that if τ is generated by a Riesz norm, then L'_c and L'_n were order dense in $L_c^\sim$ and $L_n^\sim$, respectively. This result can be extended to metrizable locally convex-solid topologies [1, Th. 15.8, p. 106]. The next theorem gives some conditions sufficient for L'_n to be order dense in $L_n^\sim$.

Theorem 3.8. *Let (L, τ) be a Hausdorff locally convex-solid Riesz space. If τ is either entire or a σ-Fatou topology, then L'_n is order dense in $L_n^\sim$.*

Proof. Let $\theta < \varphi \in L_n^\sim$. We claim that in either case there exists $\theta < u \in C_\tau$ with $\varphi(u) > 0$. Indeed, this is clearly true if τ is entire. If τ is a σ-Fatou topology, choose $\theta < u \in C_\varphi$, and note that by Lemma 3.6, $u \in C_\tau$.

https://doi.org/10.4153/CMB-1980-025-5 Published online by Cambridge University Press

Pick a normal sequence $\{V_n\}$ of solid and convex τ-neighborhoods of zero with $u \in N^d$, where $N = \bigcap_{n=1}^{\infty} V_n$. Note that $\{V_n \cap N^d\}$ defines a metrizable locally convex-solid topology τ_m on N^d, and that the restriction of φ to N^d is order continuous. By [1, Th. 15.8, p. 106], there exists a τ_m-continuous linear functional φ_1 on N^d with $\theta < \varphi_1 \leq \varphi$ on N^d. Now since τ_m is coarser than the restriction of τ to N^d, φ_1 is τ-continuous on N^d, and hence by the Hahn-Banach Theorem φ_1 has a τ-continuous extension to L, which we denote by φ_1 again. Put $\psi = |\varphi_1| \wedge \varphi$ in $L^{\sim}$, and note that $\psi \in L'_n$ and $\theta < \psi \leq \varphi$ which completes the proof.

Recall that the absolute weak topology $|\sigma|(L, A)$ determined on a Riesz space L by an ideal A of $L^{\sim}$, is the locally convex-solid topology generated by the family of Riesz seminorms $\{\rho_\varphi : \varphi \in A\}$, where $\rho_\varphi(u) = |\varphi|(|u|)$ for all $u \in L$. The next result is a converse of the previous theorem.

THEOREM 3.9. *Let τ be a Hausdorff locally convex-solid topology on a Riesz space L. If L'_n separates the points of L, then τ is entire.*

Proof. Note that $|\sigma|(L, L'_n)$ is a Hausdorff Lebesgue topology satisfying $|\sigma|(L, L'_n) \subseteq \tau$. The result now follows from $C_{|\sigma|} \subseteq C_\tau$ and the order denseness of $C_{|\sigma|}$.

Let us denote by B the disjoint complement of $L_n^{\sim}$ in $L_c^{\sim}$, that is, $L_c^{\sim} = L_n^{\sim} \oplus B$. As a Corollary of Theorem 3.8 we have the following result.

THEOREM 3.10. *If B separates the points of a Riesz space L, then $L_n^{\sim} = \{\theta\}$.*

Proof. Note that $\tau = |\sigma|(L, B)$ is a Hausdorff σ-Lebesgue locally convex-solid topology on L, satisfying $(L, \tau)' = B$, see [1, Th. 6.6, p. 40]. By Theorem 3.8 we infer that $B \cap L_n^{\sim} = \{\theta\}$ is order dense in $L_n^{\sim}$, which is impossible unless $L_n^{\sim} = \{\theta\}$.

We close this section by mentioning a Hausdorff locally solid topology that is not entire. (It should be clear, however, from the above discussion that a "reasonable" Hausdorff locally solid topology must be entire.)

Let (X, Σ, μ) be a σ-finite non-atomic measure space, and let $L = L_\infty(X, \Sigma, \mu)$. Write $L^{\sim} = L_n^{\sim} \oplus (L_n^{\sim})^d$. Then $(L_n^{\sim})^d$ separates the points of L [4, p. 348], and so $\tau = |\sigma|(L, (L_n^{\sim})^d)$ is a Hausdorff locally convex-solid topology on L. It can be shown that $C_\tau = \{\theta\}$, so that τ is not entire; for details see [2, Theorem 8].

4. **An extension of Amemiya's comparison theorem.** I. Amemiya in [3] proved that for Dedekind complete Riesz spaces a Hausdorff Fatou topology induces a finer topology on the order bounded subsets than any Lebesgue topology on the space. In [1, Th. 12.8, p. 87] this result was shown to be true without the hypothesis of Dedekind completeness. We shall generalize this result further here. To do this we need a Lemma first.

https://doi.org/10.4153/CMB-1980-025-5 Published online by Cambridge University Press

LEMMA 4.1. *Let τ be an entire topology on a Riesz space L, and let A be either an ideal or an order dense Riesz subspace of L. Then τ restricted to A is entire.*

Proof. Let τ_1 denote the restriction of τ to A, and let $\theta < u \in A$. Since τ is entire, there exists $\theta < v \in C_\tau$ with $\theta < v \le u$. In either case we can assume that $v \in A$ (if A is an ideal, then $v \in A$; and if A is order dense in L, there exists $w \in A$ with $\theta < w \le v$, and we replace v by $w \in C_\tau$).

Now choose a normal sequence $\{V_n\}$ of solid τ-neighborhoods of zero with $v \in N^d$, where $N = \bigcap_{n=1}^{\infty} V_n$. Then $\{V_n \cap A\}$ is normal sequence of solid τ_1-neighborhoods of zero of A, and v belongs to the disjoint complement of $\bigcap_{n=1}^{\infty} (V_n \cap A)$ in A. Thus $v \in C_{\tau_1}$, and therefore τ_1 is an entire topology on A.

We are now ready to generalize Amemiya's Theorem.

THEOREM 4.2. *Let τ be either entire or a Hausdorff σ-Fatou topology on L. Then τ induces a finer topology on the order bounded subsets of L than any Lebesgue topology on L.*

Proof. Let τ_1 be a Lebesgue topology on L (not necessarily Hausdorff), and let $\theta \le x \in C_{\tau_1}$. Note that τ_1 restricted to C_{τ_1} is a Hausdorff Lebesgue topology, and moreover, note that C_{τ_1} has the countable sup property. Thus if τ is a Hausdorff σ-Fatou topology, then τ restricted to C_{τ_1} is a Fatou topology. On the other hand, if τ is entire, then by Lemma 4.1 τ restricted to C_{τ_1} is also entire, and thus by Theorem 3.5 there exists a Hausdorff Fatou topology on C_{τ_1} that is coarser than the restriction of τ to C_{τ_1}. Therefore, in either case, there exists a Hausdorff Fatou topology τ^* on C_{τ_1} which is coarser than τ on C_{τ_1}. By [1, Th. 12.8, p. 87] τ^* (and hence τ) restricted to $[\theta, x]$ is finer than τ_1 on $[\theta, x]$. Thus the result holds for the order bounded subsets of C_{τ_1}.

Now assume that the order bounded net $\{u_\alpha\}$ satisfies $\theta \le u_\alpha \le u$ for all α, and $u_\alpha \xrightarrow{\tau} \theta$. To finish the proof we have to show that $u_\alpha \xrightarrow{\tau_1} \theta$. Let V be a solid τ_1-neighborhood of zero. Choose another solid τ_1-neighborhood W of zero with $W + W + W \subseteq V$. Now select a normal sequence $\{W_n\}$ of solid τ_1-neighborhoods of zero with $W_1 = W$; put $N = \bigcap_{n=1}^{\infty} W_n$. Since $N \oplus N^d$ is order dense in L and τ_1 is a Lebesgue topology, there exists $\theta \le v + w \in N \oplus N^d$ with $\theta \le v + w \le u$ and $u - (v + w) \in W$. Note that $w \in C_{\tau_1}$, and that $(u - w)^+ = (u - v - w) + v \in W + W$. According to the first part of the proof there exists β such that $u_\alpha \wedge w \in W$ for all $\alpha \ge \beta$. But then

$$\theta \le u_\alpha = (u_\alpha - w)^+ + u_\alpha \wedge w \le (u - w)^+ + u_\alpha \wedge w \in W + W + W \subseteq V$$

for $\alpha \ge \beta$, implies $u_\alpha \in V$ for $\alpha \ge \beta$. Hence, $u_\alpha \xrightarrow{\tau_1} \theta$, and we are done.

COROLLARY 4.3. *If τ is a metrizable locally solid topology on a Riesz space L, then τ induces a finer topology on the order bounded subsets of L than any Lebesgue topology on L.*

https://doi.org/10.4153/CMB-1980-025-5 Published online by Cambridge University Press

The next result is a converse of Theorem 4.2 and actually shows that Theorem 4.2 is the best possible generalization of Amemiya's theorem.

THEOREM 4.4. *Let τ^* be a Hausdorff Lebesgue topology on a Riesz space L. If τ is a locally solid topology on L that induces a finer topology than τ^* on the order bounded subsets of L, then τ must be entire.*

Proof. We claim that $C_{\tau^*} \subseteq C_\tau$. To see this, let $\theta \le u \in C_{\tau^*}$. Choose a normal sequence $\{V_n\}$ of solid τ^*-neighborhoods of zero with $u \in N^d$, where $N = \bigcap_{n=1}^{\infty} V_n$. Since $\tau^* \subseteq \tau$ holds by hypothesis on $[\theta, u]$, there exists a normal sequence $\{W_n\}$ of solid τ-neighborhoods of zero with $W_n \cap [\theta, u] \subseteq V_n \cap [\theta, u]$ for all n. Put $M = \bigcap_{n=1}^{\infty} W_n$, and note that $u \in M^d$. Indeed, since $M \oplus M^d$ is order dense in L, there exists a net $\{w_\alpha + v_\alpha\} \subseteq M \oplus M^d$ with $\theta \le w_\alpha + v_\alpha \uparrow u$ in L. Since $w_\alpha \in M \cap [\theta, u] \subseteq N \cap [\theta, u] = \{\theta\}$, we get $w_\alpha = \theta$ for all α; therefore, $\theta \le v_\alpha \uparrow u$ in L, and so $u \in M^d$. Thus $u \in C_\tau$, so that $C_{\tau^*} \subseteq C_\tau$. The result now follows by observing that by Theorem 3.2, C_{τ^*} is order dense in L.

NOTE It is interesting to observe that the Hausdorff Lebesgue topologies on a Riesz space share many properties. For example, they induce (by Theorem 4.2) the same topology on the order bounded sets, and they have the same carrier. However, in spite of all these similarities a Riesz space can admit more than one Hausdorff Lebesgue topology; see [1, Example 12.11, p. 88].

REFERENCES

1. C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, Pure and Applied Math Series, New York, 1978.

2. C. D. Aliprantis and O. Burkinshaw, *Minimal topologies and L_p-spaces.* Illinois J. Math. to appear.

3. I. Amemiya, *On ordered topological linear spaces.* In: Proc. Internat. Symp. Linear Spaces, Jerusalem, 1960, pp. 14–23. Israel Acad. Sci. Humanities, Jerusalem, 1961.

4. S. Kaplan, *The second dual of the space of continuous functions, II.* Trans. Amer. Math. Soc., **93** (1959), 329–350.

5. W. A. J. Luxemburg and A. C. Zaanen, *Notes on Banach function spaces.* Nederl. Akad. Wetensch. Proc. Ser. A, Note XII **67** (1964), 519–529.

DEPARTMENT OF MATHEMATICAL SCIENCES
INDIANA UNIVERSITY & PURDUE UNIVERSITY AT INDIANAPOLIS
INDIANAPOLIS, INDIANA 46205

https://doi.org/10.4153/CMB-1980-025-5 Published online by Cambridge University Press