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ON COMPLETING LATIN RECTANGLES

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1. **Introduction.** By an (incomplete) $r \times s$ *latin rectangle* is meant an $r \times s$ array such that (in some subset of the rs cells of the array) each of the cells is occupied by an integer from the set $1, 2, \ldots, s$ and such that no integer from the set $1, 2, \ldots, s$ occurs in any row or column more than once. This definition requires that $r \leqq s$. If $r = s$ we will replace the word rectangle by *square*. It is easy to see that for any $n \geqq 2$ there is an incomplete $n \times 2n$ latin rectangle with $2n$ cells occupied which cannot be completed to a $n \times 2n$ latin rectangle. In this paper we prove the following theorem.

THEOREM 1. *An incomplete $n \times 2n$ latin rectangle with $2n-1$ cells occupied can be completed to a $n \times 2n$ latin rectangle.*

In view of the preceding remarks, Theorem 1 gives the best possible completion for incomplete $n \times 2n$ latin rectangles with respect to the number of occupied cells. The following theorems used in the proof are stated without proof. For the necessary definitions and an expository account on these ideas, see [5].

(M. Hall [2]): An $r \times n$ latin rectangle can be completed to an $n \times n$ latin square.

(P. Hall [3]): A necessary and sufficient condition for the non-empty sets $S_1, S_2, \ldots, S_t$ to have a system of distinct representatives (SDR) is that the union of any k of them contains at least k elements.

In the final section we obtain some partial results on a conjecture due to Trevor Evans, that an incomplete $n \times n$ latin square with $n-1$ cells occupied can be completed to an $n \times n$ latin square [1].

2. **Proof of Theorem 1.** Suppose that we have an $n \times 2n$ incomplete latin rectangle satisfying the conditions of the theorem. For $n=1$ the theorem is certainly true. So we will suppose that $n \geqq 2$. We will denote by r_i the number of symbols in the ith row. Then we can permute the rows of the given rectangle so that $r_1 \geqq r_2 \geqq \cdots \geqq r_n$. If this new rectangle can be completed then so can the one that we started with (by repermuting the rows of the completed rectangle). We now note that if $1 \leqq t \leqq n-1$ then $r_1 + r_2 + \cdots + r_t \geqq 2t$ and that all empty rows, if any, have been permuted to the bottom.

To complete the first row we proceed as follows: Permute the columns so that the

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empty cells in row 1 are $(1, i)$; $i=1, 2, \ldots, 2n-r_1$. (This is not necessary but it makes the notation simpler. We repermute the columns after completing the row.) Denote by S_i, $i=1, 2, \ldots, 2n-r_1$, the set of all symbols which do not occur in row 1 or column i. We show that the sets $S_1, S_2, \ldots, S_{2n-r_1}$ have a SDR. If this is the case, then the first row can be completed. By P. Hall's theorem the sets $S_1, S_2, \ldots, S_{2n-r_1}$ have a SDR if and only if for every k, $1 \leqq k \leqq 2n-r_1$, the union of every k of them contains at least k elements. Since each of the S_i is nonempty we can take $2 \leqq k \leqq 2n-r_1$. Let $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ be any k of $S_1, S_2, \ldots, S_{2n-r_1}$. If any of the columns $i_1, i_2, \ldots, i_k$ is empty, say i_j, then $|S_{i_j}|=2n-r_1 \geqq k$. If none of these columns is empty then the maximum number of occupied cells in any one of these columns is $(2n-1)-r_1-(k-1)$ so that $|S_{i_1}| \geqq 2n-r_1-[(2n-1)-r_1-(k-1)]=k$. In any case, $|S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k}| \geqq k$. Hence the set $S_1, S_2, \ldots, S_{2n-r_1}$ has a SDR which completes the first row.

Suppose we have completed t rows, where $1 \leqq t \leqq n-1$. If all $2n-1$ of the originally occupied cells are in the first t rows the rectangle can be completed by M. Hall's theorem. Otherwise there is at least one of the originally occupied cells in row $t+1$. Again as in the case for $t=1$, we permute the columns so that empty cells in row $t+1$ are $(t+1, i)$, $i=1, 2, \ldots, 2n-r_{t+1}$. Note that the first t rows form a $t \times 2n$ latin rectangle. Denote by R_{t+1} the set of symbols in row $t+1$ and for $i=1, 2, \ldots, 2n-r_{t+1}$ let T_i be the set of symbols in column i above row $t+1$ and let C_i be the set of symbols in column i below row $t+1$, if any. Now let $S=\{1, 2, \ldots, 2n\}$ and define $S_i=S \backslash (R_{t+1} \cup T_i \cup C_i)$, $i=1, 2, \ldots, 2n-r_{t+1}$. Since $|R_{t+1}|<n$, $|T_i \cup C_i| \leqq n-1$ each S_i is nonempty. We show that the sets $S_1, S_2, \ldots, S_{2n-r_{t+1}}$ have a SDR. Since each S_i is nonempty we can take $2 \leqq k \leqq 2n-r_{t+1}$. Let $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ be any k of $S_1, S_2, \ldots, S_{2n-r_{t+1}}$. We have two cases to consider.

(1) $k \leqq 2n-(r_{t+1}+t)$:

If $C_{i_j}=\phi$ for some $i_j \in \{i_1, i_2, \ldots, i_k\}$ then $S_{i_j}=S \backslash (R_{t+1} \cup T_{i_j})$ gives $|S_{i_j}| \geqq 2n-(r_{t+1}+t) \geqq k$. If none of $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ is empty then the maximum number of elements in any one is $(2n-1)-(r_1+r_2+\cdots+r_{t+1})-(k-1)$. Hence

$$|S_{i_1}| \geqq 2n-\{r_{t+1}+t+(2n-1)-(r_1+r_2+\cdots+r_{t+1}) -(k-1)\} = (r_1+r_2+\cdots+r_t)-t+k \geqq k,$$

since $r_1+r_2+\cdots+r_t \geqq 2t$ for $1 \leqq t \leqq n-1$. In both cases $|S_{i_1} \cup \cdots \cup S_{i_k}| \geqq k$.

(2) $k>2n-(r_{t+1}+t)$:

Let $k=2n-(r_{t+1}+t)+P$, where $1 \leqq P \leqq t$. Now there are at most $(2n-1)-(r_1+r_2+\cdots r_{t+1})$ of the originally occupied cells below row $t+1$ and so at least

$$(2n-r_{t+1})-[(2n-1)-(r_1+\cdots+r_{t+1})] = r_1+r_2+\cdots+r_t+1$$

of the sets $C_1, C_2, \ldots, C_{2n-r_{t+1}}$ are empty. Hence at least

$$(r_1+r_2+\cdots+r_t+1)-(t-P) = (r_1+r_2+\cdots+r_t-t)+(P+1) = t+P+1$$

of the sets $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ are empty. For the sake of notation, let $C_{i_1}, C_{i_2}, \cdots, C_{i_{t+1}}$ be empty. If we denote $R_{t+1} \cup T_i \cup C_i$ by S'_i then

$$\begin{aligned} S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k} &= (S \backslash S'_{i_1}) \cup (S \backslash S'_{i_2}) \cup \cdots \cup (S \backslash S'_{i_k}) \\ &= S \backslash (S'_{i_1} \cap S'_{i_2} \cap \cdots \cap S'_{i_k}). \end{aligned}$$

Now,

$$S'_{i_1} \cap S'_{i_2} \cap \cdots \cap S'_{i_k} \subseteq S'_{i_1} \cap S'_{i_2} \cap \cdots \cap S'_{i_{t+1}}.$$

Claim:

$$S'_{i_1} \cap S'_{i_2} \cap \cdots \cap S'_{i_{t+1}} \subseteq R_{t+1}.$$

If $x \in S'_{i_1} \cap S'_{i_2} \cap \cdots \cap S'_{i_{t+1}}$ and $x \notin R_{t+1}$ then x occurs in the first t rows $t+1$ times which is a contradiction since the first t rows from a $t \times 2n$ latin rectangle.

This gives $S'_{i_1} \cap S'_{i_2} \cap \cdots \cap S'_{i_k} \subseteq R_{t+1}$ and since we have $R_{t+1} \subseteq S'_{i_1} \cap \cdots \cap S'_{i_k}$ by definition, it follows that $S'_{i_1} \cap S'_{i_2} \cap \cdots \cap S'_{i_k} = R_{t+1}$. Hence

$$S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k} = S \backslash R_{t+1}$$

so that

$$|S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k}| = 2n - r_{t+1} \geqq k.$$

Combining cases (1) and (2) shows that the sets $S_1, S_2, \ldots, S_{2n-r_{t+1}}$ have a *SDR* and so row $t+1$ can be added. This procedure will complete all n rows which, of course, completes the given incomplete latin rectangle.

COROLLARY. *Let $t \geqq 0$. An incomplete $n \times (2n+t)$ latin rectangle based on $1, 2, \ldots, 2n+t$ with at most $2n+(t-1)$ cells occupied can be completed to a latin rectangle based on these same symbols.*

Proof. Replace $2n$ by $2n+t$ and $2n-1$ by $2n+(t-1)$ in the proof of Theorem 1. The proof goes through as before.

3. **A conjecture due to Trevor Evans.** For any $n \geqq 2$ there is an incomplete $n \times n$ latin square with n cells occupied which cannot be completed to an $n \times n$ latin square. In [1], Trevor Evans has conjectured that an incomplete $n \times n$ latin square with $n-1$ cells occupied can always be completed to an $n \times n$ latin square. In [4] J. Marica and J. Schönheim have verified Evan's conjecture provided that the $n-1$ occupied cells are in different rows and columns. The following theorem verifies another special case of Evan's conjecture.

THEOREM 2. *Let I be an $n \times n$ incomplete latin square with $n-1$ cells occupied. Let r denote the number of rows in which the occupied cells occur and C the number of columns. If $r \leqq [n/2]$ or $C \leqq [n/2]$ then I can be completed to an $n \times n$ latin square.*

Proof. Let $n = 2m+t$ where t is 0 or 1. Without loss in generality we can assume $r \leqq [n/2] = m$. Then after a suitable permutation the first m rows of I form an $m \times (2m+t)$ incomplete latin rectangle with $2m+(t-1)$ occupied cells. By the corollary this can be completed to an $m \times (2m+t)$ latin rectangle. Using M. Hall's theorem we can add the remaining $m+t$ rows to complete I to a latin square.

References

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