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SUPPORT POINTS OF THE CLASS OF CLOSE-TO-CONVEX FUNCTIONS[1]

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Let $H(U)$ be the linear space of holomorphic functions on $U=\{z:|z|<1\}$ endowed with the topology of compact convergence, and denote by $H'(U)$ its topological dual space. Let $\mathcal{F}$ be a compact subset of $H(U)$ and $f\in\mathcal{F}$. We say f is a *support point* of $\mathcal{F}$ if there exists an $L\in H'(U)$, non-constant on $\mathcal{F}$, such that $\mathcal{Re}\, L(f)=\sup_{\mathcal{F}}\mathcal{Re}\, L$. On the other hand, f is an *extreme point* of $\mathcal{F}$ if f is not a proper convex combination of two other points of $\mathcal{F}$.

In an earlier paper Brickman-MacGregor-Wilken [1] found the extreme points for the convex closures of four subclasses of

$$S=\{f\in H(U)\colon f \text{ is univalent}, f(0)=0,\ f'(0)=1\},$$

namely for

$$\begin{aligned} K&=\{f\in S\colon f(U) \text{ is convex}\}\\ S^*&=\{f\in S\colon f(U) \text{ is starlike with respect to the origin}\}\\ S_{\mathbf{R}}&=\{f\in S\colon f(z)=\overline{f(\bar z)}\}\\ C&=\{f\in S\colon f \text{ is close-to-convex}\}.\end{aligned}$$

($f\in S$ is close-to-convex if $\mathcal{Re}\{\exp(i\alpha)(f\circ\varphi^{-1})'\}>0$ for some $\varphi\in K$ and real α). In addition, they proved for the first three classes that a support point is necessarily an extreme point of its convex closure and that for a given L there are only finitely many support points. In this note we shall extend this result to the class C.

THEOREM. *If f is a support point of C, then f is an extreme point of the convex closure of C, i.e., f is necessarily of the form*

$$f(z)=\frac{z-(\frac{1}{2})(\zeta+\eta)z^2}{(1-\eta z)^2}\qquad |\zeta|=|\eta|=1,\ \zeta\neq\eta. \tag{1}$$

Moreover, for a fixed $L\in H'(U)$ not of the form $L(g)=ag(0)+bg'(0)$ there are only finitely many support points.

Proof. Let $L\in H'(U)$, not of the form $L(g)=ag(0)+bg'(0)$, and $f\in C$ such that

$$M=\mathcal{Re}\, L(f)=\sup_{C}\mathcal{Re}\, L.$$

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By [1] f has the representation

(2)
$$f(z)=\int_T K(z;\zeta,\eta)\,d\mu$$

where

$$K(z;\zeta,\eta)=\frac{z-(\frac{1}{2})(\zeta+\eta)z^2}{(1-\eta z)^2}$$

and μ is a probability measure on $T=\{(\zeta,\eta):|\zeta|=|\eta|=1\}$. Moreover, the functions $K(z;\zeta,\eta)$ for $(\zeta,\eta)\in T$ and $\zeta\neq\eta$ are the extreme points of the convex closure of C. From the representation (2) it follows that

(3)
$$\mathcal{R}e\, L(K(z;\zeta,\eta))=M \qquad \mu-\text{a.e.}$$

We denote by E that subset of T for which equality holds in (3).

Let now

$$A(\eta)=L\left(\frac{z-\eta z^2/2}{(1-\eta z)^2}\right) \quad \text{and} \quad B(\eta)=L\left(\frac{z^2/2}{(1-\eta z)^2}\right).$$

Since L has a representing measure with compact support in U, both $A(\eta)$ *and* $B(\eta)\in H(\bar{U})$. Moreover, for $(\zeta,\eta)\in E$ we may write (3) as

(4)
$$M=\text{Re}\{A(\eta)\}-\mathcal{R}e\{\zeta B(\eta)\}.$$

If $B(\eta_0)=0$ and $(\zeta_0,\eta_0)\in E$, it is evident from (4) that $(\zeta,\eta_0)\in E$ for all $|\zeta|=1$. In particular, for $\zeta=\eta_0$ the function $z/(1-\eta_0 z)$ would be a support point of C. However, it is not even a support point of the subset S^* [1, Theorem 8]. Consequently, $B(\eta)\neq 0$ whenever $(\zeta,\eta)\in E$.

It is therefore obvious from (4) that $\zeta=-|B(\eta)|/B(\eta)$ and

(5)
$$\mathcal{R}e\, A(\eta)+|B(\eta)|=M$$

whenever $(\zeta,\eta)\in E$. In particular, ζ is uniquely determined by η.

Assume now for the purpose of contradiction that E is an infinite set. Then it follows from (5) that

(6)
$$\{\tfrac{1}{2}[A(\eta)+\overline{A(1/\bar{\eta})}]-M\}^2=B(\eta)\overline{B(1/\bar{\eta})}$$

holds for infinitely many $\eta\in\partial U$. Since the expressions in (6) are analytic in a neighbourhood of ∂U, (6) must be an identity on ∂U. That is,

(7)
$$\mathcal{R}e\, A(\eta)\pm|B(\eta)|=M$$

for each $\eta\in\partial U$. However, the minus sign in (7) cannot occur since $\mathcal{R}e\, A(\eta)\leq \mathcal{R}e\, A(\eta)+|B(\eta)|\leq M$. Consequently, (5) is an identity for all $\eta\in\partial U$.

Now define $F(\eta)=\eta B(\eta)$, $\eta\in\bar{U}$. The winding number of $F(\partial U)$ with respect to the origin is certainly non-zero. Hence, $F(\eta_1)$ is real and negative for some $\eta_1\in\partial U$; i.e., $\eta_1 B(\eta_1)=-|B(\eta_1)|$. It follows then from (5) that $(\eta_1,\eta_1)\in E$. For this η_1 the corresponding function $z/(1-\eta_1 z)$ would then be a support point of C, contradicting [1, Theorem 8] as we saw earlier.

Consequently the set E, and hence the support of μ, is a finite set. Thus (2) is of the form

$$f(z)=\sum_{k=1}^{n} \mu_k K(z;\zeta_k,\eta_k)$$

where $\mu_k>0$, $\sum_{k=1}^{n}\mu_k=1$, $|\zeta_k|=|\eta_k|=1$, and $\zeta_k=-|B(\eta_k)|/B(\eta_k)\neq\eta_k$. But since f is univalent, only one second order pole on ∂U is possible. Therefore $n=1$ and the first assertation of the theorem is proved. The second assertion follows also, for otherwise the set E would be infinite.

Remark. Extremal functions for non-trivial linear problems over C need not be unique. However, if $K(z;\zeta_0,\eta_o)$ is an extremal function, we have shown that $K(z;\zeta,\eta_0)$ for $\zeta\neq\zeta_0$ is not a solution.

Reference

1. L. Brickman, T. H. MacGregor, and D. R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. **156** (1971), 91–107.

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