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BAER ENDOMORPHISM RINGS AND CLOSURE OPERATORS

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A Baer ring is a ring in which every right (and left) annihilator ideal is generated by an idempotent. Generalizing quite naturally from the fact that the endomorphism ring of a vector space is a Baer ring, Wolfson [**5**; **6**] investigated questions such as when the endomorphism ring of a free module is a Baer ring, and when the ring of continuous linear transformations on a pair of dual vector spaces is a Baer ring. A further generalization was made in [**7**], where the question of when the endomorphism ring of a torsion-free module over a semiprime left Goldie ring is a Baer ring was treated. The results are as follows:

If ${}_RV$ is a free module, then $B = \mathrm{Hom}_R(V, V)$ is a Baer ring if and only if every closed submodule of V is a direct summand in V [**6**], Theorem 9], where closure is defined in terms of the dual module $V^* = \mathrm{Hom}_R(V, R)$; and if, in addition, V has a finite basis and R is a commutative integral domain, then the closed submodules of V are just the pure ones [**6**, Theorems 9 and 13].

If V and W are a pair of dual vector spaces over a division ring and B is the ring of all "continuous" linear transformations on (V, W), then the question of whether B is Baer reduces to the question of the existence of a certain type of complement for each closed subspace of V [**5**].

If V is a finite-dimensional (in the sense of Goldie) torsion-less module over a semiprime left Goldie ring R, then $B = \mathrm{Hom}_R(V, V)$ is a Baer ring if and only if every "annihilator-closed" submodule of V is a direct summand in V, where the annihilator-closure operator is the one obtained from the Galois connection between V and B which is given by Baer's "three-cornered Galois Theory"; and if the ring R has a (semisimple) two-sided quotient ring, then the annihilator-closed submodules of V are just the essentially-closed ones ([**7**]; this also follows from Corollary 3.7).

In the above examples, the question of whether B is Baer depends on the behavior of a certain class of closed submodules of V. With this in mind, it is natural to ask the following two questions: first, given ${}_RV$ and a subring B of $\mathrm{Hom}_R(V, V)$, is it possible to distinguish a class of submodules of V which will determine whether B is Baer? An answer to this question is given in Section 2, in terms of a collection, $\mathscr{C}_B$, of submodules which depends on B; (in case V is free and $B = \mathrm{Hom}_R(V, V)$, the elements of $\mathscr{C}_B$ are precisely the closed submodules of V when closure is defined in terms of V^*). Secondly,

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given ${}_RV$, a class $\mathscr{C}$ of closed submodules of V, and a subring B of $\mathrm{Hom}_R(V, V)$ which is "continuous with respect to $\mathscr{C}$" (in the sense that the image by $b \in B$ of the closure of a submodule of V is contained in the closure of its image), when will $\mathscr{C}$ determine whether B is Baer? In Proposition 2.4, necessary and sufficient conditions are given in order that a class $\mathscr{C}$ of closed submodules of V should be equal to $\mathscr{C}_B$. For the important case where $\mathscr{C} = \mathscr{C}_e$, the collection of essentially closed submodules of V, we show that $\mathscr{C}_e = \mathscr{C}_B$ if and only if $\mathrm{Hom}_R(V, U) \neq 0$ for each non-zero $U \in \mathscr{C}_e$ and $r_B(U) \neq 0$ for each $V \neq U \in \mathscr{C}_e$ (where $r_B(U) = \{b \in B : Ub = 0\}$) (Corollary 3.6), or, if and only if every non-zero left (right) ideal of $\mathrm{Hom}_R(E(V), E(V))$ has nonzero intersection with B (where $E(V)$ is the injective hull of V) (Corollary 3.7).

2. General closure operators. Throughout this paper, R denotes an associative ring with 1, ${}_RV$ a left R-module and B a subring of $\mathrm{Hom}_R(V, V)$ which contains 1. The action of elements of B on V will be written on the right. The right (left) annihilator in B of a subset, H, of B will be denoted by $\mathscr{R}(H)$ ($\mathscr{L}(H)$), while r and l will be used for annihilators in V of subsets of B, or in B of subsets of V, e.g.

$$l_V(H) = \{v \in H : vh = 0, \forall h \in H\}, \quad H \subseteq B \quad \text{and}$$
$$r_B(U) = \{b \in B : ub = 0 \,\forall u \in U\}, \, U \subseteq V.$$

Also, let $I_B(U) = \{b \in B : Vb \subseteq U\}$ and $UH = \{uh : u \in U \text{ and } h \in H\}$.

The following lemma is straightforward [**6**, Lemma 1].

LEMMA 2.1. *If $U \subseteq V$ and $J \subseteq B$, then*

(i) $VI_B(U) \subseteq U$.

(ii) $U \subseteq l_V r_B(U)$.

(iii) $I_B(U) r_B(U) = 0$.

(iv) $I_B l_V(J) = \mathscr{L}(J)$.

(v) $r_B(VJ) = \mathscr{R}(J)$.

Let L be a complete lattice. A *closure operator* on L is a mapping $\varphi : L \to L$, written $\varphi(a) = a^c$, such that:

(c1) $a \leqq b$ implies $a^c \leqq b^c$;

(c2) $a \leqq a^c$;

(c3) $(a^c)^c = a^c$.

An element a is *closed* under φ if $a = a^c$. In addition, we will assume that the closure operators considered here satisfy

(c4) The zero element is closed: $0^c = 0$.

Let L' be another complete lattice. A *Galois connection between* L and L' is a pair of mappings σ: $L \to L'$ and τ: $L' \to L$ satisfying:

(1) $x_1 \leqq x_2$ implies $\sigma(x_1) \geqq \sigma(x_2)$ for $x_1, x_2 \in L$.
(2) $y_1 \leqq y_2$ implies $\tau(y_1) \geqq \tau(y_2)$ for $y_1, y_2 \in L'$.
(3) $x \leqq \tau\sigma(x)$ and $y \leqq \sigma\tau(y)$ for $x \in L$, $y \in L'$.

Given a Galois connection, it can be shown that $\sigma\tau\sigma(x) = \sigma(x)$ and $\tau\sigma\tau(y) = \tau(y)$ for $x \in L$, $y \in L'$, so that the maps $\tau\sigma$ and $\sigma\tau$ are closure operators on L and L', respectively. The closed elements in L are those which are of the form $\tau(y)$ for some $y \in L'$. σ and τ induce an anti-isomorphism between the corresponding lattices of closed elements [**3**, pp. 76-78].

It is easily seen that $\mathscr{L}$ and $\mathscr{R}$ form a Galois connection between the lattice L' of right ideals of B and the lattice L'' of left ideals of B, and that the mappings r_B and l_V form a Galois connection between the lattice L of submodules of ${}_RV$ and L', giving the closure operators r_Bl_V and l_Vr_B on L' and L, respectively. Let $\mathscr{C}_a = \{U \subseteq V : U = l_Vr_B(U)\}$ be the collection of closed submodules of V with respect to the closure operator l_Vr_B, and set $\mathscr{C}_B = \{U \subseteq V : U = l_V\mathscr{R}(H)$, for $H \subseteq B\}$. The members of $\mathscr{C}_a$ will be referred to as the "annihilator-closed" submodules of V. Note that $\mathscr{C}_B \subseteq \mathscr{C}_a$, i.e. every element of $\mathscr{C}_B$ is annihilator-closed. For the closure operator r_Bl_V, the following lemma holds:

LEMMA 2.2. *If B contains an idempotent with null-space $l_V\mathscr{R}(H)$, where $H \subseteq B$, then $r_Bl_V\mathscr{R}(H) = \mathscr{R}(H)$ (i.e. $\mathscr{R}(H)$ is closed with respect to the closure operator r_Bl_V).*

Proof. Let $U = l_V\mathscr{R}(H) = l_V(e)$, where $e = e^2 \in B$. Since $1 \in B$, $1 - e \in B$, and since $U = l_V(e) = V(1 - e)$, $1 - e \in I_B(U)$; therefore, $U = V(1 - e) \subseteq VI_B(U)$, hence, by Lemma 2.1 (i), $U = VI_B(U)$. Then,

$$\begin{aligned} r_Bl_V\mathscr{R}(H) &= r_B(U) = r_B(VI_B(U)) = \mathscr{R}(I_B(U)) \quad \text{(by Lemma 2.1 (v))} \\ &= \mathscr{R}(I_Bl_V\mathscr{R}(H)) = \mathscr{R}\mathscr{L}\mathscr{R}(H) \quad \text{(by Lemma 2.1 (iv))} = \mathscr{R}(H). \end{aligned}$$

We can now show that the collection $\mathscr{C}_B$ is the one that determines whether or not B is Baer.

PROPOSITION 2.3. *B is a Baer ring if and only if, for each $U \in \mathscr{C}_B$, B contains an idempotent with null-space U.*

Proof. If B is Baer, then, given $U = l_V\mathscr{R}(H) \in \mathscr{C}_B$, we have $\mathscr{R}(H) = eB$, where $e = e^2 \in B$ and hence $U = l_V\mathscr{R}(H) = l_V(e)$.

Conversely, assume that, for each $\mathscr{R}(H)$, B contains an idempotent, e, with null-space $l_V\mathscr{R}(H)$. Then $U = l_V\mathscr{R}(H) = l_V(e)$ is a direct summand in $V : V = Ve \oplus l_V(e) = Ve \oplus V(1 - e)$. Clearly, $Ue = [l_V\mathscr{R}(H)]e = [l_V(e)]e = 0$ implies $e \in r_Bl_V\mathscr{R}(H) = \mathscr{R}(H)$, the last equality by Lemma 2.2, so that $eB \subseteq \mathscr{R}(H)$. On the other hand, if $b \in \mathscr{R}(H)$, then $[l_V\mathscr{R}(H)]b = 0$ or $[l_V(e)]b = 0$, so that, for any $v \in V$, we have $vb = [ve + v(1 - e)]b = ve\,b$. This last implies $b = eb$, and so $\mathscr{R}(H) \subseteq eB$; hence $\mathscr{R}(H) = eB$ and B is Baer.

Remarks. 1) If $B = \mathrm{Hom}_R(V, V)$, then B contains an idempotent with null-space U if and only if U is a direct summand in V, so that in this case,

B is Baer if and only if every $U \in \mathscr{C}_B$ is a direct summand in V. In particular, Theorem 6 of [**6**], namely that if ${}_RV$ is completely reducible then $\mathrm{Hom}_R(V, V)$ is a Baer ring, is an immediate corollary of Proposition 2.3.

2) If ${}_RV$ is a free module and $B = \mathrm{Hom}_R(V, V)$, let $V^* = \mathrm{Hom}_R(V, R)$ and write (v, f) for the effect of $f \in V^*$ on $v \in V$. Then $U \subseteq V$ is said to be closed if $U = {}^{\perp}U^{\perp}$, where $U^{\perp} = \{f \in V^* : (U, f) = 0\}$ and ${}^{\perp}W = \{v \in V : (v, W) = 0\}$, for $W \subseteq V^*$. By Theorem 8 of [**6**], U is closed if and only if $U = l_V r_B(U)$ and by Theorem 7 and Lemma 2 (i) of [**6**], $r_B(U) = \mathscr{R}[I_B(U)]$, i.e. in this case, U is closed if and only if $U \in \mathscr{C}_B$.

In any case, Proposition 2.3 says that whether B is Baer or not depends on the collection $\mathscr{C}_B$. Since $\mathscr{C}_B \subseteq \mathscr{C}_a$, it is natural to ask here, when does the "Baer-ness" of B depend on $\mathscr{C}_a$, i.e. when is $\mathscr{C}_B = \mathscr{C}_a$. More generally, given a closure operator $\phi(U) = U^c$ and letting $\mathscr{C} = \{U \subseteq V : U = U^c\}$, when is $\mathscr{C}_B = \mathscr{C}$? The following proposition answers these questions for a general closure operator ϕ. First, we need a definition.

Definition. If V, B and $\mathscr{C}$ are as defined above, then B is said to be *continuous with respect to* $\mathscr{C}$ *if*

$$X^c b \subseteq (Xb)^c \quad \text{for all } b \in B \text{ and } X \in L.$$

PROPOSITION 2.4. *Let* ${}_RV$, B, $\mathscr{C}_B$ *and* $\mathscr{C}$ *be as defined above. Then*

a) $\mathscr{C}_a = \mathscr{C}_B$ *if and only if* $r_B[VI_B(U)] = r_B(U)$ *for each* $U \in \mathscr{C}_a$.

b) *If* B *is continuous with respect to* $\mathscr{C}$ *then* $\mathscr{C}_B \subseteq \mathscr{C}_a \subseteq \mathscr{C}$. *Also, in this case* $\mathscr{C}_a = \mathscr{C}$ *if and only if* $l_V r_B(X) = X^c$ *for all* $X \in L$.

Proof. a) Assume $\mathscr{C}_a = \mathscr{C}_B$ and let $U \in \mathscr{C}_a$, then $U = l_V\mathscr{R}(H)$ for some $H \subseteq B$, and so $I_B(U) = I_B l_V \mathscr{R}(H) = \mathscr{L}\mathscr{R}(H)$, by Lemma 2.1 (iv). Therefore, $\mathscr{R}[I_B(U)] = \mathscr{R}\mathscr{L}\mathscr{R}(H) = \mathscr{R}(H)$, and $U = l_V\mathscr{R}(H) = l_V\mathscr{R}[I_B(U)]$, which implies

$$\begin{aligned} r_B(U) = r_B l_V \mathscr{R}[I_B(U)] &= r_B l_V r_B[VI_B(U)], \quad \text{by Lemma 2.1 (v)} \\ &= r_B[VI_B(U)]. \end{aligned}$$

Conversely, assume $r_B[VI_B(U)] = r_B(U)$ for each $U \in \mathscr{C}_a$, and let $U \in \mathscr{C}_a$. Then $U = l_V r_B(U) = l_V r_B[VI_B(U)] = l_V \mathscr{R}[I_B(U)] \in \mathscr{C}_B$ by Lemma 2.1 (v).

b) $\mathscr{C}_B \subseteq \mathscr{C}_a$ follows from Lemma 2.1 (v). To show $\mathscr{C}_a \subseteq \mathscr{C}$, let $U \in \mathscr{C}_a$, so that $U = l_V r_B(U) = l_V(J)$, with $J = r_B(U) \subseteq H$. If $x \in U^c$, then, by continuity, $xJ \subseteq U^c J \subseteq (UJ)^c = 0$; therefore, $x \in l_V(J) = U$ and $U^c = U$, i.e. $U = U^c \in \mathscr{C}$.

Now assume $l_V r_B(X) = X^c$ for all $X \in L$ and let $U \in \mathscr{C}$, so that $U = U^c$. Then $l_V r_B(U) = U$ and $U \in \mathscr{C}_a$. This implies $\mathscr{C} \subseteq \mathscr{C}_a$ so that $\mathscr{C} = \mathscr{C}_a$. Conversely, assume $\mathscr{C} = \mathscr{C}_a$. Note first that continuity of B with respect to $\mathscr{C}$ gives $r_B(X) = r_B(X^c)$ for any $X \in L$. Since $X \subseteq X^c$, we always have $r_B(X^c) \subseteq r_B(X)$; on the other hand, if $b \in r_B(X)$, then $X^c b \subseteq (Xb)^c = 0$, hence $b \in r_B(X^c)$, proving equality. Now if $X \in L$, then $X^c \in \mathscr{C} \subseteq \mathscr{C}_a$, so that $X^c = l_V r_B(X^c) = l_V r_B(X)$.

Remark. In both the free module case of [**6**] and the continuous ring case of [**5**], $\mathscr{C}_a = \mathscr{C}_B$ follows from Proposition 2.4 a) because $VI_B(U) = U$ for all submodules U ([**6**, Theorem 7] and [**5**, Lemma 1(5)]). In order to apply Proposition 2.4 b) to these two cases we show first that $B = \text{Hom}_R(V, V)$ is continuous with respect to $\mathscr{C} = \{U \in L : U = {}^\perp U^\perp\}$ i.e. that $({}^\perp X^\perp)b \subseteq {}^\perp(Xb)^\perp$, for each $X \in L$ and $b \in B$. Let $y \in {}^\perp X^\perp$, so that $(y, X^\perp) = 0$ and let $g \in (Xb)^\perp$ so that $(Xb, g) = 0$. Then $0 = (Xb, g) = (X, b^*g)$ implies $b^*g \in X^\perp$, which implies $0 = (y, b^*g) = (yb, g)$, and this last implies $(yb, (Xb)^\perp) = 0$ since g was arbitrary in $(Xb)^\perp$; i.e. $yb \in {}^\perp(Xb)^\perp$, completing the proof. Now, recalling from the proof of Proposition 2.4 b) that continuity of B with respect to $\mathscr{C}$ implies $r_B(X) = r_B({}^\perp X^\perp)$, and using the fact that $l_V r_B(U) = U$ for all $U \in \mathscr{C}$ ([**6**, Theorem 8] and [**5**, Lemma 3(1)]), we see that $l_V r_B(X) = l_V r_B({}^\perp X^\perp) = {}^\perp X^\perp$, for all $X \in L$. Hence, $\mathscr{C}_a = \mathscr{C}$ follows from Proposition 2.4 b).

3. Essential closure. A module V is an *essential extension* of a submodule U —written $U \subset' V$—if every nonzero submodule of V has nonzero intersection with U. One then says that U is *essential* in V. A submodule U of a module V is said to be *essentially closed* in V if U has no proper essential extensions in V. For any $v \in V$, set $[U : v] = \{r \in R : rv \in U\}$; it is known that, if $U \subset' V$, then, if $0 \neq v \in V$, $[U : v] \subset' R$. The *singular submodule* $Z_R(V)$ of V is defined to be $\{v \in V : [0 : v] \subset' R\}$. V is said to be *non-singular* if $Z_R(V) = 0$. If $U \subset' V$, then V is nonsingular if and only if U is nonsingular. A ring will be called (left) nonsingular if its left regular representation is nonsingular. For details on essential extensions and nonsingular modules see [**1**] or [**2**].

Let ${}_RV$ be a nonsingular module and let ${}_R\tilde{V}$ be an injective hull of ${}_RV$. If U is any submodule of V, denote by $\tilde{U}$ the unique (see [**1**, p. 61]) injective hull of U contained in $\tilde{V}$. Then the essential closure of U in V is given by $V \cap \tilde{U}$. For an injective nonsingular module, the essentially closed submodules are simply the direct summands. There is a lattice isomorphism between the lattice of essentially closed submodules of V and the lattice of essentially closed submodules of $\tilde{V}$ given by $U \to \tilde{U}$ with inverse $\tilde{U} \to \tilde{U} \cap V$ (see [**1**, p. 61], or [**3**, p. 250]).

The following known lemma will be used frequently in the sequel.

LEMMA 3.1. *If ${}_RV$ is nonsingular and U, Y are submodules of V such that $U \subset' Y$, then $r_B(U) = r_B(Y)$.*

Proof. $r_B(Y) \subset r_B(U)$ since $U \subset Y$. Let $b \in r_B(U)$, so that $Ub = 0$. For any $y \in Y$, $[U : y] \subset' R$ and $[U : y]yb = 0$. Since ${}_RV$ is nonsingular, this implies $yb = 0$ and hence $b \in r_B(Y)$. This completes the proof.

We prove now that, for a nonsingular module ${}_RV$, any subring B of $\text{Hom}_R(V, V)$ is continuous with respect to the collection of essentially closed submodules. Denote by U^e the essential closure of U, i.e. the largest essential extension of U in V, or $\tilde{U} \cap V$.

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LEMMA 3.2. *Let* ${}_RV$ *be a nonsingular module and* $\mathscr{C}_e = \{U \subseteq V : U = U^e\}$. *Then any subring, B, of* $\mathrm{Hom}_R(V, V)$ *is continuous with respect to* $\mathscr{C}_e$.

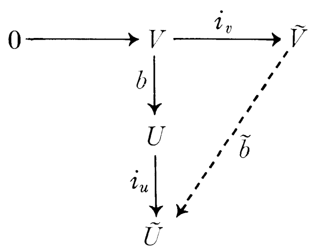
Proof. Let $x \in U^e b$ for some $U \in L$ and $b \in B$, so that $x = yb$, with $y \in U^e$. In order to show $x \in (Ub)^e$, we show $Ub \subset' Ub + Rx$. Let $0 \neq z = ub + rx \in Ub + Rx$, with $u \in U$ and $r \in R$, and show $Rz \cap Ub \neq 0$. If $rx = 0$, there is nothing to prove, so assume $rx \neq 0$, i.e. $ryb = rx \neq 0$. Since $ry \in U^e$, $[U : ry] \subset' R$, and since $[U : ry] \subset [Ub : ryb]$ $(r_1ry \in U \Rightarrow r_1ryb \in Ub)$, also $[Ub : ryb] \subset' R$. Since V is non-singular, $[Ub : ryb]z \neq 0$, but $[Ub : ryb]z = [Ub : rx]z \subseteq Rz \cap Ub$, completing the proof.

LEMMA 3.3. *If* ${}_RV$ *is non-singular, then* $D = \mathrm{Hom}_R(\tilde{V}, \tilde{V})$ *is regular, left self-injective, any subring B of* $\mathrm{Hom}_R(V, V)$ *can be embedded in D and, for each* $U \in \mathscr{C}_e$,

$$r_B(U) = r_D(\tilde{U}) \cap B \quad \text{and} \quad I_B(U) = I_D(\tilde{U}) \cap B.$$

Proof. Since $\tilde{V}$ is injective, the Jacobson radical, J, of $D = \mathrm{Hom}_R(\tilde{V}, \tilde{V})$ consists of those endomorphisms whose kernels are essential submodules of $\tilde{V}$ [**3**, Proposition XIV 1.1], and D/J is regular and left self-injective [**3**, Theorem XIV 1.2]. Here, $J = 0$ since $g \in J$ implies $l_V(g) \subset' \tilde{V}$, so that by Lemma 3.1, since $\tilde{V}$ is nonsingular, $r_Bl_V(g) = 0$, which last implies $g = 0$ [since $g \in r_Bl_V(g)$]. Hence D is regular and left self-injective. Given $b \in B$, b has an extension $\tilde{b} \in D$, since $\tilde{V}$ is injective. If $\tilde{b}_1$ is another extension of b, then $\tilde{b}_1 - \tilde{b} \in r_D(V)$, which, by Lemma 3.1, implies $\tilde{b}_1 - \tilde{b} \in r_D(\tilde{V}) = 0$. Hence each $b \in B$ has a unique extension in D and henceforth we can identify the elements of B with their extensions and consider $B \subseteq D$.

Now, with the help of Lemma 3.1, it is clear that $b \in r_B(U)$ if and only if $b \in r_D(\tilde{U}) \cap B$. For $I_B(U) = I_D(\tilde{U}) \cap B$, one uses the fact that $\tilde{U}$ is the injective hull of U so that if $b \in B$ maps V into U, then its extension $\tilde{b}$ maps $\tilde{V}$ into $\tilde{U}$; in other words, the unique extension $\tilde{b}$ is the one making the following diagram commute:



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where i_v and i_u are the natural injections. This gives $I_B(U) \subseteq I_D(\tilde{U}) \cap B$; but $b \in I_D(\tilde{U}) \cap B \Rightarrow Vb \subseteq V \cap \tilde{U} = U$, giving the reverse inclusion.

THEOREM 3.4. *Let* ${}_RV$ *be a non-singular module,* $\mathscr{C}_e$ *the collection of essentially closed submodules of V and B a subring of* $\mathrm{Hom}_R(V, V)$. *Then the following are equivalent:*

(i) $U = U^e \Rightarrow r_B(U) = \mathscr{R}[I_B(U)]$.
(ii) $r_B[VI_B(U)] = r_B(U)$ *for every* $U \in \mathscr{C}_e$.
(iii) $I_B(U) \neq 0$ *for every* $0 \neq U \in \mathscr{C}_e$.
(iv) *Every nonzero left ideal of* $\mathrm{Hom}_R(\tilde{V}, \tilde{V})$ *has nonzero intersection with* B.

Proof. (i) $\Leftrightarrow$ (ii): Let $U = U^e$; by Lemma 2.1 (v), $r_B[VI_B(U)] = \mathscr{R}[I_B(U)]$, hence, $r_B(U) = \mathscr{R}[I_B(U)] \Leftrightarrow r_B(U) = r_B[VI_B(U)]$.

(ii) $\Rightarrow$ (iii): Let $0 \neq U \in \mathscr{C}_e$. If $I_B(U) = 0$, then $VI_B(U) = 0$ and $r_B(U) = r_B[VI_B(U)] = B$. But then $U \subseteq l_V r_B(U) = l_V(B) = 0$, contradicting $U \neq 0$.

(iii) $\Rightarrow$ (ii): First note that (iii) $\Leftrightarrow$ $VI_B(U) \subset' U$, for all $U \in \mathscr{C}_e$. For, given $U \in \mathscr{C}_e$, let $0 \neq u \in U$ and let $Y = (Ru)^e$. Since U is closed, $Y \subseteq U$ and therefore $I_B(Y) \subseteq I_B(U)$. Since $0 \neq Y \in \mathscr{C}_e$, by (iii), there is $0 \neq c \in I_B(Y)$. Then $Vc \subseteq Y$ and, since $Ru \subset' Y$, there is $0 \neq x \in Vc \cap Ru$. Therefore, $0 \neq x \in VI_B(U) \cap Ru$, proving that $VI_B(U) \subset' U$. Clearly, $r_B(U) \subseteq r_B[VI_B(U)]$, since $VI_B(U) \subseteq U$. Let $b \in r_B[VI_B(U)]$; then, for any $0 \neq u \in U$, $[VI_B(U) : u] \subset' R$ and $[VI_B(U) : u]ub = 0$. Since V is non-singular, this implies $ub = 0$. Therefore $Ub = 0$ and $b \in r_B(U)$, proving (ii).

(iii) $\Rightarrow$ (iv): To prove (iv) it is sufficient to show that B intersects every nonzero principal left ideal of $D = \mathrm{Hom}_R(\tilde{V}, \tilde{V})$. By Lemma 3.3, D is regular, hence any principal ideal, K, of D is generated by an idempotent (see e.g. [**3**, Proposition I-12.1]), say $K = De$, where $e = e^2 \in D$. Consider the submodule $\tilde{V}e$; clearly, $e \in I_D(\tilde{V}e)$ and therefore $De \subseteq I_D(\tilde{V}e)$. On the other hand, $d \in I_D(\tilde{V}e) \Rightarrow \tilde{V}d \subseteq \tilde{V}e \Rightarrow$ for each $\tilde{v} \in \tilde{V}$, $\tilde{v}d = \tilde{y}e$ for some $\tilde{y} \in \tilde{V}$, $\Rightarrow \tilde{v}de = \tilde{y}e = \tilde{v}d \Rightarrow de = d$, or $d \in De$. Therefore, $De = I_D(\tilde{V}e)$.

Since $\tilde{V}e$ is a direct summand in $\tilde{V}$, and therefore closed, we have, by the lattice isomorphism between the closed submodules of $\tilde{V}$ and those of V, that $\tilde{V}e = \tilde{U}$ where $U = \tilde{V}e \cap V$ is closed in V. By (iii), $I_B(U) \neq 0$ since $K \neq 0 \Rightarrow U \neq 0$, and by Lemma 3.3, $I_D(\tilde{U}) \cap B = I_B(U)$, i.e. $K \cap B = I_B(U) \neq 0$, proving (iv).

(iv) $\Rightarrow$ (iii): If every nonzero left ideal of D intersects B, then, in particular, for any nonzero closed U, $I_B(U) = I_D(\tilde{U}) \cap B \neq 0$.

THEOREM 3.5. *Let* ${}_RV$, $\mathscr{C}_e$ *and* B *be as in the preceding theorem. Then the following are equivalent:*
(i) $U = U^e \Rightarrow U = l_V(J)$, *for some subset* J *of* B.
(ii) $X^e = l_V r_B(X)$, *for every submodule* $X \in L$.
(iii) $r_B(U) \neq 0$ *for every* $V \neq U \in \mathscr{C}_e$.
(iv) *Every nonzero right ideal of* $\mathrm{Hom}_R(\tilde{V}, \tilde{V})$ *has nonzero intersection with* B.

Proof. (i) $\Rightarrow$ (ii): Let $X \in L$; by (i), $X^e = l_V(J)$, $J \subseteq B$; then

$$l_V r_B(X) = l_V r_B(X^e), \quad \text{by Lemma 3.1, since } X \subset' X^e,$$
$$= l_V r_B l_V(J) = l_V(J) = X^e,$$

proving (ii).

(ii) $\Rightarrow$ (i) is obvious.

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(ii) $\Rightarrow$ (iii): Let $U \in \mathscr{C}_e$. Then, by (ii), $U = U^e = l_V r_B(U)$, so that $r_B(U) = 0 \Rightarrow U = l_V(0) = V$.

(iii) $\Rightarrow$ (ii): From the proof of Proposition 2.4b), we know that since, by Lemma 3.2, B is continuous with respect to $\mathscr{C}_e$, every $l_V(J)$, for $J \subseteq B$, is closed. Hence, $X \subseteq l_V r_B(X) \Rightarrow X^e \subseteq l_V r_B(X)$, so to prove (ii), it is sufficient to show that $X \subset' l_V r_B(X)$.

Recall that, if U is a submodule of W, then a relative complement for U in W is any submodule, Y, of W, which is maximal with respect to the property $U \cap Y = 0$, and in this case, $U \oplus Y \subset' W$ (see e.g. [**2**, Proposition I. 1.3]). It is known that a submodule, U, of V is essentially closed in V if and only if U is a relative complement for some $Y \subseteq V$ [**2**, Proposition I. 1.4].

Supposing X is not essential in $l_V r_B(X)$, let Y be a relative complement for X in $l_V r_B(X)$, so that $X \oplus Y \subset' l_V r_B(X)$; and let P be a relative complement for $l_V r_B(X)$ in V, so that $P \oplus l_V r_B(X) \subset' V$. Consider $l_V r_B(P \oplus X)$: we have $P \subseteq P \oplus X \subseteq l_V r_B(P \oplus X)$, and $X \subseteq P \oplus X \Rightarrow l_V r_B(X) \subseteq l_V r_B(P \oplus X)$. Therefore, $P \oplus l_V r_B(X) \subseteq l_V r_B(P \oplus X)$, and since $P \oplus l_V r_B(X) \subset' V$, also $l_V r_B(P \oplus X) \subset' V$. But then, by Lemma 3.1, $r_B l_V r_B(P \oplus X) = r_B(V) = 0$, or $r_B(P \oplus X) = 0$ and therefore $r_B((P \oplus X)^e) = 0$, again by Lemma 3.1. But, by (iii), this last implies $(P \oplus X)^e = V$ or $P \oplus X \subset' V$. Hence $Y \cap (P \oplus X) = 0$ implies $Y = 0$ and $X \subset' l_V r_B(X)$.

(iii) $\Rightarrow$ (iv): As in the proof of Theorem 3.4, $D = \operatorname{Hom}_R(\tilde{V}, \tilde{V})$ is regular, left self-injective, hence also Baer. If K is a nonzero principal right ideal of D, then, since D is regular, K is generated by an idempotent, say e, i.e. $K = eD$. Consider $l_{\tilde{V}}(e)$: this is a direct summand, hence closed, hence, as in the previous theorem, $l_{\tilde{V}}(e) = \tilde{U}$, where $U = l_{\tilde{V}}(e) \cap V$ is closed in V. Clearly, $e \in r_D l_{\tilde{V}}(e)$, so $eD \subseteq r_D l_{\tilde{V}}(e)$. And, if $d \in r_D l_{\tilde{V}}(e)$, then, for any $\tilde{v} \in \tilde{V}$, $\tilde{v}d = [\tilde{v}e + \tilde{v}(1 - e)]d = \tilde{v}ed$, since $\tilde{v}(1 - e) \in l_{\tilde{V}}(e)$; so $d = ed \in eD$ and $eD = r_D l_{\tilde{V}}(e)$, or $K = r_D(\tilde{U})$. Now $K = eD \neq 0$ implies $l_{\tilde{V}}(e) \neq \tilde{V}$ and therefore $U \neq V$, so $0 \neq r_B(U) = r_D(\tilde{U}) \cap B$, i.e. K intersects B and hence so does every right ideal in D.

(iv) $\Rightarrow$ (iii) is obvious from $r_D(\tilde{U}) \cap B = r_B(U)$, since $U \neq V$ and $U \in \mathscr{C}_e \Rightarrow \tilde{U} \neq \tilde{V} \Rightarrow r_D(\tilde{U}) \neq 0$.

Remark. If we take ${}_RV = {}_RR$, where R is a left non-singular ring, then Theorem 3.5 becomes Utumi's theorem [**4**, Theorem 2.2], giving necessary and sufficient conditions for the lattice of closed left ideals of R to be equal to the lattice of annihilator left ideals of R (see also [**3**, Proposition XII—4.7]). Here, since R is non-singular, $\operatorname{Hom}_R(\tilde{R}, \tilde{R}) \cong Q_{\max}$, the maximal left quotient ring of R.

Now, noting that condition (ii) of Theorem 3.4 is a) of Proposition 2.4, and (ii) of Theorem 3.5 is b) of Proposition 2.4, we have the following.

If ${}_RV$, $\mathscr{C}_e$ and B are as in the preceding theorems then:

COROLLARY 3.6. *$\mathscr{C}_e = \mathscr{C}_B$ if and only if*
a) $I_B(U) \neq 0$ *for every* $0 \neq U \in \mathscr{C}_e$, *and*
b) $r_B(U) \neq 0$ *for every* $V \neq U \in \mathscr{C}_e$.

COROLLARY 3.7. $\mathscr{C}_e = \mathscr{C}_B$ if and only if

a) *Every nonzero left ideal of* $\mathrm{Hom}_R(\tilde{V}, \tilde{V})$ *has nonzero intersection with* B, and

b) *Every nonzero right ideal of* $\mathrm{Hom}_R(\tilde{V}, \tilde{V})$ *has nonzero intersection with* B.

Remark. If ${}_RV$ is a finite-dimensional (in the sense of Goldie), torsionless module over a ring R which possesses a semisimple two-sided quotient ring S, and $B = \mathrm{Hom}_R(V, V)$, then $\mathrm{Hom}_R(\tilde{V}, \tilde{V})$ is a semisimple two-sided quotient ring of B ([**8**], Theorem 2.3 and 3.3 and their proofs), hence every nonzero right (respectively left) ideal of $\mathrm{Hom}_R(\tilde{V}, \tilde{V})$ has nonzero intersection with B, i.e. a) and b) of Corollary 3.7 are satisfied, and therefore B is Baer if and only if every essentially-closed submodule of V is a direct summand in V.

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