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Extension of Some Theorems of W. Schwarz

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Abstract. In this paper, we prove that a non–zero power series $F(z) \in \mathbb{C}[[z]]$ satisfying

$$F(z^d) = F(z) + \frac{A(z)}{B(z)},$$

where $d \geq 2$, $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$ is transcendental over $\mathbb{C}(z)$. Using this result and a theorem of Mahler's, we extend results of Golomb and Schwarz on transcendental values of certain power series. In particular, we prove that for all $k \geq 2$ the series $G_k(z) := \sum_{n=0}^{\infty} z^{k^n}(1 - z^{k^n})^{-1}$ is transcendental for all algebraic numbers z with $|z| < 1$. We give a similar result for $F_k(z) := \sum_{n=0}^{\infty} z^{k^n}(1 + z^{k^n})^{-1}$. These results were known to Mahler, though our proofs of the function transcendence are new and elementary; no linear algebra or differential calculus is used.

1 Introduction

Golomb proved in [4] that the values of the functions

$$\sum_{n=0}^{\infty} \frac{z^{2^n}}{1 + z^{2^n}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 - z^{2^n}}$$

are irrational at $z = \frac{1}{t}$ for $t = 2, 3, 4, \ldots$, the interesting special case of which is that the sum of the reciprocals of the Fermat numbers is irrational. Schwarz [11] gave results on series of the form

$$G_k(z) := \sum_{n=0}^{\infty} \frac{z^{k^n}}{1 - z^{k^n}}.$$

In particular, he proved that if k, t and b are integers satisfying $k \geq 2$, $t \geq 2$, and $1 \leq b < t^{1-1/k}$, then the number

$$G_k(bt^{-1}) = \sum_{n=0}^{\infty} \frac{b^{k^n}}{t^{k^n} - b^{k^n}}$$

is irrational. Schwarz also showed that for $k, t, b \in \mathbb{N}$ with $k > 2$, $t \geq 2$, and $1 \leq b < t^{1-5/2k}$ the number $G_k(bt^{-1})$ is transcendental. The case $k = 2$ proved to

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be more difficult, though he was able to show that for an integer $t \geq 2$, the number $G_2(t^{-1})$ is not algebraic of the second degree.

Schwarz also remarked [11] that, "the irrationality of

$$F_k(bt^{-1}) := \sum_{n=0}^{\infty} b^{k^n}(t^{k^n} + b^{k^n})^{-1}$$

for $k > 2$ is unsettled" (here the notation $F_k(bt^{-1})$ has been added).

Recently, Duverney [1] proved the transcendence of $G_2(t^{-1})$ for integers $t \geq 2$ and extended Schwarz's transcendence results for the case $k = 2$. He proved the following theorem.

Theorem 1.1 *Let $a \geq 2$ be an integer and let b_n be a sequence of integers satisfying $|b_n| = O(\eta^{-2^n})$ for every $\eta \in (0,1)$. Suppose that $a^{2^n} + b_n \neq 0$ for every $n \in \mathbb{N}$. Then the number*

$$S = \sum_{n=0}^{\infty} \frac{1}{a^{2^n} + b_n}$$

is transcendental.

We extend Schwarz's results further (to the best possible); in particular, we prove that for all $k \geq 2$ the series $G_k(z) = \sum_{n=0}^{\infty} z^{k^n}(1 - z^{k^n})^{-1}$ is transcendental for all algebraic numbers z with $|z| < 1$. We also prove the same result for $F_k(z) = \sum_{n=0}^{\infty} z^{k^n}(1 + z^{k^n})^{-1}$ which settles the irrationality question of Schwarz's remark. These results were known to Mahler (see [5–8]), though our proofs of the function transcendence are new and elementary, coming from the proof of our main result; no linear algebra or differential calculus is used.

Our main result is that a non–zero power series $F(z) \in \mathbb{C}[[z]]$ satisfying

$$F(z^d) = F(z) + \frac{A(z)}{B(z)},$$

where $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$ is transcendental over $\mathbb{C}(X)$. This extends a theorem of Nishioka [9] that states that $F(z)$ is either transcendental or rational.

2 A General Theorem

Nishioka [9] has shown the following.

Theorem 2.1 *Suppose that $F(z) \in \mathbb{C}[[z]]$ satisfies one of the following for an integer $d > 1$.*

(i) $F(z^d) = \varphi(z, F(z))$,

(ii) $F(z) = \varphi(z, F(z^d))$,

where $\varphi(z,u)$ is a rational function in z, u over $\mathbb{C}$. If $F(z)$ is algebraic over $\mathbb{C}(z)$, then $F(z) \in \mathbb{C}(z)$.

Nishioka's proof of Theorem 2.1 relies heavily on deep ideas from algebraic number theory. In this section we provide an elementary proof of a special case of Theorem 2.1. In this special case, we are able to refine the conclusion by eliminating the possibility of $F(z)$ being a rational function.

Theorem 2.2 *If $F(z)$ is a power series in $\mathbb{C}[[z]]$ satisfying*

$$F(z^d) = F(z) + \frac{A(z)}{B(z)},$$

where $d \geq 2$, $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$, then $F(z)$ is transcendental over $\mathbb{C}(z)$.

Proof Suppose that the power series $F(z)$ is algebraic and satisfies

$$\sum_{r=0}^{n} q_r(z) F(z)^r \equiv 0, \tag{2.1}$$

where the $q_r(z)$ are rational functions with $q_n(z) = 1$ and $n \geq 1$ is chosen minimally.

Substituting z^d into (2.1) and using the functional relation gives

$$0 \equiv \sum_{r=0}^{n} q_r(z^d) F(z^d)^r = \sum_{r=0}^{n} q_r(z^d) \Big(F(z) + \frac{A(z)}{B(z)} \Big)^r.$$

Without loss of generality, suppose $B(z)$ is monic. Multiplying by $B(z)^n$ to clear fractions as well as an application of the binomial theorem yields

$$0 \equiv \sum_{r=0}^{n} q_r(z^d) B(z)^{n-r} \big(B(z)F(z) + A(z) \big)^r = \sum_{r=0}^{n} q_r(z^d) B(z)^{n-r} \sum_{j=0}^{r} \binom{r}{j} B(z)^j F(z)^j A(z)^{r-j}. \tag{2.2}$$

Taking the difference between (2.2) and $B(z)^n$ times (2.1) gives

$$Q(z) := \sum_{r=0}^{n} q_r(z^d) B(z)^{n-r} \sum_{j=0}^{r} \binom{r}{j} B(z)^j F(z)^j A(z)^{r-j} - B(z)^n \sum_{r=0}^{n} q_r(z) F(z)^r \equiv 0. \tag{2.3}$$

Note that we may also write $Q(z) = \sum_{m=0}^{n} h_m(z) F(z)^m \equiv 0$.

We determine $h_n(z)$. The only term in $Q(z)$ that can contribute to the coefficient of $F(z)^n$ is the $r = n$ term of the sum (2.3), which, recalling that $q_n(z) = 1$, is

$$\sum_{j=0}^{n} \binom{n}{j} B(z)^j F(z)^j A(z)^{n-j} - B(z)^n F(z)^n,$$

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and only the $j = n$ term here contributes. Hence

$$h_n(z) = \binom{n}{n} B(z)^n A(z)^{n-n} - B(z)^n = 0,$$

so that $Q(z) = \sum_{m=0}^{n-1} h_m(z) F(z)^m \equiv 0$. Since n was chosen minimally, $h_m(z) \equiv 0$ for all $m = 0, 1, \ldots, n-1$.

Using (2.3), we have that

$$h_m(z) = \sum_{r=m}^{n} \binom{r}{m} q_r(z^d) B(z)^{n-r+m} A(z)^{r-m} - B(z)^n q_m(z).$$

Since $h_{n-1}(z) \equiv 0$, we have

$$\sum_{r=n-1}^{n} \binom{r}{n-1} q_r(z^d) B(z)^{n-r+(n-1)} A(z)^{r-(n-1)} = B(z)^n q_{n-1}(z),$$

so that removal of shared factors and again recalling $q_n(z) = 1$, we have the identity

$$q_{n-1}(z^d) B(z) + nA(z) = B(z) q_{n-1}(z). \tag{2.4}$$

Write $q_{n-1}(z) = \frac{\alpha(z)}{\beta(z)}$ where $\alpha(z), \beta(z) \in \mathbb{C}[z]$ with $\gcd(\alpha(z), \beta(z)) = 1$ and $\beta(z)$ monic. Then (2.4) becomes

$$\beta(z)\alpha(z^d) B(z) + n\beta(z)\beta(z^d) A(z) = \beta(z^d) B(z) \alpha(z). \tag{2.5}$$

Equation (2.5) yields $\beta(z^d) | \beta(z)\alpha(z^d) B(z)$. As $\gcd(\alpha(z^d), \beta(z^d)) = 1$, this implies that $\beta(z^d) | \beta(z) B(z)$. Therefore, $d \cdot \deg \beta(z) \le \deg \beta(z) + \deg B(z) < \deg \beta(z) + d$. Hence

$$0 \le \deg \beta(z) < 1 + \frac{1}{d-1},$$

so that since $d \ge 2$, either $\deg \beta(z) = 0$ or $\deg \beta(z) = 1$.

Suppose $\deg \beta(z) = 0$ so that $\beta(z) \in \mathbb{C}$. Hence $\beta(z) = \beta(z^d) \in \mathbb{C}$; write $\beta := \beta(z)$. Now (2.5) becomes

$$\alpha(z^d) B(z) + n\beta A(z) = B(z) \alpha(z). \tag{2.6}$$

Thus $B(z) | n\beta$, so that $\deg B(z) = 0$; write $B := B(z)$. So (2.6) becomes

$$\alpha(z^d) B + n\beta A(z) = B\alpha(z), \tag{2.7}$$

which implies that $d \cdot \deg \alpha(z) = \deg A(z) < d$, so that $\deg \alpha(z) = 0$. Equation (2.7) and $\deg \alpha(z) = 0$ imply that $A(z) = 0$, which is impossible.

Now suppose $\deg \beta(z) = 1$ and write $\beta(z) = z - \beta$. Comparing degrees in (2.5) implies that $\deg \alpha(z) \le 1$.

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Recall that $\beta(z^d)|\beta(z)B(z)$ by (2.5). As $\deg B < d$, this implies that $\deg B = d-1$. Since β and B are both monic, we conclude that $\beta(z^d) = \beta(z)B(z)$, whence

$$\frac{\beta(z^d)}{\beta(z)} = B(z).$$

Suppose that $\deg \alpha(z) = 1$. Write $\alpha(z) = \delta(z-\alpha)$ and note that $\beta \neq \alpha$. In this case, replacing $B(z)$ in (2.5) and solving for $A(z)$ gives

$$A(z) = \frac{\delta(\beta-\alpha)z(z^{d-1}-1)}{n(z-\beta)^2} \in \mathbb{C}[z].$$

Since $A(z) \in \mathbb{C}[z]$ we have that $(z-\beta)^2|(z^{d-1}-1)$, which is impossible because $z(z^{d-1}-1)$ has only simple roots; hence $\deg \alpha(z) = 0$.

If $\deg \alpha(z) = 0$, write $\alpha := \alpha(z)$. Then writing $\beta(z) = z - \beta$ and solving (2.5) for $A(z)$, we have that

$$A(z) = \frac{\alpha z(z^{d-1}-1)}{n(z-\beta)^2} \in \mathbb{C}[z],$$

which is, again, impossible. Thus the theorem is proved. ■

3 The Series $G_k(z)$ and $F_k(z)$

To prove the transcendence results surrounding $G_k(z)$ and $F_k(z)$, we apply Theorem 2.2 as well as the following theorem of Mahler [5], as taken from Nishioka's book [10]. Here $\mathbf{I}$ is the set of algebraic integers over $\mathbb{Q}$, K is an algebraic number field, $\mathbf{I}_K = K \cap \mathbf{I}$, and $f(z) \in K[[z]]$ with radius of convergence $R > 0$ satisfying the functional equation for an integer $d > 1$,

$$f(z^d) = \frac{\sum_{i=0}^m a_i(z)f(z)^i}{\sum_{i=0}^m b_i(z)f(z)^i}, \quad m < d,\ a_i(z), b_i(z) \in \mathbf{I}_K[z],$$

and $\Delta(z) := \operatorname{Res}(A,B)$ is the resultant of

$$A(u) = \sum_{i=0}^m a_i(z)u^i \quad \text{and} \quad B(u) = \sum_{i=0}^m b_i(z)u^i$$

as polynomials in u.

Theorem 3.1 ([5]) *Assume that $f(z)$ is not algebraic over $K(z)$. If α is an algebraic number with $0 < |\alpha| < \min\{1, R\}$ and $\Delta(\alpha^{d^n}) \neq 0$ ($n \geq 0$), then $f(\alpha)$ is transcendental.*

Now consider the functional equation $f(z^k) = f(z) - \frac{z}{1-z}$ with $k \geq 2$. Repeated use gives

$$f(z^{k^m}) = f(z^{k^{m-1}}) - \frac{z^{k^{m-1}}}{1-z^{k^m}} = f(z) - \sum_{n=1}^m \frac{z^{k^{m-n}}}{1-z^{k^{m-n}}}.$$

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Changing the index and setting $W_m(z) := \sum_{n=0}^{m-1} z^{k^n}/(1-z^{k^n})$ gives

$$f(z) = f(z^{k^m}) + W_m(z).$$

In the region $|z| < 1$ we have

$$f(z) = \lim_{m\to\infty} [f(z^{k^m}) + W_m(z)] = \sum_{n=0}^{\infty} \frac{z^{k^n}}{1-z^{k^n}} = G_k(z).$$

This proves the following lemma.

Lemma 3.2 *The function $G_k(z)$ satisfies the functional equation*

$$G_k(z^k) = G_k(z) + \frac{z}{z-1}.$$

As a corollary of Theorem 2.2, we have the following corollary.

Corollary 3.3 *T he function $G_k(z)$ is transcendental over $\mathbb{C}(z)$.*

To get the transcendence of the associated numbers, we use Mahler's theorem.

Proposition 3.4 *For $k \geq 2$ and $z = \alpha$ algebraic with $0 < |\alpha| < 1$, $G_k(\alpha)$ is transcendental over $\mathbb{Q}$.*

Proof Lemma 3.2 gives the functional equation

$$G_k(z^k) = \frac{(1-z)G_k(z) - z}{1-z},$$

so that, in the language of Theorem 3.1, we have $A(u) = (1-z)u - z$ and $B(u) = 1-z$, $m = 1 < k = d$, and $a_i(z), b_i(z) \in \mathbf{I}_K[z]$. Since $B(u)$ is a constant polynomial in u, $\Delta(z) := \operatorname{Res}(A, B) = 1 - z$. Let $|\alpha| < 1$ be algebraic; it is immediate that $\Delta(\alpha^{k^n}) = 1 - \alpha^{k^n} \neq 0 \quad (n \geq 0)$. Since $G_k(z)$ is not algebraic over $\mathbb{C}(z)$ (as supplied by Theorem 2.2), applying Theorem 3.1, we have that $G_k(\alpha)$ is transcendental over $\mathbb{Q}$. ∎

Corollary 3.5 *If $k, b, t \in \mathbb{N}$ with $k \geq 2$ and $0 < b < t$, then the number $G_k(bt^{-1})$ is transcendental over $\mathbb{Q}$.*

Proof Set $\alpha = b/t$ in Theorem 3.4. ∎

We turn now to the series

$$F_k(z) = \sum_{n=0}^{\infty} \frac{z^{k^n}}{1+z^{k^n}}.$$

Similar to $G_k(z)$, $F_k(z)$ satisfies a functional equation,

$$F_k(z^k) = F_k(z) - \frac{z}{z+1}.$$

Using this functional equation, we have the following corollary of Theorem 2.2.

Corollary 3.6 *The function $F_k(z)$ is transcendental over $\mathbb{C}(z)$.*

As before, Mahler's theorem gives the following proposition.

Proposition 3.7 *For $k \geq 2$ and $z = \alpha$ an algebraic number with $0 < |\alpha| < 1$, $F_k(\alpha)$ is transcendental over $\mathbb{Q}$.*

Corollary 3.8 *If $k, b, t \in \mathbb{N}$ with $k \geq 2$ and $1 \leq b < t$, then the number $F_k(bt^{-1})$ is transcendental over $\mathbb{Q}$.*

Remark 1 For some more recent work concerning results like Nishioka's Theorem 2.1, but for more general algebraic number fields, see [2] (This paper also contains a number of current references to work in this area). Also, concerning functions similar to $G_k(z)$ and $F_k(z)$ above, Duverney, Kanoko, and Tanaka [3] have given a complete classification of those series

$$f(z) := \sum_{k=0}^{\infty} \frac{a^k z^{d^k}}{1 + bz^{d^k} + cz^{2d^k}} \in C[[z]]$$

that are transcendental over $C(z)$ where C is a field of characteristic 0, $d \geq 2$, and $a, b, c \in C$ with $a \neq 0$.

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