J. Austral. Math. Soc. **21** (Series A) (1976), 368–370.

LATTICES, COMPLEMENTS AND TIGHT RIESZ ORDERS

NEIL CAMERON

(Received 19 June 1975)

Abstract

It is proved that there exists no compatible tight Riesz order on a complemented modular lattice. An example is provided of a complemented lattice with a compatible tight Riesz order.

A partially ordered set $(X, \leqq)$ is said to satisfy the (m, n) *tight Riesz interpolation property*, abbreviated TR(m, n) (where m, n are positive integers) if, for all $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ in X such that

$$x_i < y_j \text{ for } i = 1, \cdots, m \text{ and } j = 1, \cdots, n,$$

there exists an element z in X satisfying

$$x_i < z < y_j \text{ for } i = 1, \cdots, m \text{ and } j = 1, \cdots, n.$$

If $(X, \leqq)$ is a partially ordered set an *associated preorder* $\trianglelefteq$ on X is defined: for all x, y in X,

$$x \trianglelefteq Y \quad \text{iff (for all } u \text{ in } X) \quad (u < x \Rightarrow u < y)$$
$$\text{and (for all } t \text{ in } X) \quad (y < t \Rightarrow x < t).$$

If $(X, \trianglelefteq)$ is a lattice then a partial order $\leqq$ on X is a *compatible tight Riesz order* for $(X, \trianglelefteq)$ if it satisfies the TR $(1, 2)$ and TR $(2, 1)$ properties (and so also the TR $(1, 1)$ property) and has $\trianglelefteq$ as associated preorder.

Associated preorders have been studied by Cameron and Miller. In particular, if $(X, \trianglelefteq)$ is a lattice with compatible tight Riesz order $\leqq$, then $\leqq$ necessarily satisfies the stronger TR $(2, 2)$ interpolation property. Reilly (1973) and Wirth (1973) have considered the question of the existence of directed compatible tight Riesz orders on lattice ordered groups. It is easily verified that the existence of complements in a lattice prohibits the existence of a *directed* compatible tight Riesz order. In this paper we show that every modular lattice with non-trivial complements has no compatible tight Riesz order, but that there are non-modular complemented lattices with compatible tight Riesz orders.

https://doi.org/10.1017/S1446788700018668 Published online by Cambridge University Press

Theorem 1. *There exists no compatible tight Riesz order on a modular lattice with non-trivial complemented elements.*

Proof. Let $(X, \leqq)$ be a partially ordered set with associated *partial* order $\trianglelefteq$ on X. The terms 'minimal', 'maximal' will refer to $\leqq$ and a 'maxmin' is an element which is simultaneously minimal and maximal. We note first that there can be at most one maxmin, for if x and y are maxmin, we have, by vacuous implications, $x \trianglelefteq y$ and $y \trianglelefteq x$, whence $x = y$.

Next, note that is $(X, \trianglelefteq)$ has a least element 0 then x is not minimal iff $0 < x$. It is immediate that if $0 < x$, then x is not minimal, and conversely, if x is not minimal then $y < x$ for some y, but $0 \trianglelefteq y$ so, from the definition of $\trianglelefteq$, we have $0 < x$. Likewise if $(X, \trianglelefteq)$ has greatest element 1 then x is not maximal iff $x < 1$.

If $(X, \leqq)$ satisfies the TR(1, 2) property and if $x \wedge y = 0$ for some x, y, then at least one of x, y is minimal, for otherwise $0 < x, y$ and then $0 < z < x, y$ for some z, a fortiori $z \lhd x, y$. Then $z \trianglelefteq x \wedge y$ and since $0 < z$, we have $0 < x \wedge y$, a contradiction. Likewise if $(X, \leqq)$ satisfies the TR(2, 1) property and if $x \vee y = 1$ for some x, y, then at least one of x, y is maximal. (Here $\wedge, \vee$ are meet and join respectively in $(X, \trianglelefteq)$.)

Suppose $(X, \trianglelefteq)$ is a lattice with 0 and 1, that $\leqq$ is a compatible tight Riesz order for $(X, \trianglelefteq)$, and x and y are a complementary pair: $x \wedge y = 0$, $x \vee y = 1$. If x is minimal then either $x = 0$ or x is maximal, for if x is not maximal then y is maximal so $x \trianglelefteq y$ and $x = x \wedge y = 0$. Likewise if x is maximal then either $x = 1$ or x is minimal. Suppose $x, y \in X \setminus \{0, 1\}$. At least one of x, y is the unique maxmin, say x. Since y must be different from x, y is neither minimal nor maximal, so $0 < y < 1$. By the TR(1, 1) property $0 < z < y < 1$, for some z. We show that z is necessarily a complement of x. Certainly $x \wedge z \trianglelefteq z$ and $z < y$ so $x \wedge z < y$, a fortiori $x \wedge z \lhd y$. But $x \wedge z \trianglelefteq x$ so $x \wedge z \trianglelefteq x \wedge y = 0$ and $x \wedge z = 0$. If $x \vee z$ were not maximal then $x \vee z < 1$, but $x \trianglelefteq x \vee z$, so $x < 1$, which is impossible since x is maximal. Thus $x \vee z$ is maximal so either $x \vee z = 1$ or $x \vee z$ is the unique maxmin, x. But if $x \vee z = x$ then $z \trianglelefteq x$ so $z = x \wedge z = 0$, a contradiction. We deduce that $x \vee z = 1$ and z is a complement of x. The theorem is proved by noting that $z \lhd y$ and that in a modular lattice no element can have two distinct comparable complements.

Corollary 2. *There exists no compatible tight Riesz order on a complemented modular lattice.*

Proof. If $X = \{0, 1\}$, a doubleton, there is no compatible tight Riesz order. All other cases are covered by the theorem.

Corollary 3. *There exists no compatible tight Riesz order on a Boolean lattice.*

https://doi.org/10.1017/S1446788700018668 Published online by Cambridge University Press

Finally we present an example showing that the theorem is false if the qualification 'modular' is dropped.

EXAMPLE 4. Let X consist of the interval $[0, 1]$ of real numbers together with an adjoined element α and place the partial order $\leqq$ on X where $\leqq$ is the usual total order on $[0, 1]$ but α is isolated in $(X, \leqq)$. The associated partial order is $\trianglelefteq$ which coincides with $\leqq$ on $[0, 1]$ and otherwise $0 \lhd \alpha \lhd 1$. Here $(X, \trianglelefteq)$ is a complemented non-modular lattice with compatible tight Riesz order $\leqq$.

References

Neil Cameron and J. B. Miller, 'Topology and axioms of interpolation in partially ordered spaces', *J. fur die reine u. angewandte Math.* (to appear).

R. J. Loy and J. B. Miller (1972), 'Tight Riesz Groups', *J. Austral. Math. Soc.* **13**, 224–240.

N. R. Reilly (1973), 'Compatible tight Riesz orders and prime subgroups', *Glasgow Math. J.* **14**, 146–160.

A. Wirth (1973), 'Compatible tight Riesz orders', *J. Austral. Math. Soc.* **15**, 105–11.

Department of Mathematics
Monash University
Clayton
Victoria, 3168
Australia

https://doi.org/10.1017/S1446788700018668 Published online by Cambridge University Press