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POWERS OF COMPOSITION OPERATORS: ASYMPTOTIC BEHAVIOUR ON BERGMAN, DIRICHLET AND BLOCH SPACES

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Abstract

We study the asymptotic behaviour of the powers of a composition operator on various Banach spaces of holomorphic functions on the disc, namely, standard weighted Bergman spaces (finite and infinite order), Bloch space, little Bloch space, Bloch-type space and Dirichlet space. Moreover, we give a complete characterization of those composition operators that are similar to an isometry on these various Banach spaces. We conclude by studying the asymptotic behaviour of semigroups of composition operators on these various Banach spaces.

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1. Introduction

This paper can be considered as a sequel of [3], where we initiated the study of asymptotic behaviour of C_φ^n as $n \to \infty$, where C_φ is a bounded composition operator on a Banach space X which embeds continuously in $\mathrm{Hol}(\mathbb{D})$, the Fréchet space of holomorphic functions on the open unit disc $\mathbb{D}$. We write $X \hookrightarrow \mathrm{Hol}(\mathbb{D})$ for short. An equivalent formulation is that $\delta_z \in X'$ for all $z \in \mathbb{D}$, where δ_z is the evaluation at z (cf. [3, Proposition 2.1]).

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic and $X \hookrightarrow \mathrm{Hol}(\mathbb{D})$ such that $f \circ \varphi \in X$ for all $f \in X$. Then, by the closed graph theorem, $C_\varphi f = f \circ \varphi$ defines an operator $C_\varphi \in \mathcal{L}(X)$ (the algebra of all bounded operators on X). The modes of convergence we study are the following, enumerated by decreasing generality.

(U) Uniform convergence: $\lim_{n\to\infty} C_\varphi^n$ exists in $\mathcal{L}(X)$.

(S) Strong convergence: $\lim_{n\to\infty} C_\varphi^n x$ exists in X for all $x \in X$.

(W) Weak convergence: $C_\varphi^n x$ converges weakly in X for all $x \in X$.

(E) Mean ergodicity: $\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} C_\varphi^k x$ exists in X for all $x \in X$.

Recall that by [26, Theorem 1.3, page 26] and [33, Ch. VIII, Section 3]), if X is reflexive and C_φ^n is uniformly bounded, then (E) holds automatically.

Actually, the strongest mode of convergence (uniform convergence) can be characterized in the following way.

THEOREM 1.1 [3, Theorem 3.4]. *Let X be a Banach space of holomorphic functions and $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic such that $C_\varphi \in \mathcal{L}(X)$. The following assertions are equivalent:*

(i) *$(C_\varphi^n)_n$ converges in norm;*

(ii) *$r_e(C_\varphi) < 1$,*

where r_e denotes the essential spectral radius.

In [3], we studied the convergence of the iterates C_φ^n on various Banach spaces such as the Hardy spaces $H^p(\mathbb{D})$, $1 \le p \le \infty$, the Wiener algebra and the disc algebra.

Using [10], we also deduced the uniform convergence of the iterates on weighted Hardy spaces $H^2(\gamma) \hookrightarrow \mathrm{Hol}(\mathbb{D})$ defined as follows:

$$H^2(\gamma) = \Big\{ f : \mathbb{D} \to \mathbb{C} \text{ holomorphic and } \|f\|_{H^2(\gamma)}^2 := \sum_{n\ge 0} |a_n|^2 \gamma_n^2 < \infty \Big\},$$

where $(a_n)_{n\ge0}$ is the sequence of Taylor coefficients of f and where $(\gamma_n)_{n\ge0}$ is a nonincreasing sequence of positive reals.

The Banach space $H^2(\gamma)$ contains $H^2(\mathbb{D})$ and, for every holomorphic $\varphi : \mathbb{D} \to \mathbb{D}$, the composition operator C_φ is bounded on $H^2(\gamma)$. We deduced from our result on $H^2(\mathbb{D})$ that $(C_\varphi^n)_n$ converges uniformly on $H^2(\gamma)$ for every φ with a fixed point in $\mathbb{D}$ which is not an inner function (see [3, Theorem 4.15]).

It happens that this result is far from being optimal in the particular case of the so-called standard Bergman spaces A_β^2 (with $\beta > -1$), which are equal to $H^2(\gamma)$ with

$$\gamma_n^2 = \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+2)}.$$

Indeed, in this case, we can prove that the uniform convergence is true for every φ with a fixed point in $\mathbb{D}$ which is not an automorphism, even if φ is inner.

If φ is an elliptic automorphism, then φ is conjugate to a rotation and then the asymptotic behaviour of $C_\varphi^n \in \mathcal{L}(X)$ is simple to describe. Indeed, if φ is the identity map, then C_φ^n converges, and otherwise $(C_\varphi^n)_n$ does not converge weakly as soon as e_1 is in X (where e_1 is the identity map on $\mathbb{D}$).

Thus, in the following, we assume throughout that φ is a holomorphic self map of $\mathbb{D}$ which is not the identity map and not an elliptic automorphism, so that we can consider the Denjoy–Wolff point b of φ. This is the unique point $b \in \overline{\mathbb{D}}$ such that the iterates $\varphi_n(z) := \varphi \circ \cdots \circ \varphi(z)$ (n times) converge to b as $n \to \infty$ uniformly for z in compact subsets of $\mathbb{D}$. Since $C_\varphi^n = C_{\varphi_n}$, the asymptotic behaviour of C_φ^n as $n \to \infty$ is completely different in the cases $|b| = 1$ and $b \in \mathbb{D}$.

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In the paper we prove the following results on the asymptotic behaviour of C_φ^n. Note that $C_\varphi(X) \subset X$ for $X = H^p, A_\beta^p, \mathcal{B}, H_{v_q}^\infty, \mathcal{B}^\alpha$ with $1 \le p < \infty$, $q > 0$, $\beta > -1$ and $\alpha > 1$ without any restriction on φ. For $X = \mathcal{B}_0$, $C_\varphi(X) \subset X$ if and only if $\varphi \in \mathcal{B}_0$ for $X = \mathcal{B}^\alpha$ with $0 < \alpha < 1$, $\varphi(X) \subset X$ if and only if $\tau_{\varphi,\alpha}^\infty < \infty$ (defined in Section 4) and, for $X = \mathcal{D}$, a sufficient condition to get $C_\varphi(X) \subset X$ is the injectivity of φ (a necessary and sufficient condition can be expressed in terms of Carleson measures and the Nevanlinna counting function).

THEOREM 1.2. *Assume that* $|b| = 1$. *Then* C_φ *is not mean ergodic on* $X = H^p, A_\beta^p, \mathcal{B}, H_{v_q}^\infty, \mathcal{B}^\alpha$ *with* $1 \le p < \infty$, $q > 0$, $\beta > -1$ *and* $\alpha > 1$.
Assuming that $C_\varphi(X) \subset X$ *for* $X = \mathcal{B}_0$ *and* $X = \mathcal{D}$, C_φ *is not mean ergodic on these spaces either.*

Next we consider the case when $b \in \mathbb{D}$. Define $P_b : \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$ by $P_b(f) = f(b)\mathbf{1}_{\mathbb{D}}$.

THEOREM 1.3. *Assume that* $b \in \mathbb{D}$. *If* $X = H^p, A_\beta^p, H_{v_q}^\infty, \mathcal{B}^\alpha$ *with* $1 \le p < \infty$, $q > 0$, $\beta > -1$ *and* $\alpha > 1$, *then* $(C_\varphi^n)_n$ *converges to* P_b *in the operator norm. If* $X = \mathcal{B}_0$ *and* $\varphi \in \mathcal{B}_0$, *then* $(C_\varphi^n)_n$ *converges to* P_b *in the operator norm.*

On the spaces $\mathcal{B}$, $\mathcal{B}^\alpha$ with $0 < \alpha < 1$ and $\mathcal{D}$, the situation is different. First we consider the case when X is the Bloch space.

THEOREM 1.4. *Assume that* $b \in \mathbb{D}$. *If* $X = \mathcal{B}$, *then* $(C_\varphi^n)_n$ *converges to* P_b *in the operator norm if and only if* $\tau_\varphi^\infty < 1$.

The case when $X = \mathcal{B}^\alpha$, $0 < \alpha < 1$, is very different.

THEOREM 1.5. *Assume that* $b \in \mathbb{D}$. *If* $X = \mathcal{B}^\alpha$ *with* $0 < \alpha < 1$ *and* φ *such that* $\tau_\varphi^\infty < \infty$, *then* $C_\varphi(X) \subset X$ *and the following assertions are equivalent:*

(i) $(C_\varphi^n)_n$ *converges to* P_b *in the operator norm;*
(ii) C_φ *is mean ergodic;*
(iii) *there exists* $n_0 \in \mathbb{N}$ *such that* $\varphi_{n_0}(\overline{\mathbb{D}}) \subset \mathbb{D}$.

On the Dirichlet space $\mathcal{D}$, there is an example of a univalent φ such that $b \in \mathbb{D}$ and $(C_\varphi^n)_n$ is weakly convergent but not strongly convergent. However, the following theorem states that strong convergence implies uniform convergence.

THEOREM 1.6. *Assume that* φ *is univalent with a Nevanlinna counting function essentially radial. Assume also that* $b \in \mathbb{D}$. *If* $X = \mathcal{D}$, *the following assertions are equivalent:*

(i) $(C_\varphi^n)_n$ *converges to* P_b *in the operator norm;*
(ii) $(C_\varphi^n)_n$ *converges strongly to* P_b*;*
(iii) φ *is not a full map.*

In Table 1 we summarize our results which are a complete characterization of uniform convergence (U), strong convergence (S), weak convergence (W) of the

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TABLE 1. Modes of convergence.

Spaces	U	S	W	E
$H^p, 1 \leq p < \infty$	φ not inner, $\lvert b\rvert < 1$	φ not inner, $\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$
$A^p_\beta, \beta > -1, 1 \leq p < \infty$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$
$\mathcal{B}_0, \varphi \in \mathcal{B}_0$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$
$\mathcal{B}^\alpha, \alpha > 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$
$H^\infty_{v_p}, 0 < p < \infty$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$	$\lvert b\rvert < 1$

TABLE 2. Isometry and similar to an isometry.

Spaces	C_φ isometric	C_φ similar to an isometry
$H^p, 1 \leq p < \infty$	φ inner and $\varphi(0) = 0$	φ inner and there exists $b \in \mathbb{D}$ with $\varphi(b) = b$
$A^p_\beta, \beta > -1, 1 \leq p < \infty$	φ rotation	φ elliptic automorphism
$\mathcal{B}$	$\varphi(0) = 0$ and $\tau^\infty_\varphi = 1$	there exists $b \in \mathbb{D}$ with $\varphi(b) = b$ and $\tau^\infty_\varphi = 1$
$\mathcal{B}_0$ or $\mathcal{B}^\alpha, \alpha \neq 1$	φ rotation	φ elliptic automorphism
$H^\infty_{v_p}, 0 < p < \infty$	φ rotation	φ elliptic automorphism

powers of C_φ or mean ergodicity (E) of C_φ in terms of φ. For example, line 3, column 1 says:

$$(C^n_\varphi)_{n\in\mathbb{N}} \text{ converges in } \mathcal{L}(X), X = \mathcal{B}_0 \text{ if and only if } |b| < 1.$$

In this table, we assume that φ is not an elliptic automorphism and we denote by b the Denjoy–Wolff point of φ.

For $X = \mathcal{D}$, our main result is Theorem 1.6 and, for $X = \mathcal{B}^\alpha$ with $0 < \alpha < 1$, our main contribution is presented in Theorem 1.5. The case when $|b| = 1$ is not fully understood.

For $X = \mathcal{B}$, uniform and strong convergence are fully understood; moreover, mean ergodicity requires that $|b| < 1$.

As a consequence of our study of the powers of C_φ, we also get interesting results related to isometries or similarity to isometries on various Banach spaces which are not necessarily hilbertian (on a Hilbert space, Nagy's criterion for similarity to isometries can be used as in [4]). In Table 2 we summarize our results which give a complete characterization of C_φ to be isometric or similar to an isometry in terms of φ. For this problem, φ is a holomorphic self map of $\mathbb{D}$ that may be an elliptic automorphism. In the case of the Dirichlet space, we characterize when C_φ is similar to an isometry only in a special case (see Corollary 6.8).

The results in the first line of both tables refer to the previous paper [3] and the definition of τ^∞_φ is in Section 3.

In Section 7, we produce an example of a Banach space which is the quotient space of $\mathcal{D}$ modulo the constant functions. In this case, we obtain an example of a composition operator on which the strong convergence of the iterates is satisfied but the uniform convergence is false (see Theorem 7.1). Nevertheless, we have slightly

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changed our framework since this Banach space does not embed continuously in $\mathrm{Hol}(\mathbb{D})$.

We conclude this paper with the study of the asymptotic behaviour as $t \to \infty$ of semigroups of composition operators on the Banach spaces we had considered in the previous seven sections.

The results of this paper rely on previous works of many authors. For example, we are in the comfortable position that much is known on the essential spectral radius of C_φ on some of these spaces we consider. So, in some parts we just have to put together known results or, sometimes, proofs can be imitated. This is the case, for example, for the comparison of r_{e,A^p} and r_{e,A^2} for Bergman spaces due to MacCluer and Saxe [17]. We extend their result to weighted Bergman spaces.

Frequently more original proofs are needed. For example, our characterization of those C_φ on the little Bloch space $\mathcal{B}_0$ which are similar to an isometry is new. Another example is the characterization of $r_e(C_\varphi) < 1$ on $\mathcal{B}^\alpha$, $\alpha > 1$. It uses the computation of the essential norm [25] due to Montes-Rodríguez but some additional substantial arguments are needed. We prove that mean ergodicity on most of our spaces (for $X = \mathcal{B}^\alpha$, $0 < \alpha < 1$, the question remains open) forces the Denjoy–Wolff point to be in $\mathbb{D}$, assuming that φ is not an elliptic automorphism. For this latter case, we refer to [5] by Beltrán-Meneu *et al.*, where a systematic study of ergodicity on different spaces is given. Many other detailed references to known results are given throughout the paper.

2. Standard weighted Bergman spaces

We write dA for the normalized Lebesgue area measure on $\mathbb{D}$, that is, $dA(re^{i\theta}) = (1/\pi) r\,dr\,d\theta$. The *standard weighted Bergman space*, $A^p_\beta(\mathbb{D})$ (A^p_β, in short), $\beta \geq -1$, $p \geq 1$, is the space of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\beta \, dA(z) < \infty.$$

Every A^p_β is a Banach space when $1 \leq p < \infty$ with norm the pth root of the above integral, denoted by $\|f\|_{A^p_\beta}$.

The *unweighted Bergman space*, A^p, is obtained when $\beta = 0$.

The *standard Hardy space* $H^p(\mathbb{D})$ is obtained when $\beta = -1$.

Given a holomorphic self map φ of $\mathbb{D}$, the composition operator C_φ is always bounded on A^p_β space.

Theorem 2.1 [36, Theorem 11.6]. *Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and $p \geq 1$, $\beta \geq -1$. Then*

$$\frac{1}{(1 - |\varphi(0)|^2)^{(\beta+2)/p}} \leq \|C_\varphi\|_{\mathcal{L}(A^p_\beta)} \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{(\beta+2)/p}.$$

Remark 2.2. Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and $p \geq 1$, $\beta > -1$. Then C_φ is power bounded on A^p_β if and only if φ has a fixed point in $\mathbb{D}$ and, hence, C_φ is not even weakly convergent if φ has a fixed point in $\partial\mathbb{D}$.

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We are going to show now that on A^p_β, if the Denjoy–Wolff point is of modulus one, then C_φ is not even mean ergodic. To that aim we establish the following criterion.

Lemma 2.3. *Let $X \hookrightarrow \mathrm{Hol}(\mathbb{D})$ be a Banach space such that $\mathbf{1}_\mathbb{D} \in X$ and let φ be a holomorphic self map of $\mathbb{D}$ such that $C_\varphi(X) \subset X$. Suppose also that φ is not an elliptic automorphism and that 1 is the Denjoy–Wolff point of φ. Assume that there exist $r_0 \in (0,1)$ and $f \in X$ such that $f_r(z) := f(rz)$ defines functions $f_r \in X$, $r_0 \le r < 1$, satisfying:*

(a) $\lim_{r\uparrow 1} |f_r(1)| = \infty$;

(b) $\limsup_{r\uparrow 1} \|f_r\|_X < \infty$.

Then the operator C_φ is not mean ergodic on X.

Proof. Assume that $\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} C^k_\varphi$ strongly converges to P in $\mathcal{L}(X)$. For $r_0 \le r < 1$,

$$\lim_{n\to\infty} (C^n_\varphi f_r)(z) = \lim_{n\to\infty} f_r(\varphi_n(z)) = f_r(1)$$

for all $z \in \mathbb{D}$. Thus, $P f_r = f_r(1)\mathbf{1}_\mathbb{D}$ for all $r_0 \le r < 1$. It follows that

$$|f_r(1)|\|\mathbf{1}_\mathbb{D}\|_X = \|P f_r\|_X \le \|P\|\|f_r\|_X,$$

which contradicts (a) and (b). □

In Lemma 2.3, it does not suffice to have (a) alone. We show this by an example.

Example 2.4. There exists a Banach space $X \hookrightarrow \mathrm{Hol}(\mathbb{D})$ such that $\lim_{r\uparrow 1} |f_r(1)| = \infty$ for all $f \in X \backslash \{0\}$ and there exists $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic with $\lim_{z\to 1} \varphi(z) = 1$ such that $C_\varphi(X) \subset X$. But C_φ is mean ergodic on X.

Proof. Denote by $\mathbb{P}$ the upper half plane and define $\psi : \mathbb{P} \to \mathbb{P}$ by $\psi(w) = w + i$. Denote by $C : \mathbb{P} \to \mathbb{D}$ the Cayley transform, that is, $C(w) = (w - i)/(w + i)$ and $C^{-1}(z) = i(1 + z)/(1 - z)$. Let $\varphi = C \circ \psi \circ C^{-1}$. Then

$$\lim_{z\to 1, z\in\mathbb{D}} \varphi(z) = \lim_{w\to\infty, w\in\mathbb{P}} C \circ \psi(w) = C(\infty) = 1.$$

Note that $e^\psi(w) = e^i e^w$, $w \in \mathbb{P}$. Let $g := e^{C^{-1}} \in \mathrm{Hol}(\mathbb{D})$ and $X := \mathbb{C}g$. Since $g \circ \varphi = e^i g$, it follows that $C_\varphi f = e^i f$ for all $f \in X$ and then $C_\varphi(X) \subset X$ and $X \hookrightarrow \mathrm{Hol}(\mathbb{D})$. Moreover, since $C^n_\varphi f = e^{in} f$ for all $n \in \mathbb{N}$ and $f \in X$,

$$\frac{1}{n} \sum_{k=0}^{n-1} C^k_\varphi f \to 0 \quad \text{as } n \to \infty$$

for all $f \in X$. However, Condition (a) is satisfied since

$$\lim_{z\to 1, z\in\mathbb{D}} |g(z)| = \lim_{w\to\infty, w\in\mathbb{P}} |e^w| = \infty.$$ □

Proposition 2.5. *Let φ be a holomorphic self map of $\mathbb{D}$ whose Denjoy–Wolff point b is of modulus one. Then C_φ is not mean ergodic on A^p_β for $1 \le p < \infty$ and $\beta > -1$.*

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Proof. Since A^p_β is invariant by rotation, we may assume that $b = 1$. Let $\gamma > 0$ and $F(z) := (\log 2/(1-z))^\gamma$, where log is the principal branch of the logarithm. By [37, Ch. V, Theorem 2.31], the Taylor coefficients a_n of F are equivalent to $1/n(\log n)^{\gamma-1}$. It follows that $F \in \mathcal{D} \subset H^2(\mathbb{D})$ if and only if $\gamma < 1/2$. Since $\|F\|^p_{H^p(\mathbb{D})} = \|F^{p/2}\|^2_{H^2(\mathbb{D})}$, the function $G := F^{p/2}$ is in $H^p(\mathbb{D}) \subset A^p_\beta$ (for $\beta \geq -1$) as soon as $\gamma p/2 < 1/2$. Since $\|G_r\|_{A^p_\beta} \leq \|G\|_{A^p_\beta}$ and since $\lim_{r\uparrow 1} |G_r(1)| = \infty$, the conclusion follows from Lemma 2.3. □

In [3], we have already used the fact that the only isometric composition operators on any Hardy space $H^p(\mathbb{D}) = A^p_{-1}$, $1 \leq p < \infty$, are induced by inner functions φ such that $\varphi(0) = 0$ [12, Theorem 3.8].

The analogous result is really different for $\beta > -1$. It is shown in [21] that the only isometric composition operators on any weighted Bergman space are the trivial ones, that is, those whose symbol is a rotation.

Theorem 2.6 [21, Theorem 1.3(b)]. *Let $1 \leq p < \infty$, $-1 < \beta < \infty$. A composition operator C_φ is an isometry of A^p_β if and only if φ is a rotation.*

In [7], Bourdon and Shapiro proved that

$$(r_{e,H^p}(C_\varphi))^p \leq (r_{e,H^2}(C_\varphi))^2$$

for $1 \leq p < \infty$, where $r_{e,H^p}(C_\varphi)$ denotes the essential spectral radius of C_φ on H^p. They also proved that the above inequality is an equality when $p > 1$. In [17], MacCluer and Saxe proved the analogous result on Bergman space, that is,

$$(r_{e,A^p}(C_\varphi))^p \leq (r_{e,A^2}(C_\varphi))^2$$

for $1 \leq p < \infty$, with equality for $p > 1$.

This result is relevant for our purpose since it tells us that the essential spectral radius of C_φ will be strictly less than 1 on A^p or H^p as soon as it is the case on H^2 or A^2.

We establish the above formula for A^p_β, $\beta > -1$, $1 \leq p < \infty$ and then by means of Theorem 1.1 we get the uniform convergence of the iterates of C_φ.

By Theorem 2.6, we already know that the possible φ for which we may hope the uniform convergence of the iterates of C_φ on A^p_β $(\beta > -1)$ are all holomorphic self maps on the unit disc except rotations. Surprisingly, rotations are indeed the only symbols to avoid, thanks to the following result.

Theorem 2.7 [8, Proposition 2.1]. *Let φ be a holomorphic self map of $\mathbb{D}$ which is 0 at 0. If φ is not a rotation, then the essential norm of C_φ on A^2_β, with $\beta > -1$, is strictly less than 1.*

Theorem 2.8. *Suppose that φ is a holomorphic self map of $\mathbb{D}$. Then, for each $1 \leq p < \infty$ and $\beta \geq -1$,*

$$(r_{e,A^p_\beta}(C_\varphi))^p \leq (r_{e,A^2_\beta}(C_\varphi))^2.$$

Moreover, if $p > 1$, the above inequality is an equality.

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Proof. For $\beta = -1$, the result is due to Bourdon and Shapiro [7] and for $\beta = 0$ the result is proved in [17]. The proof for $\beta > -1$ can be done using technical variations of arguments given in [17]. We will just indicate the relevant changes and provide useful references for the material involved.

The elements of the proof in [17] that require a modification rely on:

- estimates of the norm and essential norm in the Bergman spaces involving the generalized Nevanlinna counting function;
- estimates of the norm of the linear functionals of evaluation at $w \in \mathbb{D}$, or evaluation of the first derivative at $w \in \mathbb{D}$ on the Banach space $A^p (:= A^p_0)$ and on the invariant subspace

$$z^m A^p := \{g \in A^p : g \text{ has a zero of at least order m at } 0\}.$$

We will now give the estimates in the standard weighted Bergman spaces in order to reconstruct the proof following exactly the strategy initiated by Bourdon and Shapiro and adapted by Saxe and MacCluer.

Formula (1.2) in [17] should be replaced by

$$\|f \circ \varphi\|^p_{A^p_\beta} \simeq |f(\varphi(0))|^p + \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 N_{\varphi,\beta+2}(z)\, dA(z),$$

which is [29, Proposition 2.4]. The symbol '$\simeq$' means that the left-hand side is bounded below and above by positive constant multiples of the right-hand side and the constants do not depend on f. Recall that the generalized Nevanlinna counting function N is defined as

$$N_{\varphi,\gamma}(w) = \sum_{z \in \varphi^{-1}\{w\}} \left(\log \frac{1}{|z|}\right)^\gamma, \quad \gamma > 0, w \in \mathbb{D} \backslash \{\varphi(0)\}.$$

The sum is taken over the preimages of w, counting multiplicities, and, when $w \notin \varphi(\mathbb{D})$, $N_{\varphi,\gamma}(w)$ is defined to be 0.

Formula (1.5) in [17] should be replaced by the following estimate proved in [28, Theorem 6.8]:

$$c(\beta)\sigma_{\beta+2}(\varphi) \le \|C_\varphi\|^2_{e,\mathcal{L}(A^2_\beta)} \le \sigma_{\beta+2}(\varphi),$$

where $\sigma_\gamma(\varphi) = \limsup_{|w|\to 1^-} (N_{\varphi,\gamma}(w)/(-\log |w|)^\gamma)$ and where $c(\beta)$ is a positive constant which depends only on β.

The assertions (a) and (b) of [17, Proposition 1] should be replaced by ($1 \le p < \infty$):

(a) for $w \in \mathbb{D}$,

$$\|\delta_w\|_{(A^p_\beta)'} = \frac{1}{(1 - |w|^2)^{(\beta+2)/p}};$$

(b) there exists a constant $c_{\beta,p}$, depending on β and p, so that, for any $w \in \mathbb{D}$ and $f \in A^p_\beta$,

$$|f'(w)| \le c_{\beta,p}(1 - |w|)^{-((p+\beta+2)/p)} \|f\|_{A^p_\beta}.$$

The assertion (a) is proved in [30, page 755] and (b) follows from (a) using the Cauchy formula.

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Proposition 2 in [17], when $1 \le p < \infty$ and $m \ge 1$, should be replaced by the following. There is a constant, $c_{\beta,p}$, depending only on β and p, so that if $f \in z^m A^p_\beta$ and $w \in \mathbb{D}$, then

$$|f(w)| \le c_{\beta,p} \frac{m^{\beta+1}}{(1-|w|^2)^{(\beta+2)/p}} |w|^m \|f\|_{A^p_\beta}.$$

This inequality follows along the same lines as the proof of [17, Proposition 2], thanks to the explicit expression of the reproducing kernel in A^2_β denoted by k^β_w (see, for example, [36, Corollary 4.20]), which is equal to

$$k^\beta_w(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+2)}{n!\Gamma(\beta+2)} z^n \overline{w}^n = \frac{1}{(1-z\overline{w})^{\beta+2}}.$$

Finally, the last modification is the assertion of Proposition 4 in [17], which should be replaced by: for $1 \le p < \infty$, $f \in z^m A^2_\beta$ and $w \in \mathbb{D}$,

$$|f'(w)| \le |w|^{m-1} 2^{(\beta+1)/2} m^{(\beta+3)/2} \sqrt{\Gamma(\beta+4)} \frac{1}{(1-|w|^2)^{(\beta+4)/2}} \|f\|_{A^2_{m,\beta}}. \qquad \square$$

Assume that φ is not the identity map to avoid a trivial statement.

THEOREM 2.9. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic with a fixed point b in $\mathbb{D}$. The following assertions are equivalent on A^p_β, $1 \le p < \infty, \beta > -1$:*

(i) *φ is not an automorphism;*
(ii) *C^n_φ converges uniformly to P_b, $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$ for all $f \in A^p_\beta$.*

PROOF. (i) $\Rightarrow$ (ii): We may assume that $b = 0$, considering $C_{\psi_b} \circ C_\varphi \circ C_{\psi_b} := C_\psi$, where $\psi_b(z) = (b-z)/(1-\overline{b}z) = \psi_b^{-1}(z)$ and $\psi := \psi_b \circ \varphi \circ \psi_b$, so that $\psi(0) = 0$. Then, by Theorem 2.7, in particular, the essential spectral radius of C_φ is strictly less than 1 on A^2_β and thus on A^p_β by Theorem 2.8. The conclusion follows from Theorem 1.1.

(ii) $\Rightarrow$ (i): If φ is an elliptic automorphism, then C_φ is similar to a composition by a rotation and therefore the sequence of iterates does not even weakly converge. $\square$

Now we can also characterize those composition operators that are similar to an isometry.

COROLLARY 2.10. *Let φ be a holomorphic self map of $\mathbb{D}$. Consider the composition operator C_φ on A^p_β ($1 \le p < \infty$, $\beta > -1$). The following assertions are equivalent:*

(i) *C_φ is similar to an isometry;*
(ii) *φ is an elliptic automorphism.*

PROOF. (i) $\Rightarrow$ (ii): Assume that there exist an isometry U on A^p_β and an invertible operator S such that

$$C_\varphi = S^{-1} U S.$$

Since $C^n_\varphi = S^{-1} U^n S$, it follows that C_φ is power bounded and, thus, by Remark 2.2, φ has a fixed point $b \in \mathbb{D}$. If φ is not an automorphism, by Theorem 2.9, $C^n_\varphi \to P_b$ as

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$n \to \infty$ and then $U^n \to Q := SP_bS^{-1}$ as $n \to \infty$. Since U^n is an isometry, also Q is an isometry. This is impossible since Q is a projection of rank one.

(ii) $\Rightarrow$ (i): If φ is an automorphism fixing a $b \in \mathbb{D}$, then $\widetilde{\varphi} := \psi_b \circ \varphi \circ \psi_b$ is a rotation. Then $C_{\widetilde{\varphi}}$ is an isometry and $C_\varphi = S^{-1}C_{\widetilde{\varphi}}S$, where $S = C_{\psi_b} = S^{-1}$. □

3. Bloch space $\mathcal{B}$ and little Bloch space $\mathcal{B}_0$

A function $f \in \mathrm{Hol}(\mathbb{D})$ is said to be a *Bloch function* if it satisfies

$$\sup_{z\in\mathbb{D}}(|1 - |z|^2)|f'(z)| < \infty.$$

The set of all Bloch functions is called the *Bloch space* $\mathcal{B}$, which becomes a Banach space under the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(|1 - |z|^2)|f'(z)|.$$

The little Bloch space $\mathcal{B}_0$ is the set of holomorphic functions f on $\mathbb{D}$ such that

$$\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0.$$

Actually $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$.

Let φ be a holomorphic self map of $\mathbb{D}$ and $f \in \mathcal{B}$. Then, for $z \in \mathbb{D}$,

$$(1 - |z|^2)|(f \circ \varphi)'(z)| = \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2}(1 - |\varphi(z)|^2)|f'(\varphi(z))). \tag{3.1}$$

Using the Schwarz–Pick lemma [12, Theorem 2.39],

$$\|f \circ \varphi\|_{\mathcal{B}} \le |f(\varphi(0))| + \|f\|_{\mathcal{B}}$$

and, thus, for every holomorphic self map φ of $\mathbb{D}$, the operator C_φ is bounded on $\mathcal{B}$.

In contrast, composition operators on $\mathcal{B}_0$ are not always bounded.

Lemma 3.1 [19, page 2680]. *Let φ be a holomorphic self map of $\mathbb{D}$. Then C_φ is bounded on $\mathcal{B}_0$ if and only if $\varphi \in \mathcal{B}_0$.*

In order to present the upper and lower bounds for the norm due to Xiong [32], we first define some quantities associated with a symbol φ.

Let

$$\tau_\varphi(z) := \frac{1 - |z|^2}{1 - |\varphi(z)|^2}|\varphi'(z)| \quad \text{and} \quad \tau_\varphi^\infty := \sup_{z\in\mathbb{D}} \tau_\varphi(z).$$

By the Schwarz–Pick lemma, $\tau_\varphi^\infty \le 1$.

Theorem 3.2 [32, Theorem 1, Corollary 1 and Theorem 2]. *Let φ be a holomorphic mapping of $\mathbb{D}$ into itself. Then the following holds:*

$$\max\Big\{1, \frac{1}{2}\log\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\Big\} \le \|C_\varphi\|_{\mathcal{L}(\mathcal{B})} \le \max\Big\{1, \frac{1}{2}\log\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} + \tau_\varphi^\infty\Big\}.$$

In particular, if $\varphi(0) = 0$, then $\|C_\varphi\|_{\mathcal{L}(\mathcal{B})} = 1$.

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The following lemma shows that $\mathcal{B}$ and $\mathcal{B}_0$ fit in the class of Banach spaces of holomorphic functions (that is, $\mathcal{B} \hookrightarrow \mathrm{Hol}(\mathbb{D})$ and $\mathcal{B}_0 \hookrightarrow \mathrm{Hol}(\mathbb{D})$).

Lemma 3.3 [32, Lemma 1]. *If $f \in \mathcal{B}$, then*

$$|f(z)| \leq |f(0)| + \frac{\|f\|_{\mathcal{B}} - |f(0)|}{2} \log \frac{1+|z|}{1-|z|}.$$

The exact formula giving the essential norm of a composition operator on $\mathcal{B}$ and $\mathcal{B}_0$ was obtained in [24].

Theorem 3.4. *Suppose that φ is a holomorphic self map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{e,\mathcal{L}(\mathcal{B})} = \inf_{0<s<1} \sup_{|\varphi(z)|>s} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|.$$

Moreover, if $\varphi \in \mathcal{B}_0$, then

$$\|C_\varphi\|_{e,\mathcal{L}(\mathcal{B}_0)} = \inf_{0<s<1} \sup_{|\varphi(z)|>s} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = \limsup_{|z|\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|.$$

An obvious corollary of Theorem 3.4 is the following inequality.

Corollary 3.5. *Suppose that φ is a holomorphic self map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{e,\mathcal{L}(\mathcal{B})} \leq \tau_\varphi^\infty \leq 1.$$

Moreover, if $\varphi \in \mathcal{B}_0$, then

$$\|C_\varphi\|_{e,\mathcal{L}(\mathcal{B}_0)} \leq \tau_\varphi^\infty \leq 1.$$

This link between the essential norm and τ_φ^∞ is of particular interest because of the complete characterization of isometric composition operators on $\mathcal{B}$ in terms of τ_φ^∞.

Theorem 3.6 [1, 23]. *Suppose that φ is a holomorphic self map of $\mathbb{D}$. Then the operator C_φ on $\mathcal{B}$ is isometric if and only if $\varphi(0) = 0$ and one of the following equivalent conditions holds.*

(i) $\tau_\varphi^\infty = 1$.

(ii) *φ either is a rotation of $\mathbb{D}$ or satisfies (M): for every $w \in \mathbb{D}$, there exists $(a_n) \subset \mathbb{D}$ such that $|a_n| \to 1$, $\varphi(a_n) \to w$ and $\tau_\varphi(a_n) \to 1$ as $n \to \infty$.*

(iii) *φ either is a rotation of $\mathbb{D}$ or the zeros of φ form an infinite sequence $(z_k)_k$ in $\mathbb{D}$ such that* $\limsup_{k\to\infty}(1-|z_k|^2)|\varphi'(z_k)| = 1$.

(iv) *φ either is a rotation of $\mathbb{D}$ or $\varphi = gB$, where g is a nonvanishing holomorphic function mapping $\mathbb{D}$ into itself and B is an infinite Blaschke product whose zero set Z contains a sequence $(z_k)_k$ such that $|g(z_k)| \to 1$ when $k \to \infty$ and*

$$\lim_{k\to\infty} \prod_{\xi \in Z, \xi \neq z_k} \left| \frac{z_k - \xi}{1 - \bar{\xi} z_k} \right| = 1.$$

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Easily constructible examples which are not rotations are *thin Blaschke products*, that is, Blaschke products whose set of zeros $(z_k)_k$ satisfies

$$\lim_{k\to\infty}\prod_{n\neq k}\left|\frac{z_k - z_n}{1-\overline{z_n}z_k}\right| = 1.$$

We have now all the material to prove the convergence theorem for the Bloch space.

Theorem 3.7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic with a fixed point b in $\mathbb{D}$. The following assertions are equivalent on $\mathcal{B}$:*

(i) $\tau_\varphi^\infty < 1$;
(ii) C_φ^n *converges uniformly to P_b as $n \to \infty$, where $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$;*
(iii) C_φ^n *converges strongly.*

Proof. (i) ⇒ (ii): Since $\|C_\varphi\|_{e,\mathcal{L}(\mathcal{B})} \le \tau_\varphi^\infty$, (i) implies in particular that $r_e(C_\varphi) < 1$. Then (ii) follows from Theorem 1.1.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i): Let P be the strong limit of C_φ^n as $n \to \infty$. Then, for $f \in \mathcal{B}$, $(Pf)(z) = \lim_{n\to\infty} f(\varphi_n(z)) = f(b)$ for all $z \in \mathbb{D}$. This implies that $P = P_b$. Let $\psi = \psi_b \circ \varphi \circ \psi_b$, where $\psi_b(z) = (b-z)/(1-\bar{b}z)$. Note that $\psi(0) = 0$. Suppose that $\tau_\psi^\infty = 1$. Then, by Theorem 3.6, C_ψ is an isometry. Then $C_\psi^n = C_{\psi_b} C_\varphi^n C_{\psi_b}$ converges strongly to $C_{\psi_b} P C_{\psi_b} =: Q$. Thus, $Q^2 = Q$ is isometric. Hence, $Q = \mathrm{Id}$, which implies that $P = \mathrm{Id}$. Therefore, we have $\tau_\psi^\infty < 1$. It remains to check that $\tau_\varphi^\infty < 1$. Since $\varphi = \psi_b \circ \psi \circ \psi_b$,

$$\varphi'(z) = \psi_b'(z)\psi'(\psi_b(z))\psi_b'(\psi\circ\psi_b(z)).$$

It follows that

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = \frac{1-|z|^2}{1-|\psi_b\circ\psi\circ\psi_b(z)|^2}|\psi_b'(z)||\psi'(\psi_b(z))||\psi_b'(\psi\circ\psi_b(z))|,$$

which is equal to the product of three terms, say A, B and C, with

$$A = \frac{1-|z|^2}{1-|\psi_b(z)|^2}|\psi_b'(z)|, \quad B = \frac{1-|\psi_b(z)|^2}{1-|\psi(\psi_b(z))|^2}|\psi'(\psi_b(z))|$$

and

$$C = \frac{1-|\psi(\psi_b(z))|^2}{1-|\psi_b(\psi(\psi_b(z)))|^2}|\psi_b'(\psi(\psi_b(z)))|.$$

The Schwarz–Pick lemma applied to ψ_b asserts that $A \le 1$ and $C \le 1$. It follows that $\tau_\varphi^\infty \le \tau_\psi^\infty < 1$, which concludes the proof. □

Remark 3.8. When φ has no fixed point in $\mathbb{D}$, its Denjoy–Wolff point is on the unit circle and, therefore, by Theorem 3.2, $\lim_{n\to\infty}\|C_\varphi^n\| = \infty$ since

$$\|C_\varphi^n\|_{\mathcal{L}(\mathcal{B})} \ge \max\left\{1, \frac{1}{2}\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|}\right\}$$

with $|\varphi_n(0)| \to 1$. In this case, even the weak convergence of the iterates is impossible. We will see below that C_φ is not even mean ergodic (Proposition 3.12).

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Now we obtain on the Bloch space the following characterization of similarity to an isometry.

Corollary 3.9. *Let φ be a holomorphic self map of $\mathbb{D}$. Consider the composition operator C_φ on $\mathcal{B}$. The following assertions are equivalent:*

(i) *C_φ is similar to an isometry;*
(ii) *φ has a fixed point $b \in \mathbb{D}$ and $\tau_\varphi^\infty = 1$.*

Proof. (i) $\Rightarrow$ (ii): φ has a fixed point $b \in \mathbb{D}$ by Remark 3.8. Since P_b is not isometric, C_φ^n cannot even converge strongly to P_b as $n \to \infty$. Thus, Theorem 3.7 implies that $\tau_\varphi^\infty = 1$.

(ii) $\Rightarrow$ (i): Let $\widetilde{\varphi} := \psi_b \circ \varphi \circ \psi_b$. Then we have $\widetilde{\varphi}(0) = 0$. If $\tau_{\widetilde{\varphi}}^\infty < 1$, by Theorem 3.7, $C_{\widetilde{\varphi}}^n \to P_0$ as $n \to \infty$ and thus $C_\varphi^n \to C_{\psi_b} \circ P_0 \circ C_{\psi_b} = P_b$ as $n \to \infty$. This is absurd since C_φ is similar to an isometry. Therefore, $\tau_{\widetilde{\varphi}}^\infty = 1$ and, then, by Theorem 3.7, $C_{\widetilde{\varphi}}$ is isometric. □

The convergence result of the iterates of C_φ on the little Bloch space is different from the case of the Bloch space since there are many fewer isometric composition operators on $\mathcal{B}_0$.

Theorem 3.10. *Let φ be a holomorphic self map of $\mathbb{D}$ such that $\varphi \in \mathcal{B}_0$. The following assertions are equivalent:*

(i) *φ is a rotation;*
(ii) *C_φ is isometric;*
(iii) *$\varphi(0) = 0$ and $\tau_\varphi^\infty = 1$.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii): First we show that $\varphi(0) = 0$. Let $a \in \mathbb{D}$ be such that $\varphi(0) = a$. The automorphism ψ_a is in $\mathcal{B}_0$ and $\|\psi_a\|_{\mathcal{B}} = |a| + 1$. Moreover, by (3.1) and since $\psi_a(\varphi(0)) = 0$, it follows that $|a| + 1 \le 1$ and then $a = 0$.

Now assume that $\tau_\varphi^\infty < 1$. Then $\|C_\varphi\|_{e,\mathcal{B}_0} < 1$ by Corollary 3.5. Hence, C_φ converges to P in $\mathcal{L}(\mathcal{B}_0)$ by Theorem 1.1. Since $\varphi(0) = 0$, it follows that $P = P_0$, which is a contradiction since P_0 is not isometric.

(iii) $\Rightarrow$ (i): If φ is not a rotation, it follows from Theorem 3.6 that the zeros of φ form an infinite sequence $(z_k)_k$ such that

$$\limsup_{k\to\infty} (1 - |z_k|^2)|\varphi'(z_k)| = 1.$$

This is impossible since $\varphi \in \mathcal{B}_0$. □

We can now easily prove a convergence theorem on $\mathcal{B}_0$.

Theorem 3.11. *Let φ be a holomorphic self map of $\mathbb{D}$ such that $\varphi \in \mathcal{B}_0$ and such that φ fixes a point $b \in \mathbb{D}$. The following assertions are equivalent:*

(i) *φ is not an automorphism;*
(ii) *C_φ^n converges uniformly to P_b as $n \to \infty$, where $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$;*
(iii) *C_φ^n weakly converges to P_b as $n \to \infty$, where $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$.*

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Proof. Since $\mathcal{B}_0$ is Möbius invariant, we may assume that $b = 0$.

(i) ⇒ (ii) It follows from Theorem 3.10 that $\tau_\varphi^\infty < 1$. Now Corollary 3.5 implies that $\|C_\varphi\|_{e,\mathcal{L}(\mathcal{B}_0)} < 1$. Then (ii) follows from Theorem 1.1.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (i): Since P_b is a projection different from the identity, C_φ cannot be isometric. Thus, φ is not a rotation by Theorem 3.10. Since $\varphi(0) = 0$, this is the same as (i). □

If the Denjoy–Wolff point b is of modulus one, then we show that C_φ is not mean ergodic.

Proposition 3.12. *Let φ be a holomorphic self map of $\mathbb{D}$ which is not an elliptic automorphism and whose Denjoy–Wolff point b is of modulus one. Consider C_φ on $\mathcal{B}$ or on $\mathcal{B}_0$ assuming that $\varphi \in \mathcal{B}_0$. Then C_φ is not mean ergodic.*

Proof. Since $\mathcal{B}$ and $\mathcal{B}_0$ are invariant by rotation, we may assume that $b = 1$. Let $f(z) = \log(\log 2/(1-z))$. Since $2/(1-z) \in \mathbb{P}_1 := \{z \in \mathbb{C} : Re(z) > 1\}$ when $z \in \mathbb{D}$, the function f is well defined (choosing the principal branch of the logarithm). Moreover, since $f'(z) = 1/(1-z)\log(2/(1-z))$,

$$(1-|z|^2)|f'(z)| \le (1+|z|)\left|\frac{1}{\log(2/(1-z))}\right| \to 0 \quad \text{as } |z| \to 1.$$

In other words, $f \in \mathcal{B}_0 \subset \mathcal{B}$. Since $\|f_r\|_{\mathcal{B}} = \|f_r\|_{\mathcal{B}_0} \le \|f\|_{\mathcal{B}_0} = \|f\|_{\mathcal{B}}$ and $\lim_{r\uparrow 1} |f_r(1)| = \infty$, the conclusion follows from Lemma 2.3. □

Also on $\mathcal{B}_0$ we obtain a characterization of those composition operators which are similar to an isometry.

Corollary 3.13. *Let φ be a holomorphic self map of $\mathbb{D}$ such that $\varphi \in \mathcal{B}_0$. Consider the composition operator C_φ on $\mathcal{B}_0$. The following assertions are equivalent:*

(i) *C_φ is similar to an isometry;*

(ii) *φ is an elliptic automorphism.*

Proof. This equivalence follows from Remark 3.8, Theorems 3.10 and 3.11, using the arguments given in the proof of Corollary 3.9. □

4. Bloch-type space $\mathcal{B}^\alpha$, $\alpha > 0$

For $\alpha > 0$, the *Bloch-type space* $\mathcal{B}^\alpha$ is the space of all functions f in $\mathrm{Hol}(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty.$$

Each $\mathcal{B}^\alpha$ is a Banach space with a norm given by

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha.$$

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For $\alpha > 0$ and φ a holomorphic self map of $\mathbb{D}$, let

$$\tau_{\varphi,\alpha}(z) := \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}, \qquad \tau_{\varphi,\alpha}^\infty := \sup_{z\in\mathbb{D}} \tau_{\varphi,\alpha}(z).$$

Then we have the following characterization of boundedness of a composition operator.

Theorem 4.1 [18, 19, 31]. *For $\alpha > 0$ and φ a holomorphic self map of $\mathbb{D}$, the composition operator C_φ is a bounded operator on $\mathcal{B}^\alpha$ if and only if*

$$\tau_{\varphi,\alpha}^\infty < \infty.$$

Indeed, there exist positive constants k_α and K_α depending only on α such that

$$k_\alpha \tau_{\varphi,\alpha}^\infty \le \|C_\varphi\|_{\mathcal{L}(\mathcal{B}^\alpha)} \le K_\alpha \tau_{\varphi,\alpha}^\infty.$$

We recall the following result from [25] on the essential norm of C_φ on $\mathcal{B}^\alpha$, $\alpha > 0$, which is a generalization of Theorem 3.4.

Theorem 4.2. *Suppose that φ is a holomorphic self map of $\mathbb{D}$. If C_φ defines a bounded operator on $\mathcal{B}^\alpha$, $\alpha > 0$, then*

$$\|C_\varphi\|_{e,\mathcal{B}^\alpha} = \lim_{s\to 1-} \sup_{|\varphi(z)|>s} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\alpha |\varphi'(z)|.$$

Note that we have in particular that

$$\|C_\varphi\|_{e,\mathcal{L}(\mathcal{B}^\alpha)} \le \tau_{\varphi,\alpha}^\infty.$$

The following result by Zorboska describes all the isometric composition operators on $\mathcal{B}^\alpha$, $\alpha \neq 1$. Surprisingly, the situation here is very different from the case of the classical Bloch space ($\alpha = 1$). It can be compared to the fact that on the classical Hardy spaces and the weighted Bergman spaces the classes of symbols defining isometric composition operators are different.

Theorem 4.3 [34]. *Let $\alpha > 0$, $\alpha \neq 1$ and let φ be a holomorphic self map of $\mathbb{D}$. Then the composition operator C_φ is an isometry on $\mathcal{B}^\alpha$ if and only if φ is a rotation.*

Very often, in order to use the Schwarz lemma, one considers a self map φ of $\mathbb{D}$ such that $\varphi(0) = 0$. In order to deal with the general case, the following lemma will be very useful.

Lemma 4.4 [35, Section 3]. *For $a \in \mathbb{D}$, $a \neq 0$ and $\alpha > 0$,*

$$\tau_{\psi_a,\alpha}^\infty \le \left(\frac{1+|a|}{1-|a|}\right)^{|1-\alpha|},$$

where ψ_a is the automorphism (equal to its inverse) defined by $\psi_a(z) = (a-z)/(1-\overline{a}z)$. Thus, C_{ψ_a} defines an isomorphism on $\mathcal{B}^\alpha$.

The following lemma shows that, for $\alpha > 0$ and for all $z \in \mathbb{D}$, the evaluation functionals δ_z are bounded on $\mathcal{B}^\alpha$. See [16, Lemma 1.2] for a similar estimate. In other words, $\mathcal{B}^\alpha \hookrightarrow \mathrm{Hol}(\mathbb{D})$.

Lemma 4.5. *Suppose that $f \in \mathcal{B}^\alpha$, $\alpha > 0$. Then, for each $z \in \mathbb{D}$,*

$$|f(z)| \le |f(0)| + \frac{\|f\|_{\mathcal{B}^\alpha}}{2} \int_0^{|z|^2} \frac{1}{\sqrt{x}(1-x)^\alpha}\,dx.$$

Proof. Let $f \in \mathcal{B}^\alpha$, $\alpha > 0$. Then, for $z \in \mathbb{D}$,

$$\begin{aligned}|f(z) - f(0)| &= \left| z \int_0^1 f'(zt)\,dt \right| \le |z| \int_0^1 |f'(zt)|\,dt \\ &= |z| \int_0^1 \frac{1}{(1-|z|^2t^2)^\alpha}(1-|z|^2t^2)^\alpha |f'(zt)|\,dt \\ &\le |z| \|f\|_{\mathcal{B}^\alpha} \int_0^1 \frac{1}{(1-|z|^2t^2)^\alpha}\,dt \\ &= \frac{\|f\|_{\mathcal{B}^\alpha}}{2} \int_0^{|z|^2} \frac{1}{\sqrt{x}(1-x)^\alpha}\,dx,\end{aligned}$$

using the change of variable $x = |z|^2t^2$. □

Note that the integral in the previous lemma exists for all $z \in \mathbb{D}$, no matter what α is, whereas it exists for all $z \in \overline{\mathbb{D}}$ if and only if $0 < \alpha < 1$.

Lemma 4.6. *Let $\alpha > 0$, $\alpha \neq 1$ and let φ be a holomorphic self map of $\mathbb{D}$ such that C_φ is bounded on $\mathcal{B}^\alpha$. Then*

$$\|C_\varphi\|_{\mathcal{L}(\mathcal{B}^\alpha)} \ge \max\left\{1, \frac{1}{2^\alpha|\alpha-1|}\left(\frac{1}{(1-|\varphi(0)|)^{\alpha-1}} - 1\right)\right\}.$$

Proof. Let $\varphi(0) = e^{i\theta}|\varphi(0)|$, $F(z) = 1/2^\alpha|\alpha-1|((1/(1-z)^{(\alpha-1)}) - 1)$ and $g(z) = F(e^{-i\theta}z)$. Then $F(0) = 0$ and $\|g\|_{\mathcal{B}^\alpha} = 1$. We then have

$$\begin{aligned}\|C_\varphi\|_{\mathcal{B}^\alpha} &\ge \|g \circ \varphi\|_{\mathcal{B}^\alpha} \ge |g(\varphi(0))| \\ &= |F(e^{-i\theta}(\varphi(0)))| = |F(|\varphi(0)|)| \\ &= \frac{1}{2^\alpha|\alpha-1|}\left(\frac{1}{(1-|\varphi(0)|)^{(\alpha-1)}} - 1\right).\end{aligned}$$

Taking $f = \mathbf{1}_{\mathbb{D}}$, we have $\|C_\varphi(f)\|_{\mathcal{L}(\mathcal{B}_\alpha)} = 1$ and so $\|C_\varphi\|_{\mathcal{L}(\mathcal{B}_\alpha)} \ge 1$. This completes the proof. □

Remark 4.7. When φ has no fixed point in $\mathbb{D}$, its Denjoy–Wolff point is on the unit circle and, therefore, by Lemma 4.6, when $\alpha > 1$, $\lim_{n\to\infty} \|C_\varphi^n\| = \infty$ since

$$\|C_\varphi^n\|_{\mathcal{L}(\mathcal{B}^\alpha)} \ge \frac{1}{2^\alpha(\alpha-1)}\left(\frac{1}{(1-|\varphi_n(0)|)^{\alpha-1}} - 1\right)$$

with $|\varphi_n(0)| \to 1$. In this case, even the weak convergence of the iterates is impossible.

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4.1. Bloch-type space $\mathcal{B}^\alpha$, $\alpha > 1$. First recall that in this case, for every holomorphic self map φ of $\mathbb{D}$, C_φ is bounded [19, 31, 35]. This follows from Lemma 4.4 combined with the following lemma, which will be useful later.

Lemma 4.8. *Let φ be a holomorphic self map of $\mathbb{D}$ such that $\varphi(0) = 0$. Then $\tau^\infty_{\varphi,\alpha} \le 1$.*

Proof. The Schwarz–Pick lemma [12, Theorem 2.39] asserts that

$$\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \le 1.$$

It follows that

$$\tau_{\varphi,\alpha}(z) := \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\frac{(1-|z|^2)^{\alpha-1}}{(1-|\varphi(z)|^2)^{\alpha-1}} \le \frac{(1-|z|^2)^{\alpha-1}}{(1-|\varphi(z)|^2)^{\alpha-1}}.$$

Since $\varphi(0) = 0$, using the Schwarz lemma, $|\varphi(z)| \le |z|$ and thus

$$\tau_{\varphi,\alpha}(z) \le 1,$$

which obviously implies that $\tau^\infty_{\varphi,\alpha} \le 1$. □

So, by Lemma 4.8, C_φ is bounded on $\mathcal{B}^\alpha$ under the assumption that $\varphi(0) = 0$. To deal with the case when $\varphi(0) = a \ne 0$, note that $C_\varphi = C_{\varphi_a}C_{\psi_a}$, where $\psi_a = (a-z)/(1-\overline{a}z)$ and where $\varphi_a = \psi_a \circ \varphi$ fixes 0. By Lemma 4.4, C_φ is bounded on $\mathcal{B}^\alpha$ for every self map φ of $\mathbb{D}$.

Lemma 4.9. *Let φ be a holomorphic self map of $\mathbb{D}$ such that $\varphi(0) = 0$ and such that φ is not a rotation. Then, for each $s \in (0,1)$, there exists $r \in (0,1)$ such that*

$$\frac{1-|z|^2}{1-|\varphi_n(z)|^2} \le 2r^n$$

for all $n \ge 1$ and every $z \in \mathbb{D}$ such that $|\varphi_n(z)| > s$.

Proof. Note that

$$\frac{1-|z|^2}{1-|\varphi_n(z)|^2} = \frac{1-|z|}{1-|\varphi_n(z)|}\frac{1+|z|}{1+|\varphi_n(z)|} \le 2\frac{1-|z|}{1-|\varphi_n(z)|}. \tag{4.1}$$

By [12, Lemma 7.33], since $\varphi(0) = 0$ and φ is not a rotation, for each $0 < s < 1$, there exists $r \in (0,1)$ such that

$$\frac{1-|u|}{1-|\varphi(u)|} < r$$

for all u with $|u| > s$. Then we use the identity

$$\frac{1-|z|}{1-|\varphi_n(z)|} = \frac{1-|z|}{1-|\varphi(z)|}\frac{1-|\varphi(z)|}{1-|\varphi_2(z)|}\cdots\frac{1-|\varphi_{n-1}(z)|}{1-|\varphi_n(z)|}.$$

By the Schwarz lemma, since $\varphi(0) = 0$,

$$|z| \ge |\varphi(z)| \ge \cdots \ge |\varphi_n(z)|$$

and thus $|\varphi_n(z)| > s$ implies that $|z| > s, \ldots, |\varphi_{n-1}(z)| > s$. Finally,

$$\frac{1-|z|}{1-|\varphi_n(z)|} \leq r^n$$

for all z such that $|\varphi_n(z)| > s$. The lemma follows from (4.1). □

We are now able to obtain the key result for the essential spectral radius.

Theorem 4.10. *Let φ be a holomorphic self map of $\mathbb{D}$ such that $\varphi(0) = 0$ and such that φ is not a rotation. Then $r_e(C_\varphi) < 1$, where $r_e(C_\varphi)$ is the essential spectral radius of C_φ on $\mathcal{B}^\alpha$.*

Proof. Fix $s_a \in (0, 1)$. By Theorem 4.2,

$$\|C_{\varphi_n}\|_{e,\mathcal{L}(\mathcal{B}^\alpha)} \leq \sup_{|\varphi_n(z)|>s_a} \left(\frac{1-|z|^2}{1-|\varphi_n(z)|^2}\right)^\alpha |\varphi_n'(z)|.$$

For $\alpha > 1$,

$$\left(\frac{1-|z|^2}{1-|\varphi_n(z)|^2}\right)^\alpha |\varphi_n'(z)| = \left(\frac{1-|z|^2}{1-|\varphi_n(z)|^2}\right)^{\alpha-1} \frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi_n'(z)|$$

with $1-|z|^2/1-|\varphi_n(z)|^2|\varphi_n'(z)| \leq 1$ once more using the Schwarz–Pick lemma. By Lemma 4.9, there exists $r \in (0, 1)$ such that

$$A(s) := \sup_{|\varphi_n(z)|>s} \left(\frac{1-|z|^2}{1-|\varphi_n(z)|^2}\right)^\alpha |\varphi_n'(z)| \leq 2^{\alpha-1} r^{n(\alpha-1)}.$$

It follows that

$$\|C_{\varphi_n}\|_{e,\mathcal{L}(\mathcal{B}^\alpha)} \leq 2^{\alpha-1} r^{n(\alpha-1)}$$

for all $n \in \mathbb{N}$. The conclusion follows from the equality $r_e(C_\varphi) = \lim_{n\to\infty} \|C_{\varphi_n}\|_{e,\mathcal{L}(\mathcal{B}_\alpha)}^{1/n}$, which implies that $r_e(C_\varphi) \leq r^{\alpha-1} < 1$. □

We have now all the material to prove the convergence theorem of $\mathcal{B}^\alpha$ for $\alpha > 1$.

Theorem 4.11. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic and with a fixed point b in $\mathbb{D}$. The following assertions are equivalent on $\mathcal{B}^\alpha$:*

(i) *φ is not an automorphism;*
(ii) *C_φ^n converges uniformly to P_b, where $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$;*
(iii) *C_φ^n converges strongly to P_b, where $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$.*

Proof. (i) ⇒ (ii): By the usual trick involving the isomorphism C_{ψ_b}, we may assume that $b = 0$ and that φ is not a rotation. It follows from Theorem 4.10 that $r_e(C_\varphi) < 1$. Then (ii) follows from Theorem 1.1.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i): If φ is an elliptic automorphism, C_φ is similar to an isometry and cannot converge strongly to a rank-one projection. □

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When φ has no fixed point in $\mathbb{D}$, its Denjoy–Wolff point is on the unit circle and, therefore, by Lemma 4.6, $\lim_{n\to\infty} \|C_\varphi^n\| = \infty$ since

$$\|C_\varphi^n\|_{\mathcal{L}(\mathcal{B}^\alpha)} \geq \frac{1}{2^\alpha(\alpha-1)}\Big(\frac{1}{(1-|\varphi_n(0)|)^{\alpha-1}} - 1\Big)$$

with $|\varphi_n(0)| \to 1$. In this case, even the weak convergence of the iterates is impossible. We can say even more. Since $\mathcal{B} \subset \mathcal{B}^\alpha$, as a corollary of Proposition 3.12, we can prove that the weakest convergence of the iterates is not true when φ has no fixed point in $\mathbb{D}$.

Proposition 4.12. *Let φ be a holomorphic self map of $\mathbb{D}$ whose Denjoy–Wolff point b is of modulus one. Then C_φ is not mean ergodic on $\mathcal{B}^\alpha$, $\alpha > 1$.*

With the help of Theorem 4.3 and Remark 4.7, as in the the sections on A^p_α and $\mathcal{B}$, we obtain the following characterization of composition operators similar to isometries.

Corollary 4.13. *Let φ be a holomorphic self map of $\mathbb{D}$ and consider the composition operator C_φ on $\mathcal{B}^\alpha$ with $\alpha > 1$. The following assertions are equivalent:*

(i) *C_φ is similar to an isometry;*
(ii) *φ is an elliptic automorphism.*

4.2. Bloch-type space $\mathcal{B}^\alpha$, $0 < \alpha < 1$. When $0 < \alpha < 1$, $\mathcal{B}^\alpha = \mathrm{Lip}_{1-\alpha}$, the holomorphic Lipschitz space consisting of all holomorphic functions f on $\mathbb{D}$, satisfies

$$|f(z) - f(w)| \leq C|z - w|^{1-\alpha}$$

for some constant $C > 0$ and all $z, w \in \mathbb{D}$. Thus, for $0 < \alpha < 1$,

$$\mathcal{B}^\alpha \subset A(\mathbb{D}) \subset H^\infty(\mathbb{D}),$$

where $A(\mathbb{D})$ is the disc algebra. By Lemma 4.4, C_φ is bounded for all automorphic self map φ of $\mathbb{D}$, but it is not the case for an arbitrary self map φ of $\mathbb{D}$. Indeed, since $e_1 : z \mapsto z$ is in $\mathcal{B}^\alpha$, a necessary condition is that $\varphi \in \mathcal{B}^\alpha$.

We will show that in $\mathcal{B}^\alpha$, mean ergodicity implies uniform convergence, provided the Denjoy–Wolff point lies in the interior.

Theorem 4.14. *Let $0 < \alpha < 1$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic such that $\tau^\infty_{\varphi,\alpha} < \infty$. Suppose that there exists $b \in \mathbb{D}$ such that $\varphi(b) = b$. The following assertions are equivalent:*

(i) *C_φ^n converges strongly to P_b, where $P_b f = f(b)\mathbf{1}_{\overline{\mathbb{D}}}$;*
(ii) *there exists $n_0 \in \mathbb{N}$ such that $\varphi_{n_0}(\overline{\mathbb{D}}) \subset \mathbb{D}$;*
(iii) *C_φ^n converges uniformly to P_b, where $P_b f = f(b)\mathbf{1}_{\overline{\mathbb{D}}}$;*
(iv) *C_φ is mean ergodic.*

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PROOF. Once more, using Lemma 4.4, we may assume that $\varphi(0) = 0$. (i) $\Rightarrow$ (ii): For $e_1(z) = z$, $\|C_\varphi^n e_1 - P_0 e_1\|_{\mathcal{B}^\alpha} = \|\varphi_n\|_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$. Using the fact that $\mathcal{B}^\alpha = \mathrm{Lip}_{1-\alpha}$ with equivalent norm and since $\varphi_n(0) = 0$, it follows that, for all $z \in \overline{\mathbb{D}}$,

$$|\varphi_n(z)| \le c\|\varphi_n\|_{\mathcal{B}^\alpha}|z|^{1-\alpha} \le c\|\varphi_n\|_{\mathcal{B}^\alpha}.$$

Consequently, φ_n converges to 0 uniformly on $\overline{\mathbb{D}}$ and then (ii) follows.

(ii) $\Rightarrow$ (iii): By [31], (ii) implies that $C_{\varphi_{n_0}} = C_\varphi^{n_0}$ is compact. It follows that $r_e(C_\varphi^{n_0}) = 0$ and thus, by the spectral mapping theorem, $r_e(C_\varphi) = 0$. Then (iii) follows from Theorem 1.1.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (ii): Let $C_n := (1/n)\sum_{k=0}^{n-1} C_\varphi^k$ be the nth Cesaro sum. By assumption, $\lim_{n\to\infty} C_n f = P_0 f = f(0)\mathbf{1}_{\overline{\mathbb{D}}}$ in $\mathcal{B}^\alpha$ for all $f \in \mathcal{B}^\alpha$. Since $\mathcal{B}^\alpha \hookrightarrow A(\mathbb{D})$, it follows that $\lim_{n\to\infty} C_n f = f(0)\mathbf{1}_{\overline{\mathbb{D}}}$ in $A(\mathbb{D})$ for all $f \in \mathcal{B}^\alpha$. Note that $\mathcal{B}^\alpha$ is dense in $A(\mathbb{D})$ and $\|C_n\|_{\mathcal{L}(A(\mathbb{D}))} \le 1$ for all $n \in \mathbb{N}$. Therefore, we obtain $\lim_{n\to\infty} C_n f = f(0)\mathbf{1}_{\overline{\mathbb{D}}}$ in $A(\mathbb{D})$ for all $f \in A(\mathbb{D})$. Since $\varphi(0) = 0$, it follows from [5, Theorem 3.4 (i) $\Rightarrow$ (iii)] that $\varphi_n(z) \to 0$ for all $z \in \overline{\mathbb{D}}$. Dini's theorem implies uniform convergence on $\overline{\mathbb{D}}$.

(iii) $\Rightarrow$ (i) is obvious and the proof is complete. □

We can now give a complete characterization of composition operators which are similar to isometries on $\mathcal{B}^\alpha$, $\alpha \in (0, 1)$.

THEOREM 4.15. *Let $\varphi \in \mathcal{B}^\alpha$, $0 < \alpha < 1$. The following assertions are equivalent:*

(i) *φ is an elliptic automorphism;*
(ii) *C_φ is similar to an isometry.*

PROOF. (i) $\Rightarrow$ (ii): Let $b \in \mathbb{D}$ be such that $\varphi(b) = b$. Then $C_{\psi_b} C_\varphi C_{\psi_b} = C_{\widetilde{\varphi}}$, where $\widetilde{\varphi} := \psi_b \circ \varphi \circ \psi_b$ is a rotation. Note that $C_{\psi_b} = C_{\psi_b}^{-1}$. By Theorem 4.3, (ii) follows.

(ii) $\Rightarrow$ (i): Let S be invertible and U isometric such that $C_\varphi = S^{-1}US$. Since $\mathcal{B}^\alpha$ and $\mathrm{Lip}_{1-\alpha}$ have equivalent norms, there exists $C_1 > 0$ such that for all $z, w \in \mathbb{D}$,

$$|\varphi_n(z) - \varphi_n(w)| \le C_1\|\varphi_n\|_{\mathcal{B}^\alpha}|z - w|^{1-\alpha}. \tag{4.2}$$

Since $\|\varphi_n\|_{\mathcal{B}^\alpha} = \|C_\varphi^n e_1\|_{\mathcal{B}^\alpha} = \|S^{-1}U^n S e_1\|_{\mathcal{B}^\alpha}$, where U^n is isometric and S^{-1} bounded below,

$$\|\varphi_n\|_{\mathcal{B}^\alpha} \simeq \|S e_1\|_{B^\alpha}. \tag{4.3}$$

So, by (4.2), $|\varphi_n(z) - \varphi_n(w)| \le C_2|z - w|^{1-\alpha}$ for some positive constant C_2. It follows that $(\varphi_n)_n$ is an equicontinuous family in the set of continuous functions on $\overline{\mathbb{D}}$. Using the Arzela–Ascoli theorem, there exists a subsequence $(\varphi_{n_k})_k$ which converges uniformly on $\overline{\mathbb{D}}$. If φ is not an elliptic automorphism, denote by b its Denjoy–Wolff point. If $|b| < 1$, for k large enough, $\varphi_{n_k}(\overline{\mathbb{D}}) \subset \mathbb{D}$. By Theorem 4.14,

$$C_\varphi^n \to P_b \in \mathcal{L}(\mathcal{B}^\alpha) \quad \text{as } n \to \infty,$$

which is a contradiction if C_φ is similar to an isometry.

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Suppose now that $|b| = 1$. Since $\|C_\varphi^n\| = \|S^{-1}U^nS\| \le \|S\|\|S^{-1}\|$ and since $\|C_\varphi^n\| \simeq \tau^\infty_{\varphi_n,\alpha}$, there exists $C_3 > 0$ such that, for all $n \in \mathbb{N}$ and $z \in \mathbb{D}$,

$$\frac{(1-|z|^2)^\alpha|\varphi_n'(z)|}{(1-|\varphi_n(z)|^2)^\alpha} \le C_3. \tag{4.4}$$

By (4.3), there exists $C_4 > 0$ such that

$$C_4 \le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\varphi_n'(z)|. \tag{4.5}$$

Now, combining (4.4) and (4.5),

$$C_4 \le C_3 \sup_{z\in\mathbb{D}}(1-|\varphi_n(z)|^2)^\alpha. \tag{4.6}$$

Since $\|\varphi_n - b\|_\infty \to 0$ with $|b| = 1$, (4.6) is impossible and therefore φ is necessarily an elliptic automorphism. □

5. Standard weighted Bergman space of infinite order, $H^\infty_{v_p}(\mathbb{D})$

For $p > 0$, the *standard weighted Bergman space of infinite order*, $H^\infty_{v_p}(\mathbb{D})$ (or $H^\infty_{v_p}$), is the Banach space of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^\infty_{v_p}} := \sup_{z\in\mathbb{D}} v_p(z)|f(z)| < \infty$$

with the norm as defined above, where $v_p(z) = (1-|z|^2)^p$.

For $w \in \mathbb{D}$, it follows from the definition of the norm that

$$|f(w)| \le \frac{1}{(1-|w|^2)^p}\|f\|_{H^\infty_{v_p}} \quad \text{for all } f \in H^\infty_{v_p}$$

and the norm of evaluation at w is $1/(1-|w|^2)^p$ and is attained for the function $f_w(z) = (1-\overline{w}z)^{-p}$. In other words, $H^\infty_{v_p} \hookrightarrow \mathrm{Hol}(\mathbb{D})$ for all $p > 0$.

Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is a self map of $\mathbb{D}$. By [6, Theorem 2.3], C_φ is always bounded on $H^\infty_{v_p}$ and, by [11, Proposition 3.1],

$$\|C_\varphi\|_{\mathcal{L}(H^\infty_{v_p})} = \sup_{z\in\mathbb{D}}\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^p.$$

It follows that

$$\text{if } \varphi(0) = 0 \text{ then } \|C_\varphi\|_{\mathcal{L}(H^\infty_{v_p})} = 1. \tag{5.1}$$

Moreover, Montes-Rodríguez [25] proved that the essential norm is

$$\|C_\varphi\|_{e,\mathcal{L}(H^\infty_{v_p})} = \inf_{0<s<1}\sup_{|\varphi(z)|>s}\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^p.$$

This formula for the essential norm provides a bound for the essential spectral radius.

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Theorem 5.1. *Let φ be a self map of $\mathbb{D}$ such that $\varphi(0) = 0$. If φ is not a rotation, then $r_e(C_\varphi) < 1$, where $r_e(C_\varphi)$ denotes the essential spectral radius of C_φ on $H^\infty_{\nu_p}$, $p > 0$.*

Proof. Fix $s_a \in (0, 1)$. By Lemma 4.9, there exists $r \in (0, 1)$ such that, for all $n \geq 1$,

$$\sup_{|\varphi_n(z)|>s_a} \left(\frac{1 - |z|^2}{1 - |\varphi_n(z)|^2}\right)^p \leq 2^p r^{np}.$$

It follows that for all $n \geq 1$,

$$\|C_{\varphi_n}\|_{e,H^\infty_{\nu_p}} \leq 2^p r^{np}.$$

The conclusion follows from the equality

$$r_e(C_\varphi) = \lim_{n\to\infty} \|C_{\varphi_n}\|^{1/n}_{e,\mathcal{L}(H^\infty_{\nu_p})},$$

which implies that $r_e(C_\varphi) \leq r^p < 1$. □

We next prove the convergence result for the standard weighted Bergman space of infinite order, $H^\infty_{\nu_p}$, $p > 0$.

Theorem 5.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic with a fixed point $b \in \mathbb{D}$. The following assertions are equivalent on $H^\infty_{\nu_p}$, $p > 0$:*

(i) *C^n_φ converges strongly to P_b, $P_b f = f(b)\mathbf{1}_\mathbb{D}$ for all $f \in H^\infty_{\nu_p}$;*
(ii) *φ is not an automorphism;*
(iii) *C^n_φ converges uniformly to P_b, $P_b f = f(b)\mathbf{1}_\mathbb{D}$ for all $f \in H^\infty_{\nu_p}$.*

Proof. The proof goes along the same lines of the proof of Theorem 4.11, using Theorem 5.1. □

As in the previous sections on weighted Bergman spaces and Bloch and Bloch-type spaces, one can deduce a complete characterization of composition operators which are similar to isometries, using Remark 5.4 and Theorem 5.2.

Corollary 5.3. *Let φ be a holomorphic self map of $\mathbb{D}$. Consider C_φ to be a composition operator on $H^\infty_{\nu_p}$ ($p > 0$). The following assertions are equivalent:*

(i) *C_φ is similar to an isometry;*
(ii) *φ is an elliptic automorphism.*

Remark 5.4. When φ has no fixed point in $\mathbb{D}$, its Denjoy–Wolff point is on the unit circle and, therefore, by [6, Theorem 2.3], $\lim_{n\to\infty} \|C^n_\varphi\| = \infty$ since

$$\|C^n_\varphi\|_{\mathcal{L}(H^\infty_{\nu_p})} \geq \left(\frac{1}{1 - |\varphi_n(0)|}\right)^p$$

with $|\varphi_n(0)| \to 1$. In this case, even the weak convergence of the iterates is impossible.

In fact, we can even prove that mean ergodicity is impossible when φ has no fixed point in $\mathbb{D}$.

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Proposition 5.5. *Let φ be a holomorphic self map of $\mathbb{D}$ whose Denjoy–Wolff point b is of modulus one. Then C_φ is not mean ergodic on $H^\infty_{v_p}$.*

Proof. Since $H^\infty_{v_p}$ is invariant by rotation, we may assume that $b = 1$. Let $f(z) = (2/(1-z))^p$. Then $f \in H^\infty_{v_p}$, $\|f_r\|_{H^\infty_{v_p}} \le \|f\|_{H^\infty_{v_p}}$ and $\lim_{r\uparrow 1} |f_r(1)| = \infty$. The conclusion follows from Lemma 2.3. □

6. The classical Dirichlet space $\mathcal{D}$

The *classical Dirichlet Space*, denoted by $\mathcal{D}$, is the Hilbert space of holomorphic functions f on $\mathbb{D}$ with norm given by

$$\|f\|_{\mathcal{D}} := \Big(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)\Big)^{1/2},$$

where $dA(z) = (1/\pi)\, dx\, dy = (1/\pi) r\, dr\, d\theta$ $(z = x + iy = re^{i\theta})$ denotes the normalized Lebesgue area measure of the unit disc $\mathbb{D}$. It is a well-known fact that $\mathcal{D} \subset H^2(\mathbb{D})$.

If φ is a holomorphic self map of $\mathbb{D}$, then the composition operator C_φ is not necessarily bounded on $\mathcal{D}$. Indeed, since e_1 is in $\mathcal{D}$, where $e_1(z) = z$ for all $z \in \mathbb{D}$, C_φ bounded on $\mathcal{D}$ implies that $\varphi \in \mathcal{D}$.

Since the Dirichlet norm measures the area of the image counting multiplicity, it is easy to construct $\varphi : \mathbb{D} \to \mathbb{D}$ and φ not in $\mathcal{D}$ (for example, infinite Blaschke products). However, if φ is univalent, then C_φ is bounded, using a change of variables in the definition using the integral on $\mathbb{D}$.

A necessary and sufficient condition for φ to induce a bounded composition operator on $\mathcal{D}$ is given in terms of *counting functions* and *Carleson measures* (see [15]):

$$\int_{S(\zeta,h)} n_\varphi \, dA \le C \int_{S(\zeta,h)} dA = Kh^2$$

for all $\zeta \in \partial\mathbb{D}$ and all $h \in (0, 1)$. Here $S(\zeta, h) = \{z \in \mathbb{D} : |z - \zeta| < h\}$ is a typical Carleson set and the counting function, $n_\varphi(w)$, $w \in \mathbb{D}$, associated with φ is defined as the cardinality of the set $\{z \in \mathbb{D} : \varphi(z) = w\}$ when the latter is finite and as $+\infty$ otherwise.

Under the restriction that φ is univalent, we have the following estimation of the norm in terms of $|\varphi(0)|$.

Theorem 6.1 [20, Theorem 2]. *Let φ be a univalent self map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{\mathcal{L}(\mathcal{D})} \le \sqrt{\frac{L + 2 + \sqrt{L(4 + L)}}{2}},$$

where $L = -\log(1 - |\varphi(0)|^2)$. In particular, $\|C_\varphi\|_{\mathcal{L}(\mathcal{D})} = 1$ if $\varphi(0) = 0$.

A lower bound of the norm follows from the following observation.

It is well known that for each $w \in \mathbb{D}$, the function

$$k_w(z) = 1 + \log \frac{1}{1 - \overline{w}z}$$

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is the *reproducing kernel* at w in the Dirichlet space, whose norm is

$$\|k_w\|_{\mathcal{D}} = \sqrt{1 + \log \frac{1}{1 - |w|^2}}.$$

The existence of the reproducing kernels implies that $\mathcal{D} \hookrightarrow \mathrm{Hol}(\mathbb{D})$.

Since $C_\varphi^*(k_w) = k_{\varphi(w)}$,

$$\|C_\varphi\| = \|C_\varphi^*\| \geq \frac{\|k_{\varphi(0)}\|_{\mathcal{D}}}{\|k_0\|_{\mathcal{D}}} = \sqrt{1 + \log \frac{1}{1 - |\varphi(0)|^2}}. \tag{6.1}$$

This lower bound of the norm of a composition operator implies that if φ has its Denjoy–Wolff point on the unit circle, then $|\varphi_n(0)| \to 1$ as $n \to \infty$, which implies that $\|C_\varphi^n\| \to \infty$ as $n \to \infty$. Thus, as in the previous cases, the weak convergence of C_{φ_n} is impossible when φ has no fixed point in $\mathbb{D}$. In fact, C_φ is not even mean ergodic.

Theorem 6.2. *Let φ be a holomorphic self map of $\mathbb{D}$ such that C_φ is bounded on $\mathcal{D}$. If the Denjoy–Wolff point b of φ is of modulus one, then C_φ is not mean ergodic.*

Proof. Let $\beta > 0$ and $F(z) := (\log 2/(1 - z))^\beta$, where log is the principal branch of the logarithm. By [37, Ch. V, Theorem 2.31], the Taylor coefficients a_n of F are equivalent to $1/n(\log n)^{\beta-1}$. It follows that $F \in \mathcal{D}$ if and only if $\beta < 1/2$. Since $\|F_r\|_{\mathcal{D}} \leq \|F\|_{\mathcal{D}}$ and since $\lim_{r\uparrow 1} |F_r(1)| = \infty$, the conclusion follows from Lemma 2.3. □

Recall that a self map of $\mathbb{D}$ is a *univalent full map* if it is one-to-one and $A[\mathbb{D}\backslash\varphi(\mathbb{D})] = 0$. The following result characterizes the isometric composition operators on $\mathcal{D}$.

Theorem 6.3 [22]. *A composition operator C_φ on $\mathcal{D}$ is an isometry if and only if φ is a univalent full map of $\mathbb{D}$ that fixes the origin.*

In our previous results, provided $\varphi(0) = 0$, we proved that the uniform and strong convergence of the iterates is equivalent to the fact that C_φ is not an isometry.

The space $\mathcal{D}$ is of particular interest since the bounded nonisometric composition operators whose symbol fixes zero is different from the set of composition operators for which we can guarantee the uniform and strong convergence of the iterates. This follows from the next proposition.

Proposition 6.4. *Let φ be a fractional linear self map of $\mathbb{D}$ defined with two fixed points, namely 0 and $\beta \in \mathbb{T}$ (for example, $\varphi(z) = \beta z/(2\beta - z)$). Then C_φ is a contraction which is not isometric, but C_φ^n does not strongly converge as $n \to \infty$.*

Proof. By Theorem 6.1, the norm of C_φ is 1. By Theorem 6.3, since φ is a linear fractional map fixing 0, C_φ is isometric if and only if $\varphi(\mathbb{D}) = \mathbb{D}$, that is, if and only if φ is a nontrivial rotation. Since $\varphi(\beta) = \beta$, the function φ cannot be a rotation and therefore C_φ is not an isometry.

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Since 0 is the Denjoy–Wolff point of φ, if it exists, the strong limit of C_φ is P_0 defined by $(P_0 f)(z) = f(0)$.

Moreover, since for all $n \geq 1$, $\varphi_n(0) = 0$ and $\varphi_n(\beta) = \beta$, $\varphi_n(\mathbb{D})$ contains the smallest disc D_0 containing 0 and β.

Hence,

$$\|C_\varphi^n e_1 - Pe_1\|_{\mathcal{D}}^2 = \|\varphi_n\|_{\mathcal{D}}^2 = A(\varphi_n(\mathbb{D}) \geq A(D_0) > 0,$$

where A denotes the normalized Lebesgue area measure. Therefore, $(C_\varphi^n)_n$ does not strongly converge as $n \to \infty$. □

Remark 6.5. When φ satisfies the hypothesis of Proposition 6.4, since $\mathcal{D}$ is reflexive and since $(\|C_\varphi^n\|)_n$ is bounded by 1, for all $f \in \mathcal{D}$, $(C_\varphi^n f)_n$ is weakly convergent to $f(0)\mathbf{1}_{\mathbb{D}}$ (by [3, Theorem 4.4]).

Nevertheless, we can get a positive result concerning the uniform convergence of (C_φ^n), with an extra hypothesis on φ, which implies that the essential spectral radius of C_φ is strictly less than 1.

Definition 6.6. A function $h : \mathbb{D} \to [0, \infty)$ is *essentially radial* if for almost all $r \in [0, 1)$, $h(re^{i\theta}) = h(r)$ for almost all $\theta \in [0, 2\pi)$.

Theorem 6.7. *Let φ be a univalent and holomorphic self map of $\mathbb{D}$ with a fixed point b in $\mathbb{D}$ and such that n_φ is essentially radial. The following assertions are equivalent:*

(i) *C_φ^n converges strongly to P_b, $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$ for all $f \in \mathcal{D}$;*

(ii) *φ is not a full map of $\mathbb{D}$;*

(iii) *C_φ^n converges uniformly to P_b, $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$ for all $f \in \mathcal{D}$.*

Proof. Since, for all $b \in \mathbb{D}$, C_{ψ_b} is invertible on $\mathcal{D}$ and since $\psi := \psi_b \circ \varphi \circ \psi_b$ which fixes 0 is a full map if and only φ is a full map, we may assume that $b = 0$.

(i) ⇒ (ii) follows from the characterization of the isometric composition operators on $\mathcal{D}$ in Theorem 6.3 and the fact that $e_1 \in \mathcal{D}$ and $C_\varphi^n e_1 = \varphi_n$ cannot converge to 0 in norm. Indeed, since $C_\varphi^n = C_{\varphi_n}$ is isometric, φ_n is a univalent full map and then $\|\varphi_n\|_{\mathcal{D}} = A(\varphi_n(\mathbb{D})) = 1$.

(ii) ⇒ (iii): since $\varphi(0) = 0$ and φ is not a full map of $\mathbb{D}$, by Theorem 3.1 in [9], $\|C_\varphi\|_{\mathcal{L}(\mathcal{D}_0)} < 1$, where $\mathcal{D}_0$ is the subspace of $\mathcal{D}$ of those functions that vanish at 0. It follows that

$$r_{e,\mathcal{D}}(C_\varphi) = r_{e,\mathcal{D}_0}(C_\varphi) \leq \|C_\varphi\|_{\mathcal{L}(\mathcal{D}_0)} < 1.$$

The uniform convergence of $(C_\varphi^n)_n$ follows from Theorem 1.1.

(iii) ⇒ (i) is obvious. □

We conclude this section with an obvious corollary of Theorem 6.7 concerning the similarity of a composition operator to an isometry.

Corollary 6.8. *Let φ be a univalent and holomorphic self map of $\mathbb{D}$. Suppose that n_φ is essentially radial. The following assertions are equivalent:*

(i) *φ is a full map with a fixed point $b \in \mathbb{D}$;*

(ii) *C_φ is similar to an isometry.*

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Proof. (i) ⇒ (ii): Since $\widetilde{\varphi} := \psi_b \circ \varphi \circ \psi_b$ fixes 0 and is a full map if φ is a full map, (ii) follows from Theorem 6.3 and the identity

$$C_\varphi = C_{\psi_b} C_{\widetilde{\varphi}} C_{\psi_b}.$$

(ii) ⇒ (i): If φ has no fixed point in $\mathbb{D}$, its Denjoy–Wolff point b is of modulus one and, therefore, by (6.1), C_φ^n is not power bounded. It follows that C_φ cannot be similar to an isometry and thus (ii) implies that φ has a fixed point in $\mathbb{D}$. If φ is not a full map, by Theorem 6.7, $C_\varphi^n \to P_b$ in $\mathcal{L}(\mathcal{D})$ as $n \to \infty$. As detailed previously, this property is clearly in contradiction with the fact that C_φ is similar to an isometry. Thus, φ must be a full map. □

7. The quotient space of $\mathcal{D}$

In this section we slightly change the framework. Let $\widetilde{\mathcal{D}} := \mathcal{D}/\mathbb{C}\mathbf{1}_{\mathbb{D}}$ be the space of $\mathcal{D}$ modulo the constant functions. It is a Banach space for the norm

$$\|[f]\|_{\widetilde{\mathcal{D}}}^2 = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z),$$

where $[\ ] : \mathcal{D} \to \mathcal{D}/\mathbb{C}\mathbf{1}_{\mathbb{D}}$ denotes the quotient map.

Let $\varphi \in \mathcal{D}$ be a univalent map. If $f, g \in \mathcal{D}$ and $f - g = c$ is a constant, then $f \circ \varphi - g \circ \varphi = c$. It follows that $[f \circ \varphi] = [g \circ \varphi]$ and thus

$$\widetilde{C}_\varphi([f]) = [f \circ \varphi]$$

defines a linear operator on $\widetilde{\mathcal{D}}$.

This operator may also be obtained in a different way. Let $\mathcal{D}_0 := \{f \in \mathcal{D} : f(0) = 0\}$. Then $\mathcal{D}_0$ is a closed subspace of $\mathcal{D}$. The mapping

$$S : \mathcal{D}_0 \to \widetilde{\mathcal{D}}, \quad Sf = [f]$$

is an isometric isomorphism. Then $\widehat{C}_\varphi := S^{-1}\widetilde{C}_\varphi S$ is given by

$$(\widehat{C}_\varphi f)(z) = f(\varphi(z)) - f(\varphi(0)).$$

Now let $\varphi(z) = (az + b)/(cz + d)$ be a linear fractional map with $ad - bc \neq 0$. Then φ has two fixed points in $\mathbb{C} \cup \{\infty\}$. One calls φ *parabolic* if they coincide. If in addition $\varphi(\mathbb{D}) \subset \mathbb{D}$, the unique fixed point lies on the unit circle.

The spectrum of $\widetilde{C}_\varphi$ on $\mathcal{D}_0$ has been investigated in [13] and [14]. Using these results, we obtain the following theorem.

Theorem 7.1. *Let φ be a parabolic linear fractional self map of $\mathbb{D}$ which is not an automorphism of $\mathbb{D}$. Then, for all $f \in \mathcal{D}$,*

$$\lim_{n\to\infty} \widetilde{C}_\varphi^n([f]) = 0,$$

but $(\widetilde{C}_\varphi^n)_n$ does not converge uniformly.

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Proof. By [13, Theorem 4.3], $\widetilde{C}_\varphi$ is unitarily equivalent to the multiplication operator T on $L^2((0,\infty), t\,dt)$ given by

$$Tf(t) = e^{iat} f(t),$$

where $a \in \mathbb{C}$ and $\mathrm{Im}(a) > 0$. It follows from the dominated convergence theorem that

$$\|T^n f\|_{L^2}^2 = \int_0^\infty |e^{inat} f(t)|^2 t\,dt = \int_0^\infty e^{-2\mathrm{Im}(a)nt} |f(t)|^2 t\,dt \to 0$$

as $n \to \infty$. On the other hand, $\|T^n\| = \sup_{t>0} |e^{inat}| = 1$. □

However, as already noticed in Section 6, the operator C_φ defined on the whole space $\mathcal{D}$ is not even power bounded since its Denjoy–Wolff point is on the unit circle.

The space $\widetilde{\mathcal{D}}$ does not exactly meet our general assumptions since the point evaluations are not defined on this space. Thus, so far, for composition operators on Banach spaces X such that $X \hookrightarrow \mathrm{Hol}(\mathbb{D})$, we merely know situations where strong convergence of the powers implies uniform convergence.

8. Semigroups of composition operators

The aim of this section is to study the asymptotic behaviour of semigroups of composition operators on the Banach spaces of holomorphic functions considered in the previous sections.

Let X be a Banach space. By a *semigroup* we understand a family $S = (S_t)_{t>0}$ in $\mathcal{L}(X)$ satisfying $S_{t+s} = S_t S_s$ for all $s, t > 0$ without any further topological condition. A semigroup S is called a C_0-*semigroup* if in addition

$$\lim_{t \downarrow 0} S_t x = x$$

for all $x \in X$. We say that S is *locally bounded* if $\sup_{0<t\le 1} \|S_t\| < \infty$ and *bounded* if $\sup_{t>0} \|S_t\| < \infty$. Each C_0-semigroup is locally bounded by the uniform boundedness principle. If S is locally bounded, its *growth bound*

$$w(S) := \inf\{w \in \mathbb{R} : \exists M > 0, \|S_t\| \le Me^{wt} \text{ for all } t > 0\}$$

is finite. By the proof of [2, Proposition 5.1.1],

$$w(S) = \lim_{t\to\infty} \log \|S_t\| = \inf_{t>0} \log \|S_t\|. \tag{8.1}$$

In particular, if $\|S_{t_0}\| < 1$ for some $t_0 > 0$, then S is *exponentially stable*, that is,

$$\|S_t\| \le Me^{-\varepsilon t}$$

for some $\varepsilon > 0$ and $M \ge 0$.

Assume that S is a locally bounded semigroup. Here we are interested in uniform convergence of S_t as $t \to \infty$.

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LEMMA 8.1. *Assume that* $\lim_{t\to\infty} S_t =: P$ *exists in* $\mathcal{L}(X)$. *Then*

$$S_tP = PS_t = P = P^2. \tag{8.2}$$

Moreover, there exist $\varepsilon > 0$ *and* $M \geq 0$ *such that*

$$\|S_t - P\| \leq Me^{-\varepsilon t} \quad \textit{for all } t > 0.$$

PROOF. It is obvious that P is a projection and (8.2) holds and, for that, strong convergence would be sufficient. Let $Y = (I - P)X$. Then Y is invariant by S and $U_t := S_{t|Y}$ defines a locally bounded semigroup U such that $\lim_{t\to\infty} \|U_t\| = 0$. It follows from (8.1) that $w(U) < 0$. This implies that $\|U_t\| \leq Me^{-\varepsilon t}$ for some $\varepsilon > 0$ and $M \geq 0$. Therefore,

$$\|S_t - P\| = \|S_tP - P + S_t(I - P)\| = \|S_t(I - P)\| \leq Me^{-\varepsilon t}. \qquad \square$$

Let S be a locally bounded semigroup. If $\lim_{t\to\infty} S_t = P$, then $\lim_{n\to\infty} S_{t_0}^n = P$ for all $t_0 > 0$. Now assume conversely that there exists $t_0 > 0$ such that $\lim_{n\to\infty} S_{t_0}^n = P$ exists. Then P is clearly a projection and $S_{t_0}P = PS_{t_0} = P$. However, in general, (8.2) may fail for $t \neq t_0$. In particular, S_t might not converge as $t \to \infty$. Periodic C_0-semigroups give a counterexample of this sort. However, if we assume (8.2) to hold, then the converse implication holds.

LEMMA 8.2. *Let S be a locally bounded semigroup and let $t_0 > 0$. Assume that:*

(i) $\lim_{n\to\infty} S_{t_0}^n = P$ *exists in* $\mathcal{L}(X)$*; and*
(ii) $S_tP = PS_t = P$ *for all* $t > 0$.

Then $\lim_{n\to\infty} S_t = P$ *in* $\mathcal{L}(X)$.

PROOF. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\|S_{t_0}^n - P\| \leq \varepsilon$ for all $n \geq n_0$. Let $t \geq n_0$. There exist $n \in \mathbb{N}$, $s \in [0, t_0)$ such that $t = nt_0 + s$. Hence,

$$\|S_t - P\| = \|S_sS_{t_0}^n - P\| = \|S_sS_{t_0}^n - S_sP\| \leq \Big(\sup_{s\in(0,t_0]} \|S_s\|\Big)\varepsilon. \qquad \square$$

Now we consider semigroups which are associated with holomorphic semiflows $(\varphi_t)_{t\geq 0}$ defined as follows.

DEFINITION 8.3. A family $(\varphi_t)_{t\geq 0}$ is called a *holomorphic semiflow* on $\mathbb{D}$ if:

(1) $\varphi_t : \mathbb{D} \to \mathbb{D}$ is holomorphic for all $t \geq 0$;
(2) $\varphi_0(z) = z$ for all $z \in \mathbb{D}$;
(3) $\varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z)$ for all $t, s \geq 0$, $z \in \mathbb{D}$;
(4) $\lim_{t\to 0} \varphi_t(z) = z$ for all $z \in \mathbb{D}$.

The following properties of holomorphic semiflows are well known.

(1) For all $t \geq 0$, φ_t is injective.
(2) $\lim_{t\to 0} \varphi_t = \mathrm{id}$ uniformly on every compact subset of $\mathbb{D}$.

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(3) If there exists $t_0 > 0$ such that φ_{t_0} is an automorphism (respectively elliptic automorphism), then φ_t is an automorphism (respectively elliptic automorphism) for all $t > 0$.

(4) For all semiflows which are not elliptic automorphisms, there exists a unique $\alpha \in \overline{\mathbb{D}}$ such that $\lim_{t\to\infty} \varphi_t(z) = \alpha$ uniformly on every compact subset of $\mathbb{D}$. This α is called the *Denjoy–Wolff point* of the semiflow.

We will consider the spaces $X = A^p_\beta, \mathcal{B}, \mathcal{B}_0, \mathcal{B}^\alpha(\alpha > 1), B^\alpha(0 < \alpha < 1), H^\infty_{v_p}$ and $\mathcal{D}$ as above.

If $b \in \mathbb{D}$, $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$ defines a projection $P_b \in \mathcal{L}(X)$ for all these spaces.

Let $\varphi = (\varphi_t)_{t\geq 0}$ be a holomorphic semiflow of the open unit disc with Denjoy–Wolff point $b \in \mathbb{D}$. Thus, $\varphi_t(b) = b$ for all $t > 0$.

Proposition 8.4. *Let $X = A^p_\beta, \mathcal{B}, \mathcal{B}_0, \mathcal{B}^\alpha$ with $0 < \alpha < 1$, $H^\infty_{v_p}$ or $\mathcal{D}$. Then*

$$C_t f := f \circ \varphi_t$$

defines a bounded semigroup on X.

Proof. The spaces we consider are all invariant by automorphisms (see Lemma 4.4 for $\mathcal{B}^\alpha$). This allows us to assume that $b = 0$. Indeed, otherwise, replace the semiflow φ by ψ, where $\psi_t = \psi_b \circ \varphi_t \circ \psi_b$, where $\psi_b(z) = (b - z)/(1 - \overline{b}z)$. Now, for $X = A^p_\alpha$, Theorem 2.1 shows that $C_t f = f \circ \varphi_t$ defines a bounded operator on X and $\sup_{t>0} \|C_t\| < \infty$. For $X = \mathcal{B}$ and $\mathcal{B}_0$, the assertion follows from Theorem 3.2. For $\mathcal{B}^\alpha$ with $\alpha > 1$, it follows from Lemma 4.8 that the semigroup is bounded. If $X = H^\infty_{v_p}$, then the estimate (5.1) shows that $\|C_t\| = 1$ for all $t > 0$. Finally, since φ_t is univalent for all $t > 0$, when $X = \mathcal{D}$, Theorem 6.1 shows that $\|C_t\| = 1$ for all $t > 0$. □

Next we consider the asymptotic behaviour.

Theorem 8.5. *Let $X = A^p_\alpha$, $\mathcal{B}_0$, $\mathcal{B}^\alpha$ for $\alpha > 1$ or $H^\infty_{v_p}$. The following assertions are equivalent:*

(i) $\lim_{t\to\infty} C_t = P_b$ *in* $\mathcal{L}(X)$*;*

(ii) φ_{t_0} *is not an automorphism for some* $t_0 > 0$*;*

(iii) φ_t *is not an automorphism for any* $t > 0$.

Proof. Since $P_b f = f(b)\mathbf{1}_{\mathbb{D}}$ and $\varphi_t(b) = b$ for all $t > 0$, it follows that $C_t P_b = P_b C_t = P_b$ for all $t > 0$. Thus, by Lemma 8.2, C_t converges to P_b as $t \to \infty$ in $\mathcal{L}(X)$ as soon as $C^n_{t_0} \to P_b$ in $\mathcal{L}(X)$ as $n \to \infty$ for some t_0.

For $X = A^p_\alpha$, it is Theorem 2.9 which implies that $C_t \to P_b$ as $t \to \infty$ if and only if φ_t is not an automorphism for some (equivalently all) $t > 0$.

The same assertion follows for $\mathcal{B}_0$ from Theorem 3.11 and for $\mathcal{B}^\alpha$ with $\alpha > 1$ from Theorem 4.11. For $X = H^\infty_{v_p}$, it follows from Theorem 5.2. □

Note that the equivalence of (ii) and (iii) is well known, but also follows directly from our description of the asymptotic behaviour.

The spaces $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{B}^\alpha$ $(0 < \alpha < 1)$ play a special role.

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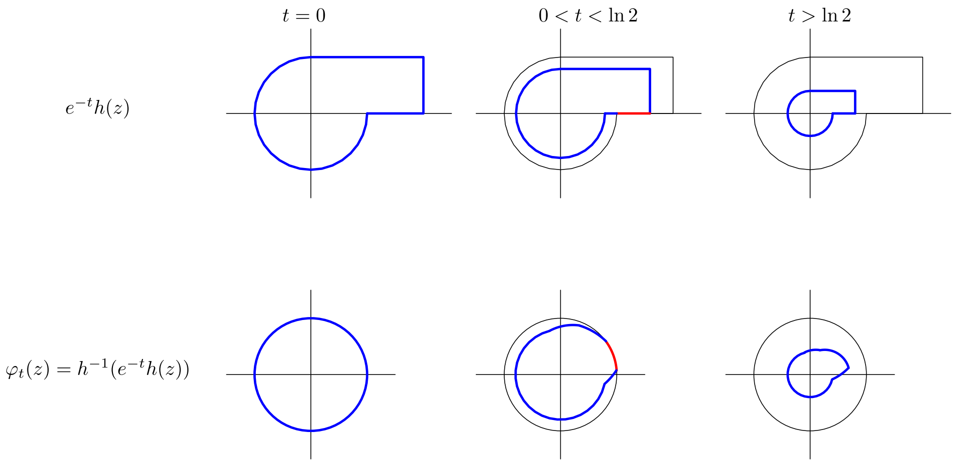

FIGURE 1. An example of holomorphic semiflow.

THEOREM 8.6. *Let $X = \mathcal{B}$. The following assertions are equivalent:*

(i) $\lim_{t\to\infty} C_t = P_b$ *in* $\mathcal{L}(X)$*;*
(ii) *there exists* $t_0 > 0$ *such that* $\tau^{\infty}_{\varphi_{t_0}} < 1$*;*
(iii) *for all* $t > 0$*, one has* $\tau^{\infty}_{\varphi_t} < 1$.

PROOF. This follows from Theorem 3.7 and Lemma 8.2. □

For the Dirichlet space, the following result is a consequence of Theorem 6.7.

THEOREM 8.7. *Let $X = \mathcal{D}$. Assume that φ_{t_0} is not a full map. Then*

$$\lim_{t\to\infty} C_t = P_b \quad \text{in } \mathcal{L}(X).$$

Finally, we consider the space $\mathcal{B}^\alpha$ for $0 < \alpha < 1$. The asymptotic behaviour follows from Theorem 4.14.

THEOREM 8.8. *Let $X = \mathcal{B}^\alpha$, where $0 < \alpha < 1$. Assume that there exists $t_0 > 0$ such that $\varphi_{t_0}(\mathbb{D}) \subset \{z \in \mathbb{C} : |z| \le r\}$ for some $r < 1$. Then $C_t f = f \circ \varphi_t$ defines a bounded operator on $\mathcal{B}^\alpha$ for all $t \ge t_0$ and $\lim_{t\to\infty} C_t = P_b$ in $\mathcal{L}(X)$.*

The following example of a semiflow illustrates our last theorem.

Let h be the Riemann map from $\mathbb{D}$ onto the starlike region

$$\Omega := \mathbb{D} \cup \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 2 \text{ and } 0 < \mathrm{Im}(z) < 1\}$$

with $h(0) = 0$. Since $\partial\Omega$ is a Jordan curve, the Carathéodory theorem [27, Theorem 2.6, page 24] implies that h extends continuously to $\partial\mathbb{D}$.

Let $\phi_t(z) = h^{-1}(e^{-t}h(z))$. Note that for $0 < t < \log 2$, $\phi_t(\mathbb{T})$ intersects $\mathbb{T}$ on a set of positive measure. Moreover, for $t > \log 2$, $\|\phi_t\|_\infty < 1$ and therefore $\phi_t(\mathbb{D})$ is included in a compact subset of $\mathbb{D}$ for all $t > \ln 2$. Figure 1 represents the image of φ_t for different values of t.

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