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CONTINUITY CHARACTERISATIONS OF DIFFERENTIABILITY OF LOCALLY LIPSCHITZ FUNCTIONS

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Recently David Preiss contributed a remarkable theorem about the differentiability of locally Lipschitz functions on Banach spaces which have an equivalent norm differentiable away from the origin. Using his result in conjunction with Frank Clarke's non-smooth analysis for locally Lipschitz functions, continuity characterisations of differentiability can be obtained which generalise those for convex functions on Banach spaces. This result gives added information about differentiability properties of distance functions.

For a continuous convex function ϕ on an open convex subset A of a real normed linear space X where ϕ is Gateaux differentiable on a dense subset D of A, it is known that ϕ is Gateaux (Fréchet) differentiable at $x \in A$ if and only if $\phi'(x_n)$ is weak* (norm) convergent for all $x_n \in D$ converging to x.

In [5, Corollary 2, p.64] it was shown that a similar result holds for a distance function d on a normed linear space X with uniformly Gateaux (uniformly Fréchet) differentiable norm.

Here we prove that such a continuity characterisation of differentiability applies with locally Lipschitz functions on a real Banach space with equivalent norm Gateaux differentiable away from the origin. This result in turn contributes further to our knowledge of differentiability properties of distance functions. But it is also of use in non-smooth optimisation relying as it does on the Clarke generalised subdifferential which is not in itself a good indicator of differentiability.

A real function ϕ on an open subset A of a normed linear space X is said to be *locally Lipschitz* on A if for each $x \in A$ there exists a $K > 0$ and a $\delta > 0$ such that

$$|\phi(y) - \phi(z)| \leqslant K \|y - z\| \text{ for all } y, z \in B(x; \delta).$$

Such a function ϕ is said to be *Gateaux differentiable* at $x \in A$ if there exists a continuous linear functional $\phi'(x)$ on X where, given $\varepsilon > 0$ and $\|y\| = 1$ there exists a $\delta(\varepsilon, x, y) > 0$ such that

$$\left| \frac{\phi(x + ty) - \phi(x)}{t} - \phi'(x)(y) \right| < \varepsilon \text{ when } 0 < |t| < \delta.$$

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The function ϕ is said to be *Fréchet differentiable* at x if there exists a $\delta(\varepsilon, x) > 0$ such that the inequality holds for all $\|y\| = 1$. Given a locally Lipschitz function ϕ on an open subset A of a normed linear space X, the *Clarke generalised subdifferential* of ϕ at $x \in A$ is

$$\partial\phi(x) = \{f \in X^*\colon f(y) \leqslant \limsup_{\substack{z \to x \\ t \to 0+}} \frac{\phi(z+ty) - \phi(z)}{t} \text{ for all } y \in X\}.$$

The function ϕ is said to be *strictly differentiable* at $x \in A$ if there exists a continuous linear functional $F(x)$ on X such that

$$\lim_{\substack{z \to x \\ t \to 0+}} \frac{\phi(z+ty) - \phi(z)}{t} = F(x)(y) \text{ for all } y \in X.$$

The function ϕ is said to be *uniformly strictly differentiable* at x if this limit is approached uniformly for all $\|y\| = 1$.

Now for a locally Lipschitz function ϕ, $\partial\phi(x)$ is singleton at $x \in A$ if and only if ϕ is strictly differentiable at x, [**2**, p.33].

To compute with Clarke's generalised subdifferential we need the following proposition.

Proposition 1. The Lebourg Mean Value Theorem, [**2**, p.41]. *For a locally Lipschitz function ϕ on an open set A in a normed linear space X, given $x, y \in A$ where $\{\lambda x + (1-\lambda)y : 0 \leqslant \lambda \leqslant 1\} \subseteq A$, there exists a $0 < \lambda_0 < 1$ such that*

$$\phi(y) - \phi(x) = f(y - x) \text{ for some } f \in \partial\phi(\lambda_0 x + (1-\lambda_0)y).$$

We now explore the relation between strict and uniformly strict differentiability and Gateaux and Fréchet differentiability.

Lemma 2. *For a locally Lipschitz function ϕ on an open subset A of a normed linear space X, if ϕ is strictly (uniformly strictly) differentiable at $x \in A$ then ϕ is Gateaux (Fréchet) differentiable at x and*

$$\lim_{\substack{z \to x \\ t \to 0+}} \frac{\phi(z+ty) - \phi(z)}{t} = \phi'(x)(y) \text{ for all } y \in X.$$

Proof: We may suppose that $x = 0$ and $\phi(0) = 0$ and that

$$\lim_{\substack{z \to x \\ t \to 0+}} \frac{\phi(z+ty) - \phi(z)}{t} = 0 \text{ for all } y \in X.$$

As both cases are proved with a similar argument we will only consider the uniformly strictly differentiable case.

Suppose that there exists an $r > 0$ and sequences $t_n \to 0$ and $\|y_n\| = 1$ such that

$$\frac{\phi(t_n y_n)}{t_n} > r \text{ for all } n.$$

By the Lebourg Mean Value Theorem, for each n there exists a $0 < \lambda_n < t_n$ such that

$$\frac{\phi(t_n y_n)}{t_n} = f_n(y_n) \text{ for some } f_n \in \partial\phi(\lambda_n y_n).$$

But there exists $\|z_n - \lambda_n y_n\| < 1/n$ and $0 < t'_n < 1/n$ such that

$$f_n(y_n) \leqslant \frac{(z_n + t'_n y_n) - \phi(z_n)}{t'_n} + \frac{1}{n}$$

from which we deduce that

$$\lim_{n\to\infty} \frac{\phi(z_n + t'_n y_n) - \phi(z_n)}{t'_n} \geqslant r,$$

which implies that ϕ is not uniformly strictly differentiable at $x = 0$. ◻

In general, if ϕ is Gateaux differentiable at $x \in A$ then $\phi'(x) \in \partial\phi(x)$, but $\partial\phi(x)$ is not necessarily singleton. So the converse for the strict differentiability case does not hold in general. That the converse for the uniformly strict differentiability case does not hold in general is shown by the following example of a locally Lipschitz function which is Fréchet differentiable and strictly differentiable but not uniformly strictly differentiable.

EXAMPLE 3. Consider the Banach sequence space $\left(\ell p, \|\ \|_p\right)$ with $1 < p < \infty$ and standard basis $\{e_1, e_2, \dots, e_n, \dots\}$. Given $K > 0$, we define a real function ϕ on ℓp by

$$\phi(te_n) = \begin{cases} 0 & \text{if } |t| \leqslant \frac{1}{n} \\ K\left(|t| - \frac{1}{n}\right) & \text{if } \frac{1}{n} < |t| < \frac{K}{nK-1} \\ \frac{1}{n}|t| & \text{if } \frac{K}{nK-1} \leqslant |t|. \end{cases}$$

and for

$$x = x_1 e_1 + x_2 e_x + \dots + x_n e_n + \dots \in \ell_p,$$

$$\phi(x) = \left(\sum (\phi(x_n e_n))^p\right)^{1/p}.$$

Now ϕ is locally Lipschitz at $x = 0$ since

$$\begin{aligned} |\phi(y) - \phi(z)| &= \left| \left(\sum (\phi(y_n e_n))^p\right)^{1/p} - \left(\sum (\phi(z_n e_n))^p\right)^{1/p} \right| \\ &\leqslant \left(\sum |\phi(y_n e_n) - \phi(z_n e_n)|^p\right)^{1/p} \quad \text{by Minkowski's inequality,} \\ &\leqslant K \left(\sum |y_n - z_n|^p\right)^{1/p} = K\|y - z\|_p. \end{aligned}$$

Also ϕ is Fréchet differentiable at $x = 0$ with $\phi'(0) = 0$ since, given $\varepsilon > 0$ there exists a $\nu > 1/\varepsilon$ such that for all $|t| < 1/\nu$ and all $\|y\|_p = 1$,

$$\left|\frac{\phi(ty)}{t}\right| \leqslant \left(\sum_{n=\nu}^{\infty}\left|\frac{ty_n}{nt}\right|^p\right)^{1/p} \leqslant \frac{\|y\|_p}{\nu} < \varepsilon.$$

Now ϕ is strictly differentiable at $x = 0$ since, for $\|y\|_p = 1$, given $\varepsilon > 0$ there exists a $\nu > 1/\varepsilon$ such that $\left(\sum\limits_{n=\nu}^{\infty}|y_n|^p\right)^{1/p} < \varepsilon$ and so for all $0 < t < 1/2\nu$ and $\|z\|_p < 1/2\nu$ we have

$$\begin{aligned}\left|\frac{\phi(x+ty)-\phi(z)}{t}\right| &\leqslant \frac{1}{t}\left|\left(\sum_{n=\nu}^{\infty}(\phi(z_n+ty_n)e_n)^p\right)^{1/p} - \left(\sum_{n=\nu}^{\infty}(\phi(z_ne_n))^p\right)^{1/p}\right| \\ &\leqslant K\left(\sum_{n=\nu}^{\infty}|y_n|^p\right)^{1/p} \qquad \text{by Minkowski's inequality.} \\ &\leqslant K\varepsilon.\end{aligned}$$

However, ϕ is not uniformly strictly differentiable at $x = 0$ since for all $0 < t < 1/(n(nK-1))$ we have

$$\frac{\phi\left(\frac{1}{n}e_n + te_n\right) - \phi\left(\frac{1}{n}e_n\right)}{t} = K.$$

□

For a continuous convex function ϕ on an open convex subset A of a normed linear space X, if ϕ is Fréchet differentiable at $x \in A$ then the subdifferential mapping $x \to \partial\phi(x)$ is upper semi-continuous at x, [**3**, p.147]; that is, given any open set $W \supseteq \partial\phi(x)$ there exists a $\delta > 0$ such that

$$\partial\phi(y) \subseteq W \text{ for all } \|y - x\| < \delta.$$

But further, if ϕ is strictly differentiable at $x \in A$ and the subdifferential mapping is upper semi-continuous at x then ϕ is Fréchet differentiable at x.

We show that for a locally Lipschitz function ϕ, uniformly strict differentiability can be similarly characterised by upper semi-continuity of the subdifferential mapping.

Theorem 4. *A locally Lipschitz function ϕ on an open subset A of a normed linear space X is uniformly strictly differentiable at $x \in A$ if and only if ϕ is strictly differentiable at x and the subdifferential mapping $x \to \partial\phi(x)$ is upper semi-continuous at x.*

Proof: If ϕ is uniformly strictly differentiable at $x \in A$ then $\partial\phi(x)$ is singleton and from Lemma 2 we have that ϕ is Fréchet differentiable at x. Suppose that the

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subdifferential mapping $x \to \partial\phi(x)$ is not upper semi-continuous at x. Then there exists an $r > 0$ and a sequence $\{x_n\}$ in A where $x_n \to x$ and $f_n \in \partial\phi(x_n)$ such that

$$\|f_n - \phi'(x)\| > r \text{ for all } n.$$

So there exists a sequence $\{y_n\}$, $\|y_n\| = 1$ such that

$$f_n(y_n) - \phi'(x)(y_n) > r \text{ for all n.}$$

For each x, since $f_n \in \partial\phi(x_n)$ there exists $z_n \in A$ where $\|z_n - x_n\| < 1/n$ and $0 < t_n < 1/n$ such that

$$f_n(y_n) < \frac{\phi(z_n + t_n y_n) - \phi(z_n)}{t_n} + \frac{1}{n}.$$

But then

$$\begin{aligned} r &< f_n(y_n) - \phi'(x)(y_n) \\ &< \frac{\phi(z_n + t_n y_n) - \phi(z_n)}{t_n} - \phi'(x)(y_n) + \frac{1}{n} \end{aligned}$$

which implies that ϕ is not uniformly strictly differentiable at x.

Conversely, suppose that ϕ is strictly differentiable at $x \in A$ and the subdifferential mapping $x \to \partial\phi(x)$ is upper semi-continuous at x. Then from Lemma 2, ϕ is Gateaux differentiable at x and given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|f_z - \phi'(x)\| < \varepsilon \text{ for all } f_z \in \partial\phi(z) \text{ where } \|z - x\| < \delta.$$

The Lebourg Mean Value Theorem implies that for given $\|y\| = 1$ and $z \in B(x;\delta/2)$ and $0 < t < \delta/2$,

$$\inf\{f_w(ty) \colon w \in B(x;\delta)\} \leqslant \phi(z + ty) - \phi(z) \leqslant \sup\{f_w(ty) \colon w \in B(x;\delta)\}.$$

So

$$\left| \frac{\phi(z + ty) - \phi(z)}{t} - \phi'(x)(y) \right| \leqslant 2\varepsilon \text{ for all } \|z - x\| < \frac{\delta}{2} \text{ and all } \|y\| = 1;$$

that is, ϕ is uniformly strictly differentiable at x. ∎

Theorem 4 reveals some important inherent differentiability properties of convex functions.

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COROLLARY 5. *For a continuous convex function ϕ on an open convex subset A of a normed linear space X, ϕ is strictly (uniformly strictly) differentiable at $x \in A$ if and only if it is Gateaux (Fréchet) differentiable at x.*

PROOF: The equivalence of Gateaux differentiability and strict differentiability is well-known, [4, p.122]. However, it is also known that if ϕ is Fréchet differentiable at $x \in A$ then this subdifferential mapping is upper semi-continuous at x, [4, p.147] so we deduce from Theorem 4 that ϕ is uniformly strictly differentiable at x. Lemma 2 completes the equivalence. ∎

Our final characterisation theorem depends on the significant advance in our knowledge of the differentiability properties of locally Lipschitz functions given in the following proposition.

PREISS' THEOREM 6. [6, Theorem 2.4]. *Let X be a Banach space with an equivalent norm Gateaux (Fréchet) differentiable away from the origin. Then every locally Lipschitz function ϕ on an open subset A of X is Gateaux (Fréchet) differentiable on a dense subset D of A.*

In both cases, and for the appropriate derivative, for every open ball B in A and every $y, z \in B$

$$\inf\{\phi'(x)(y-z) : x \in B \cap D\} \leqslant \phi(y) - \phi(z) \leqslant \sup\{\phi'(x)(y-z) : x \in B \cap D\}.$$

THE CHARACTERISATION THEOREM 7. *Let X be a Banach space with an equivalent norm Gateaux differentiable away from the origin. Then a locally Lipschitz function ϕ on an open subset A of X is strictly (uniformly strictly) differentiable at $x \in A$ if and only if $\phi'(x_n)$ is weak* (norm) convergent as $x_n \to x$ for all $x_n \in D$ the dense subset of A on which ϕ is Gateaux differentiable.*

PROOF: It follows from Preiss' Theorem, that $\partial\phi(x)$ is the weak* closed convex hull of the weak* cluster points of $\{\phi'(x_n)\}$ for $x_n \in D$ as $x_n \to x$. So if $\phi'(x_n)$ is weak* convergent as $x_n \to x$ then $\partial\phi(x)$ is singleton and so ϕ is strictly differentiable.

Conversely, if ϕ is strictly differentiable at $x \in A$ then $\phi'(x_n)$ is weak* convergent to $\phi'(x)$ as $x_n \to x$.

We now turn to the uniformly strictly differentiable case. Suppose that $\phi'(x_n)$ is norm convergent as $x_n \to x$. Then ϕ is Gateaux differentiable at x and given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|\phi'(y) - \phi'(x)\| < \varepsilon \text{ for } y \in D \text{ and } \|y - x\| < \delta.$$

From Preiss' Theorem, for given $\|y\| = 1$ and $z \in B(x; \delta/2)$ and $0 < t < \delta/2$

$$\inf\{\phi'(w)(ty) : w \in D \cap B(x;\delta)\} \leqslant \phi(z+ty) - \phi(z) \leqslant \sup\{\phi'(w)(ty) : w \in D \cap B(x;\delta)\}.$$

So

$$\left|\frac{\phi(z+ty)-\phi(z)}{t}-\phi'(x)(y)\right|\leqslant 2\varepsilon \text{ for all } \|z-x\|<\frac{\delta}{2} \text{ and } 0<t<\frac{\delta}{2} \text{ and } \|y\|=1;$$

that is, ϕ is uniformly strictly differentiable at x.

Conversely, suppose that ϕ is uniformly strictly differentiable at $x \in A$. Then by Lemma 2, ϕ is Fréchet differentiable at x and from Theorem 4, the subdifferential mapping $x \to \partial\phi(x)$ is upper semi-continuous at x. But this implies that $\phi'(x_n)$ is norm convergent to $\phi'(x)$ as $x_n \to x$ for all $x_n \in D$. ∎

Notice that, for a Banach space X with equivalent norm Fréchet differentiable away from the origin (or indeed for any Asplund space), given a locally Lipschitz function ϕ on an open subset A of X, to obtain uniformly strict differentiability of ϕ at $x \in A$ we need only have $\phi'(x_n)$ norm convergent for $x_n \to x$ at points $x_n \in D$ the dense subset of A where ϕ is Fréchet differentiable.

We now explore the consequences of this characterisation for distance functions. Given a non-empty closed set K in a normed linear space X the distance function d generated by K is defined by

$$d(x)=d(x,K)\equiv\inf\{\|x-y\|:y\in K\}.$$

1. It is known that on a normed linear space X with uniformly Gateaux differentiable norm, a distance function d as generated by a non-empty closed set K is strictly differentiable at $x \in X\setminus K$ if and only if it is Gateaux differentiable at x, [**1**, p.525]. However, from the continuity characterisation of differentiability of distance functions, [**5**, p.64] we deduce from Theorem 7 that when X is a Banach space with uniformly Fréchet differentiable norm, it is uniformly strictly differentiable at $x \in A$ if and only if it is Fréchet differentiable at x.

2. But further, Theorem 7 improves the continuity characterisation of differentiability of distance functions given in [**5**, p.64] by establishing that in a Banach space with equivalent norm Gateaux differentiable away from the origin, d is Gateaux (Fréchet) differentiable at $x \in X\setminus K$ if $d'(x_n)$ is weak* (norm) convergent for all $x_n \to x$ where $x_n \in D$ the dense subset of A on which d is Gateaux differentiable.

In paper [**3**], Preiss' Theorem was used to establish generic differentiability properties for certain locally Lipschitz functions. Using Theorem 7, Corollaries 2.4 and 2.5, [**3**, p.44] can be restated in stronger form. To pursue this we need the following result.

LEMMA 8. *Consider a Banach space X with equivalent norm Gateaux (Fréchet) differentiable away from the origin and a locally Lipschitz function ϕ on an open subset*

A of X with the property that wherever ϕ is Gateaux (Fréchet) differentiable it is strictly differentiable. At any point $x \in A$ where the real mapping Ψ on A defined by

$$\Psi(x) = \inf\{\|f\| : f \in \partial\phi(x)\}$$

is continuous, we have $\|f_n\| \to \|f\|$ as $x_n \to x$ for all $f_n \in \partial\phi(x_n)$ and $f \in \partial\phi(x)$.

PROOF: From the weak* upper semi-continuity of the subdifferential mapping $x \to \partial\phi(x)$ and the weak* lower semi-continuity of the dual norm we have that Ψ is lower semi-continuous. But also the local Lipschitz mapping λ on A defined by

$$\lambda(x) = \lim_{\delta\to 0} \sup\{\frac{|\phi(y) - \phi(z)|}{\|y - z\|} : y, z \in B(x;\delta),\ y \neq z\}$$

is upper semi-continuous. So as $x_n \to x$,

$$\begin{aligned} \lambda(x) \geqslant \limsup \lambda(x_n) &\geqslant \limsup \|f_n\| \text{ for all } f_n \in \partial\phi(x_n) \\ &\geqslant \liminf \|f_n\| \geqslant \liminf \Psi(x_n) \geqslant \Psi(x). \end{aligned}$$

But at $x \in A$ where Ψ is continuous, we have from [**3**, Theorem 2.3(c), p.43] that $\lambda(x) = \Psi(x) = \|f\|$ for all $f \in \partial\phi(x)$. We conclude that at such a point x,

$$\|f_n\| \to \|f\| \text{ as } x_n \to x \text{ for all } f_n \in \partial\phi(x_n) \text{ and } f \in \partial\phi(x).$$

□

A normed linear space X is said to have *sequentially weak* Kadec dual norm* if whenever a sequence $\{f_n\}$ is weak* convergent to f in X^* and $\|f_n\| \to \|f\|$ then $\{f_n\}$ is norm convergent to f. A normed linear space with a rotund sequentially weak* Kadec dual norm has norm Fréchet differentiable away from the origin, [**4**, p.152].

We are now ready to present the generic differentiability properties of locally Lipschitz functions.

COROLLARY 9. *On a Banach space X which can be equivalently renormed to have a rotund dual (and a sequentially weak* Kadec dual norm), a locally Lipschitz function ϕ on an open subset A of X with the property that wherever ϕ is Gateaux (Fréchet) differentiable it is strictly differentiable is strictly (uniformly strictly) differentiable on a dense G_δ subset of A.*

PROOF: Consider X so renormed. The strictly differentiable case is given in [**3**, Theorem 2.3, p.42] so we concern ourselves with the uniformly strictly differentiable case. At a point x of continuity of Ψ we have that ϕ is strictly differentiable at x. From Theorem 7, we have that $\phi'(x_n)$ is weak* convergent to $\phi'(x)$ and from Lemma 8, that $\|\phi'(x_n)\| \to \|\phi'(x)\|$ as $x_n \to x$ for all $x_n \in D$ the dense subset of A where ϕ

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is Fréchet differentiable. But since the dual norm is sequentially weak* Kadec we have that $\phi'(x_n)$ is norm convergent to $\phi'(x)$ as $x_n \to x$ for all $x_n \in D$. From Theorem 7, we deduce that ϕ is uniformly strictly differentiable at x. But the set of points of continuity of Ψ is a dense G_δ subset of A. ∎

On a finite dimensional normed linear space X a locally Lipschitz function has the following satisfying properties.

THEOREM 10. *For a locally Lipschitz function ϕ on an open subset A of a finite dimensional normed linear space X_n,*

(i) *if ϕ is Gateaux differentiable at x then it is Fréchet differentiable at x, and*

(ii) *if ϕ is strictly differentiable at x then it is uniformly strictly differentiable at x.*

PROOF: (i) Consider the unit sphere in X_n and for any given $\varepsilon > 0$ consider a cover by open balls centres $y_k \in X$, $\|y_k\| = 1$ with radius ε. Since X_n is finite-dimensional, the unit sphere is compact, so it has a finite subcover by such balls with centres $y_1, y_2, \dots y_m$. Since ϕ is Gateaux differentiable at x, given y_k, where $k \in \{1, 2, \dots, m\}$, there exists a $\delta_k(\varepsilon, y_k) > 0$ such that

$$\left|\frac{\phi(x+ty_k)-\phi(x)}{t} - \phi'(x)(y_k)\right| < \varepsilon \text{ for all } 0 < |t| < \delta_k.$$

Since ϕ is locally Lipschitz there exists a $K > 0$ and a $\delta'(x) > 0$ such that

$$|\phi(x+ty) - \phi(x+ty_k)| \leqslant K\,|t|\,\|y-y_k\| \text{ for all } y_k,\ k \in \{1, 2, \dots, m\} \text{ and } 0 < |t| < \delta'.$$

Therefore, given $\|y\| = 1$,

$$\begin{aligned}\left|\frac{\phi(x+ty)-\phi(x)}{t} - \phi'(x)(y)\right| &\leqslant \left|\frac{\phi(x+ty)-\phi(x)}{t} - \frac{\phi(x+ty_k)-\phi(x)}{t}\right| \\ &\quad + \left|\frac{\phi(x+ty_k)-\phi(x)}{t} - \phi'(x)(y_k)\right| + |\phi'(x)(y) - \phi'(x)(y_k)| \\ &\leqslant K\,\|y-y_k\| + \varepsilon + \|\phi'(x)\|\,\|y-y_k\| \text{ when } 0 < |t| < \min\{\delta', \delta_k\} \\ &< (K + \|\phi'(x)\| + 1)\varepsilon \text{ for } y_k \text{ chosen such that } \|y-y_k\| < \varepsilon.\end{aligned}$$

and this holds for all $\|y\| = 1$ where $0 < |t| < \min\{\delta', \delta_1, \dots, \delta_m\}$. That is, ϕ is Fréchet differentiable at x.

(ii) Since X_n is a finite dimensional normed linear space it has an equivalent norm Gateaux differentiable away from the origin. From Theorem 7, we have that if ϕ is strictly differentiable at $x \in A$ then $\phi'(x_n)$ is weak* convergent to $\phi'(x)$ as $x_n \to x$

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for all $x_n \in D$ the dense subset of A on which ϕ is Gateaux differentiable. But again since X_n is finite-dimensional we have that $\phi'(x_n)$ is norm convergent to $\phi'(x)$ which by Theorem 7, implies that ϕ is uniformly strictly differentiable at x. □

Theorem 10(i) is a special case of a more general result given in [7, Theorem 3.3.3, p.43]. We note that Theorem 10 (i) does not hold under the less restrictive condition that ϕ is Lipschitz at x.

References

[1] J.M. Borwein, S.P. Fitzpatrick and J.R. Giles, 'The differentiability of real functions on normed linear spaces using generalised subgradients', *J. Math. Anal. Appl.* **128** (1987), 512–534.

[2] F.H. Clarke, *Optimisation and non-smooth analysis*, Canad. Math. Soc. Ser. Monographs Adv. Texts (Wiley, New York, 1983).

[3] G. de Barra, Simon Fitzpatrick and J.R. Giles, 'On generic differentiability of locally Lipschitz functions on Banach spaces', *Proc. Centre Math. Anal. Austral. Nat. Univ.* **20** (1988), 39–49.

[4] J.R. Giles, *Convex analysis with application in the differentiation of convex functions*, Res. Notes in Math.**58** (Pitman, Boston, London, 1982).

[5] J.R. Giles, 'A distance function property implying differentiability', *Bull. Austral. Math. Soc* **39** (1989), 59–70.

[6] D. Preiss, 'Differentiability of Lipschitz functions on Banach spaces', *J. Func. Anal.* (to appear).

[7] Bernice Sharp, *New directions in convex analysis, the differentiability of convex functions on topological linear spaces* (Ph.D. Thesis, University of Newcastle, Newcastle, N.S.W., 1988).

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