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INVARIANT NEUTRAL SUBSPACES FOR SYMMETRIC AND SKEW REAL MATRIX PAIRS

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ABSTRACT. Real matrix pairs (A, H) satisfying $\det H \neq 0$, $H^T = \xi H$, and $HA = \eta A^T H$, where ξ, η take the values +1 or -1, are considered. It is shown that maximal A-invariant H-neutral subspaces have the same dimension (depending on ξ and η), called the *order of neutrality* of the pair (A, H). The order of neutrality of definitizable pairs is investigated. In particular, this concept is used to obtain lower bounds for the number of pure imaginary eigenvalues of low rank perturbations of definitizable pairs when $(\xi, \eta) = (1, -1)$ and when $(\xi, \eta) = (-1, -1)$.

1. **Introduction.** Consider a nonsingular hermitian $H \in \mathbb{C}^{n\times n}$, the space of complex $n \times n$ matrices, and the (generally) indefinite scalar product $[\,.\,,\,.\,]$ defined on $\mathbb{C}^n$ by

$$[x, y] = (Hx, y) = y^* Hx$$

for all $x, y \in \mathbb{C}^n$ (and the star denotes conjugate transpose). If $A \in \mathbb{C}^{n\times n}$ and satisfies $(HA)^* = HA$, A is said to be *H-selfadjoint*. A subspace $\mathcal{S}$ of $\mathbb{C}^n$ is A-invariant if $A\mathcal{S} \subseteq \mathcal{S}$ and is said to be *H-neutral* if $[x, y] = 0$ for all $x, y \in \mathcal{S}$. It has been shown in the recent paper [4] that all maximal A-invariant and H-neutral subspaces have the same dimension, say $\gamma(A; H)$, called the *order of neutrality* of A with respect to H. This number is not less than the number of eigenvalues of A in the open upper half of the complex plane plus the number of distinct real eigenvalues, λ, with an isotropic eigenvector x (*i.e.* for which $[x, x] = 0$, in which case λ is necessarily a multiple eigenvalue). When $\gamma(A; H) = 0$ all eigenvalues of A are real and "definite", in the sense that they have no isotropic eigenvector, and A is said to be "definitizable with respect to H" (see also reference [5]).

These ideas can be applied to find lower bounds for the number of real eigenvalues of an H-selfadjoint matrix A when A, H are low-rank perturbations of a pair A_0, H_0 and A_0 is definitizable with respect to H_0. There are also applications to estimates for the degrees of factors in symmetric factorizations of polynomial and rational matrix functions.

Now this discussion has counter-parts in the context of *real* spaces and matrices, and these counter-parts are the subject of the present paper. In the complex case analysis depends on the fine-structure of multiple eigenvalues and their spectral subspaces, and

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this information is crystallized in the canonical form for transformations of pairs:

$$(A, H) \longrightarrow (S^{-1}AS, S^*HS). \tag{1.1}$$

For real pairs there are significant differences in these canonical structures giving rise to different evaluations for $\gamma(A; H)$ and different conclusions for some of the applications.

The analysis over $\mathbb{R}$ is further complicated by the presence of four cases, which are all included in the present work. Thus, we have $A, H \in \mathbb{R}^{n\times n}$ with H nonsingular and admit

$$H^T = \pm H, \quad HA = \pm A^T H.$$

The two positive signs give the immediate restriction of the analysis over $\mathbb{C}$ described above. The two negative signs involve "symplectic" symmetries arising in the analysis of real Riccati equations, for example, and the $(\ ,+)$, $(+, \ )$ pairs also have their applications (see [8], for example).

The canonical forms under the transformation (1.1) for the four separate cases are significantly different and are collected under one heading as Theorem A.1 of the appendix to this paper. This uniform format facilitates comparisions between the four cases, although all of this information is available in the literature. In Section 2 the central result is formulated: that maximal A-invariant H-neutral subspaces have the same dimension. Then Sections 3–6 are devoted to proofs of this fact in the four cases described above, together with evaluation of the corresponding order of neutrality, $\gamma(A; H)$.

In Section 7 the relationship of the condition $\gamma(A; H) = 0$ with the definitizable property of A with respect to H is examined, and in Section 8 some spectral properties are outlined for low rank perturbations of A and H when A is definitizable with respect to H.

Section 9 contains a brief description of some problems of applied analysis giving rise to real pairs discussed in this paper. Subsequent papers will deal with the relationship between the degrees of factors appearing in symmetric factorizations of polynomial and rational matrix functions, and the order of neutrality defined in terms of their realizations.

2. **Maximal invariant neutral subspaces.** For $\xi = \pm 1$ and $\eta = \pm 1$, let $L_n(\xi, \eta)$ be the class of all ordered pairs of $n \times n$ real matrices (A, H) with H invertible, $H^T = \xi H$, and $HA = \eta A^T H$. When $\xi = -1$ we assume always that n is even; otherwise H fails to be invertible.

If $H^T = H$ or $H^T = -H$, and H is invertible, a subspace $\mathcal{M} \subseteq \mathbb{R}^n$ is said to be H-neutral if $x^T Hy = 0$ for all $x, y \in \mathcal{M}$. Note that, in the partial ordering on subspaces determined by inclusion, the following well-known result holds (see [3], for example, where the corresponding result for complex spaces is obtained).

PROPOSITION 2.1. *If $H^T = H$ and $\det H \neq 0$ then the maximal dimension of an H-neutral subspace is*

$$\nu(H) := \min\{\#\textit{ of positive eigenvalues of } H, \#\textit{ of negative eigenvalues of } H\}.$$

where the eigenvalues are counted with multiplicities. If $H^T = -H$ and $\det H \neq 0$ then the maximal dimension of an H neutral subspace is $n/2$.

Now let $\mathrm{IN}(A,H)$ be the set of subspaces of $\mathbb{R}^n$ which are both A-invariant and H-neutral. The first theorem is an important property for our investigations and applies for any of the four distinct pairs ξ, η.

THEOREM 2.2. *All maximal elements in* $\mathrm{IN}(A,H)$ *have the same dimension.*

The proof of Theorem 2.2 will be given in Sections 3–6 for the four different pairs (ξ,η). Furthermore, the dimension of the maximal elements will be obtained and will be seen to depend on the choice of ξ and η. The proofs depend on the detailed structure of the four pairs of canonical forms for H and A which are presented in Appendix A.

3. **The case $L_n(-1,1)$.** We start with the case giving the least technical difficulty.

THEOREM 3.1. *Let* $(A,H) \in L_n(-1,1)$. *Then the dimension of every maximal A-invariant H-neutral subspace is equal to* $\frac{n}{2}$, *i.e.* $\gamma(A;H) = \frac{n}{2}$.

PROOF. Let $\mathcal{M}$ be a maximal A-invariant H-neutral subspace. By Proposition 2.1, $\dim \mathcal{M} \leq \frac{n}{2}$. Assume $d := \dim \mathcal{M} < \frac{n}{2}$, and consider the subspace

$$\mathcal{M}^{[\perp]} = \{x \in \mathbb{R}^n : x^T H y = 0 \text{ for all } y \in \mathcal{M}\}$$

(the "H-orthogonal companion" of $\mathcal{M}$). As $\mathcal{M}$ is H-neutral, clearly we have $\mathcal{M} \subseteq \mathcal{M}^{[\perp]}$. Furthermore, $\dim \mathcal{M}^{[\perp]} = n-d$ (because $\mathcal{M}^{[\perp]}$ coincides with the (euclidean) orthogonal complement to the d-dimensional subspace $H\mathcal{M}$). The subspace $\mathcal{M}^{[\perp]}$ is also A-invariant. Indeed, if $x \in \mathcal{M}^{[\perp]}$ then for every $y \in \mathcal{M}$ we have

$$(Ax)^T H y = x^T A^T H y = x^T H(Ay) = 0$$

because $\mathcal{M}$ is A-invariant. Hence $Ay \in \mathcal{M}$. So, choosing a (euclidean) orthonormal basis $\{y_1, \ldots, y_n\}$ in $\mathbb{R}^n$, where $y_1, \ldots, y_d$ form a basis in $\mathcal{M}$, $y_{d+1}, \ldots, y_{n-d}$ form a basis in the euclidean orthogonal complement to $\mathcal{M}$ in $\mathcal{M}^{[\perp]}$, and $y_{n-d+1}, \ldots, y_n$ form a basis in the euclidean orthogonal complement to $\mathcal{M}^{[\perp]}$ in $\mathbb{R}^n$, we represent A in the following form

$$A' = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}. \tag{3.1}$$

The corresponding representation for H is given by the matrix $H' = [y_i^T H y_j]_{i,j=1}^n$, which has the form

$$H' = \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ -H_{13}^T & -H_{23}^T & H_{33} \end{bmatrix}. \tag{3.2}$$

The matrix H_{22} is skew-symmetric and invertible (because H' has these properties), and $H_{22}A_{22} = A_{22}^T H_{22}$.

We now apply the canonical form (Case 1 of Theorem A.1) to the pair $(A_{22}, H_{22}) \in L_{n-2s}(-1,1)$. The canonical form clearly shows that there exists a one, or two-dimensional

A_{22}-invariant H- neutral subspace, call it $\mathcal{M}'$. Indeed, for A_i, H_i in the form (A.2) one can take $\mathcal{M}' = \operatorname{span}\{e_{n_1}\}$, for A_i, H_i in the form (A.3) one can take

$$\mathcal{M}' = \operatorname{span}\{e_{2n_1}, e_{2n_1-1}\}.$$

(Here and elsewhere e_k denotes the column vector whose k-th component is 1 and all other components are 0; the number of components is understood from the context.) Embed this subspace in $\mathbb{R}^n$ in the natural way to obtain a subspace $\mathcal{M}''$ in $\mathbb{R}^n$, (*i.e.* dilate the vectors of $\mathcal{M}'$ with zeros). Then the subspace $\mathcal{M} + \mathcal{M}''$ is clearly A'=invariant, H'-neutral and strictly bigger than $\mathcal{M}$. This contradicts the maximality of $\mathcal{M}$. ∎

4. **The case $L_n(1,1)$.** For a real eigenvalue ν_j of a real matrix A, let $\mathcal{R}(A;\nu_j)$ be the corresponding spectral subspace. For a non-real eigenvalue pair $\lambda_j \pm i\mu_j$ of A we write $\mathcal{R}(A;\lambda_j \pm i\mu_j)$ for the spectral subspace associated with the *two* eigenvalues $\lambda_j \pm i\nu_j$. Then we may write

$$\mathbb{R}^n = \sum_{j=1}^{p} \mathcal{R}(A;\lambda_j \pm i\mu_j) + \sum_{j=1}^{q} \mathcal{R}(A;a_j)$$

where $\lambda_j + i\mu_j$, $j = 1,2,\ldots,p$, are all the distinct eigenvalues of A in the open upper halfplane, and $a_1, a_2, \ldots, a_q$ are all the distinct real eigenvalues of A. Note also that, for a real eigenvalue a_j, $m_j := \dim \mathcal{R}(A;a_j)$ is the algebraic multiplicity of a_j and, for a non-real eigenvalue $\lambda_j + i\mu_j$ the algebraic multiplicity is

$$m_j = \frac{1}{2}\dim \mathcal{R}(A;\lambda_j \pm i\mu_j).$$

For a non-real eigenvalue we also define

$$\alpha_j = \begin{cases} m_j & \text{if } m_j \text{ is even} \\ m_j - 1 & \text{if } m_j \text{ is odd.} \end{cases} \tag{4.1}$$

Now let $(A,H) \in L_n(1,1)$. For a real eigenvalue a_i let r_i be the dimension of a maximal H-neutral subspace in $\mathcal{R}(A;a_i)$. Using Proposition 2.1 this can be expressed as

$$r_i = \nu([y_j^T H y_k]_{j,k=1}^{m_i})$$

where $y_1, \ldots, y_m$ is an orthonormal basis for $\mathcal{R}(A;a_i)$.

The number r_i can also be expressed in terms of the parameters defining the canonical blocks (A.4) associated with the real eigenvalue a_i. As these blocks are the same for hermitian pencils over $\mathbb{C}^n$ the evaluation carried out in Theorem 3.1 of [4] applies. Let p_j be the number of blocks J_{n_j} in (A.4) of size j, and let $p_j = p_j^+ + p_j^-$ where p_j^+, p_j^- are the numbers of corresponding ϵ_j's that equal +1 and −1, respectively. Also, define $\mu_i = \sum_{j=1}^{p} [\frac{1}{2}j] p_j$. Then

$$r_i = \mu_i + \min\Big(\sum_r p_{2r-1}^+, \sum_r p_{2r-1}^-\Big) \tag{4.2}$$

and is called the *order of neutrality* of a_i.

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THEOREM 4.1. *Let $(A, H) \in L_n(1,1)$ and let the numbers α_j, r_i be defined as above (equations (4.1) and (4.2)). Then every maximal A- invariant H-neutral subspace has dimension*

$$\gamma(A;H) = \sum_{j=1}^{p} \alpha_j + \sum_{i=1}^{q} r_i \tag{4.3}$$

PROOF. Every A-invariant subspace is a direct sum of its intersections with $\mathcal{R}(A;\lambda_j \pm i\mu_j)$ and $\mathcal{R}(A;\gamma_j)$. Indeed, it is easily seen from the canonical forms of (A.4) and (A.5) that these subspaces are H-orthogonal. Consequently, it may be assumed without loss of generality that either, (1) A has only one eigenvalue (possibly of high multiplicity) and it is real or, (2) A has only two eigenvalues and they form a non-real conjugate pair.

For case (1) let ν be the dimension of a maximal H-neutral subspace and $\mathcal{M}$ be a maximal A-invariant H-neutral subspace. Let $d = \dim \mathcal{M} \leq \nu$. Suppose $d < \nu$ and proceed as in the proof of Theorem 3.1 to obtain representations A' and H' as in (3.1), (3.2) with $(A_{22}, H_{22}) \in L_{n-2d}(1,1)$. Clearly, $d < \nu$ implies that H_{22} is indefinite. But then the canonical forms of (A.4) for (A', H') show that there is a nonzero A_{22}-invariant and H_{22}-neutral vector and this would admit the dilation of $\mathcal{M}$ to a strictly bigger A'-invariant and H'-neutral subspace contradicting the maximality of $\mathcal{M}$. Hence $d = \dim \mathcal{M} = \nu$.

Now consider case (2) above. In this case every A-invariant subspace has even dimension. Let $\mathcal{M}$ be a maximal A-invariant and H-neutral subspace and suppose that $d := \dim \mathcal{M} < m_0$ where we define

$$m_0 = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is divisible by 4,} \\ \frac{1}{2}n - 1 & \text{otherwise.} \end{cases}.$$

Thus, both d and m_0 are even. Again, proceed as in the proof of Theorem 3.1 to obtain representations (3.1) and (3.2). Now A_{22} and H_{22} have size $n - 2d$ and, because $d < m_0 \leq \frac{1}{2}n$ and d is even, $n - 2d \geq 4$. Now we obtain a contradiction, and prove the theorem, by applying the following lemma (note that α_j of (4.1) and (4.3) is $\frac{1}{2}n$ in this context). ∎

LEMMA 4.2. *Let $(A, H) \in L_n(1,1)$ and assume that $\sigma(A) = \{\lambda + i\mu, \lambda - i\mu\}$ where $\mu > 0$, $\lambda \in \mathbb{R}$. Assume also that $n \geq 4$. Then there exists a two-dimensional A-invariant H-neutral subspace.*

PROOF. Without loss of generality it may be assumed that A and H have the canonical forms of Case 2 in Theorem A.1. Say,

$$A = \bigoplus_{j=1}^{p} J_{n_j}\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}, \quad H = \bigoplus_{j=1}^{p} K_{2n_j}.$$

If at least one n_j exceeds one, say $n_1 > 1$, define

$$\mathcal{M}_0 = \operatorname{span}\{e_{2n_1}, e_{2n_1-1}\},$$

and verify that $\mathcal{M}_0$ has the desired properties.

If, on the other hand, all n_j are equal to one then $n \geq 4$ implies $p \geq 2$. Now define

$$\mathcal{M}_0 = \operatorname{span}\{e_1 + e_4, e_2 - e_3\}.$$ ∎

5. **The case** $L_n(1,-1)$. When (A,H) is in $L_n(1,-1)$ the canonical forms of Case 3 in Theorem A.1 suggest that we first consider separately the zero eigenvalue (when it exists) and any pure imaginary (nonzero) eigenvalues. As before, $\nu(X)$ will denote the dimension of a maximal X-neutral subspace for a real symmetric or skew-symmetric matrix X.

For a pair of pure imaginary eigenvalues $\pm ib$ of A $(b>0)$, let $y_1,\ldots,y_p$ be a real orthonormal basis in $\mathcal{R}(A;\pm ib)$ and write

$$\nu(\pm ib)=\nu([y_j^T H y_k]_{j,k=1}^p).$$

This number can be determined by careful examination of the formula for H_i in (A.8). It is found that, if J is the set of indices j for which n_j is odd (in (A.8)), then

$$\nu(\pm ib)=\frac{1}{2}\dim\mathcal{R}(A;\pm ib)-\Big|\sum_J\epsilon_j\Big|. \tag{5.1}$$

When $0\in\sigma(A)$, let $z_1,\ldots,z_p$ be a real orthonormal basis in $\mathcal{R}(A;0)$ and define

$$\nu(0)=\nu([z_j^T H z_k]_{j,k=1}^p).$$

This number can be determined by examination of equation (A.6). It is found that, with $\epsilon_1,\ldots,\epsilon_p$ as in (A.6),

$$\nu(0)=\frac{1}{2}\dim\mathcal{R}(A;0)-\frac{1}{2}\Big|\sum_{j=1}^p\epsilon_j\kappa_j\Big| \tag{5.2}$$

where $\kappa_j=+1$, or $\kappa_j=-1$ according as $2n_j+1$ is an odd integer of the form $4k+1$, or $4k-1$, respectively. Observe that, although $\dim\mathcal{R}(A;0)$ can be odd, $\nu(0)$ is always a nonnegative integer.

THEOREM 5.1. *Let $(A,H)\in L_n(1,-1)$ and $\nu(\pm ib_j)$, $\nu(0)$ be defined as in (5.1) and (5.2). If $\mathcal{S}$ is the spectral subspace of A corresponding to all eigenvalues of A not lying on the imaginary axis, then the dimension of all maximal A-invariant H-neutral subspaces is*

$$\begin{aligned}\gamma(A;H)&=\nu(0)+\sum_{k=1}^r\nu(\pm ib_k)+\frac{1}{2}\dim\mathcal{S}\\&=\frac{1}{2}n-\sum_{k=1}^r\Big|\sum_{J_k}\epsilon_j\Big|-\frac{1}{2}\Big|\sum_{j=1}^p\epsilon_j\kappa_j\Big|.\end{aligned} \tag{5.3}$$

Of course the term $\nu(0)$ does not appear in the formula if $0\notin\sigma(A)$ and similarly for the other terms. Observe also that, from the canonical forms (A.6)–(A.9), a nonzero number $a+ib\in\sigma(A)$ if and only if $-(a+ib)\in\sigma(A)$ and both eigenvalues have the same partial multiplicities. Consequently, $\dim\mathcal{S}$ is even and formula (5.3) is well-defined.

PROOF. The line of argument is that of Theorem 4.1 in the context of an H- orthogonal decomposition $\mathbb{R}^n = \mathcal{S}_0 \dot{+} \mathcal{S}_1 \dot{+} \mathcal{S}$ where $\mathcal{S}_0$, $\mathcal{S}_1$ are spectral subspaces associated with the zero eigenvalue of A, and the nonzero pure imaginary eigenvalues, respectively. It will be shown that a maximal A-invariant H- neutral subspace in $\mathcal{S}_0$ is, in fact, maximal H-neutral in $\mathcal{S}_0$, and similarly for $\mathcal{S}_1$ and $\mathcal{S}$.

In the case of subspace $\mathcal{S}$ equations (A.7) and (A.9) apply and our assertion is obvious. In the case of $\mathcal{S}_1$ it is sufficient to verify our assertion for just one pair of nonzero pure-imaginary eigenvalues $\pm ib$.

The proof now follows the same line as the proof of Theorem 4.1 and is based on the following lemma. ∎

LEMMA 5.2. *Let $(A,H) \in L_n(1,-1)$ and assume that either A has only the zero eigenvalue, or A has just one pair of eigenvalues $\pm ib$ where $b > 0$. If H is indefinite then there exists a nonzero A-invariant H-neutral subspace $\mathcal{M}_0$.*

PROOF. Suppose $\sigma(A) = \{0\}$. It is apparent from equation (A.6) that, if $q \geq 1$ or one of $n_1, \ldots, n_p$ is positive then such a subspace $\mathcal{M}_0$ exists. Similarly, if $\sigma(A) = \{\pm ib\}$ and one of $n_1, \ldots, n_p$ exceeds one in (A.8), such a subspace $\mathcal{M}_0$ can be constructed using (A.8).

These observations leave only two cases to be considered. They derive from (A.6) and (A.8), respectively.

(i) $A = 0$ and $H = \operatorname{diag}[\epsilon_1, \ldots, \epsilon_p]$ with $\epsilon_j = \pm 1$ and not all equal.

(ii) $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$, $H = \epsilon_1 I_2 \oplus \cdots \oplus \epsilon_p I_2$ where $b > 0$, $\epsilon_j = \pm 1$ and not all ϵ_j's are equal.

In case (i) we may assume $\epsilon_1 = -\epsilon_2$ and choose

$$\mathcal{M}_0 = \operatorname{span}\{e_1 + e_2\}.$$

In case (ii) we may also assume $\epsilon_1 = -\epsilon_2$ and then take

$$\mathcal{M}_0 = \operatorname{span}\{e_1 + e_4, e_2 - e_3\}.$$ ∎

6. **The case $L_n(-1,-1)$.** Our argument follows the now familiar pattern, but using canonical forms (A.10)–(A.13) for a pair $(A,H) \in L_n(-1,-1)$. Consider equations (A.12) first of all, and note that $b_k > 0$, (it is convenient to replace i by k). Let the eigenvalue ib_k have partial multiplicities $\{n_j\}$. Let J be the set of all indices j for which n_j is odd, and define the nonnegative integer

$$m(A; \pm ib_k) = \frac{1}{2} \dim \mathcal{R}(A; \pm ib_k) - \Big|\sum_J \epsilon_j\Big|. \tag{6.1}$$

THEOREM 6.1. *Let $(A,H) \in L_n(-1,-1)$, define $m(A; \pm ib_k)$ as in (6.1) and let $\mathcal{S}$ be the spectral subspace of A corresponding to all eigenvalues of A except the nonzero* *pure imaginary eigenvalues. Then every maximal A-invariant H-neutral subspace has dimension equal to*

$$\gamma(A;H) := \sum_{k=1}^{r} m(A;\pm ib_k) + \frac{1}{2}\dim \mathcal{S} = \frac{1}{2}n - \sum_{k=1}^{r}\Big|\sum_{J_k}\epsilon_j\Big| \tag{6.2}$$

PROOF. It follows from the structure of the canonical forms (A.10)–(A.13) that we can consider each of the following four cases separately: (1) $\sigma(A) = \{0\}$; (2) $\sigma(A) = \{a, -a\}$, where a is real and positive; (3) $\sigma(A) = \{\pm ib\}$, where b is real and positive; (4) $\sigma(A) = \{a+ib, a-ib, -a+ib, -a-ib\}$, where $a > 0$, $b > 0$.

In cases (1), (2) and (4) one argues as in the proof of Theorem 4.1, using the fact (easily observed from the canonical form) that if H is indefinite and one of (1), (2) or (4) holds, then there exists a nontrivial A-invariant H-neutral subspace.

Assume now that the case (3) holds. Observe that $(A, HA) \in L_n(1,-1)$ and that (because A is invertible) a subspace is H-neutral if and only if it is HA-neutral. So, applying Theorem 5.1 to the pair (A, HA) the dimension of a maximal A-invariant H-neutral subspace coincides with

$$\nu\big([y_i^T(HA)y_j]_{i,j=1}^{n}\big),$$

where $y_1, \ldots, y_n$ is an orthonormal basis in $\mathbb{R}^n$. Using equation (A.12) it is found that the matrix HA is congruent to $Q := Q_1 \oplus \cdot \oplus Q_p$, where

$$Q_j = \epsilon_j \begin{bmatrix} 0 & 0 & \cdots & 0 & F_2^{n_j} & bF_2^{n_j+1} \\ 0 & \cdots & 0 & -F_2^{n_j} & -bF_2^{n_j+1} & 0 \\ \vdots & & & & \vdots & \vdots \\ (-1)^{n_j-2}F_2^{n_j} & (-1)^{n_j-2}bF_2^{n_j+1} & 0 & \cdots & 0 & 0 \\ (-1)^{n_j-1}bF_2^{n_j+1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{6.3}$$

It is clear from (6.3) that for even n_j the matrix Q_j has n_j positive and n_j negative eigenvalues (counted with multiplicities). Also, when n_j is odd the matrix Q_j has $n_j - \epsilon_j$ positive and $n_j + \epsilon_j$ negative eigenvalues (because of the 2×2 block $\epsilon_j \begin{bmatrix} -b & 0 \\ 0 & -b \end{bmatrix}$ present in the middle of Q_j). Consequently, the dimension of a maximal Q-neutral subspace is $\min\{\#\text{ of positive eigenvalues of } Q, \#\text{ of negative eigenvalues of } Q\} = \sum_{j=1}^{p} n_j - |\sum_J \epsilon_j|$, where J is the set of indices j for which n_j is odd. This proves Theorem 6.1. ∎

We can use equations (6.1) and (6.2) to describe those situations in which $\gamma(A;H)$ is maximal, *i.e.* in which $\gamma(A;H) = \frac{1}{2}n$. First, equation (6.1) shows that $m(A;\pm ib_k) = m_k$, the algebraic multiplicity of ib_k if and only if $\sum_J \epsilon_j = 0$. If this is the case for all nonzero pure imaginary eigenvalues then it follows that $\gamma(A;H) = \frac{1}{2}n$. In particular, this result holds if all nonzero pure imaginary eigenvalues have only even partial multiplicities for then J is empty.

COROLLARY 6.2. *If* $(A, H) \in L_n(-1,-1)$ *and all the nonzero pure imaginary eigenvalues of* A *have only even partial multiplicities then* $\gamma(A; H) = \frac{1}{2}n$.

EXAMPLE. This example illustrates that the converse of Corollary 6.2 is not true. Let

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

then $(A, H) \in L_4(-1,-1)$. The two-dimensional subspace

$$\mathcal{S} = \left\{ x \mid x = \begin{bmatrix} \alpha \\ \beta \\ \alpha \\ \beta \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

is maximal A-invariant and H-neutral, but the eigenvalues $+i$ and $-i$ of A each have partial multiplicities equal to one. For the sign characteristic we have $\epsilon_1 = +1$, $\epsilon_2 = -1$. ■

7. **Definitizable pairs.** If $(A, H) \in L_n(\xi, \eta)$ we say that A is *definitizable with respect to* H if there is a real scalar polynomial p such that $Hp(A) > 0$, *i.e.* $Hp(A)$ is symmetric and positive definite. In this section relationships between the condition $\gamma(A, H) = 0$, the definitizable property, and properties of the spectrum of A are established. Note first of all that if (A, H) have $(\hat{A}, \hat{H})$ as a canonical pair under the transformation (A.1) then, clearly, $Hp(A) > 0$ if and only if $\hat{H}p(\hat{A}) > 0$. Thus the definitizable property can be examined in terms of the appropriate canonical pairs appearing in Theorem A.1.

The case $(\xi, \eta) = (-1, 1)$ is quickly disposed of because equations (A.2) and (A.3) show that $H_i p(A_i) > 0$ is impossible. Thus, *if* $H^T = -H$ *and* $HA = A^T H$, *A cannot be definitizable with respect to H.*

The case $(\xi, \eta) = (1, 1)$ offers an interesting contrast with the corresponding case of complex matrices. Thus, we compare the case $A, H \in \mathbb{R}^{n\times n}$, $H^T = H$, $HA = A^T H$, with the case $A, H \in \mathbb{C}^{n\times n}$, $H^* = H$, $HA = A^* H$. The latter case has been examined in [5] and [4] and it is known that then $\gamma(A, H) = 0$ and "A definitizable with respect to H" are equivalent and, in turn, these are equivalent to the condition that A have only real eigenvalues and that they have definite type (*i.e.* the eigenspace of each eigenvalue is either H-positive or H-negative). For real $(1, 1)$ pairs these equivalent statements are lost.

The essential difference is that for complex matrices the (generic) *simple* non-real eigenvalues contribute to $\gamma(A; H)$ while for real matrices they do not do so (see equations (4.1) and (4.3)).

THEOREM 7.1. *If* $(A, H) \in L_n(1, 1)$ *then* $\gamma(A, H) = 0$ *if and only if all non-real eigenvalues of A (if any) are simple and all real eigenvalues have definite type.*

PROOF. This is a direct consequence of equations (4.1) and (4.3). ■

Note the conditions on $\sigma(A)$ described here are stable under small perturbations of A and H which retain their symmetries. Thus, $\gamma(A,H) = 0$ ensures a "strongly stable" system in this sense. Note also that the definitizable property implies $\gamma(A,H) = 0$ but not conversely. Indeed, it may be said that, in the $(1,1)$ case, $\gamma(A,H) = 0$ is "generically" satisfied, as it is satisfied if all the eigenvalues of A are simple. Conversely, if A is definitizable with respect to H, then $\sigma(A) \subset \mathbb{R}$ and all eigenvalues have definite type. This follows from equation (4.2). Thus, when $(\xi,\eta) = (1,1)$, A definitizable with respect to H implies $\gamma(A,H) = 0$, but not conversely.

For the cases $(\xi,\eta) = (1,-1)$ and $(\xi,\eta) = (-1,-1)$ we need a notion of "definite type" for pure imaginary eigenvalues. Define this in terms of real quantities by saying that an eigenvalue ib of A ($b \in \mathbb{R}$) has definite type if any elementary divisor of A (over $\mathbb{R}$) of the form $(\lambda^2 + b^2)^k$ has $k = 1$ and, if there are several such divisors (with the same b), then the corresponding numbers ϵ_j (of equations (A.8) or (A.12)) are all equal.

THEOREM 7.2. *If $(A,H) \in L_n(1,-1)$ the following are equivalent:*

(i) $\gamma(A,H) = 0$.

(ii) A is definitizable with respect to H using a polynomial which is an even function.

(iii) All eigenvalues of A are on the imaginary axis and have definite type.

PROOF. The equivalence of (i) and (iii) follows from Theorem 5.1. Let us verify that (ii) and (iii) are equivalent: Given (ii) there is a real polynomial p such that $H_i p(A_i) > 0$ for all basic canonical pairs (A_i, H_i) of (A,H) in (A.6)–(A.9). We see immediately that cases (A.7) and (A.9) cannot occur, so $\sigma(A)$ is on the imaginary axis. The remaining properties follow from examination of (A.6) and (A.8). So (ii) $\Rightarrow$ (iii).

Conversely, given (iii), let $\tilde{A} = iA$ and consider the pair $(\tilde{A}, H)$ in the context of complex matrices, noting that $H^* = H$ and $H\tilde{A} = \tilde{A}^* H$. Using the main result of [5] the spectral properties of $\tilde{A}$ imply that $\tilde{A}$ is definitizable with respect to H. Thus, there is a real polynomial p such that $Hp(\tilde{A}) > 0$ on $\mathbb{C}^n$. If $p(\lambda) = \sum_{j=0}^{d} p_j \lambda^j$ define $\hat{p}(\lambda) = \sum_j p_{2j}(-1)^j \lambda^{2j}$ and it is easily seen that

$$H\hat{p}(A) = \operatorname{Re}\big(Hp(\tilde{A})\big) > 0$$

on $\mathbb{R}^n$. This gives condition (ii). ■

Similar arguments apply in the $(-1,-1)$ case and yield the following result:

THEOREM 7.3. *If $(A,H) \in L_n(-1,-1)$ the following are equivalent:*

(i) $\gamma(A,H) = 0$,

(ii) A is definitizable with respect to H by a polynomial which is an odd function,

(iii) A is nonsingular, all eigenvalues are on the imaginary axis, and all eigenvalues have definite type.

8. **Low rank perturbations of definitizable pairs.** In the light of the results of Section 7, let us now examine some properties of pairs $(A, H) \in L_n(\xi, \eta)$ which are low rank perturbations of pairs (A_0, H_0) and for which A_0 is definitizable with respect to H_0. The arguments used to prove Theorem 5.2 of [4] apply verbatim in the context of real matrix pairs and yield the next theorem. Note first that A_0 is said to be *d-definitizable with respect to* H_0 if A_0 is definitizable with respect to H_0 and, among all real polynomials p for which $H_0 p(A_0) > 0$, the least degree is $d - 1$.

THEOREM 8.1. *Let* (A, H), $(A_0, H_0) \in L_n(\xi, \eta)$ *and* A_0 *be d-definitizable with respect to* H_0. *If* $a = \operatorname{rank}(A - A_0)$ *and* $h = \operatorname{rank}(H - H_0)$, *then*

$$\gamma(A, H) \leq (d - 1)a + h. \tag{8.1}$$

Observe first of all that the case $(\xi, \eta) = (-1, 1)$ is excluded from this statement as there are no definitizable pairs (A_0, H_0) in this case. The significance of the bound (8.1) is different in the three remaining cases.

If $A \in \mathbb{R}^{n\times n}$ has r real eigenvalues and $2c$ non-real eigenvalues then $r = n - 2c$. In the $(1, 1)$ case over $\mathbb{C}$ considered in [4] it is shown that $\gamma(A; H) \geq c$ and hence $n - 2\gamma(A; H)$ provides a lower bound for r. In the $(1, 1)$ case over $\mathbb{R}$, $\gamma(A; H) \geq c$ no longer obtains and, for the purpose of estimating r from below, the result obtained for complex pairs, *i.e.*

$$r \geq n - 2\{(d - 1)a + h\}$$

cannot be improved. However, in the $(1, -1)$ and $(-1, -1)$ cases over $\mathbb{R}$ the same idea gives lower bounds for the number of eigenvalues on the imaginary axis.

COROLLARY 8.2. *For the cases* $(\xi, \eta) = (1, -1)$ *and* $(\xi, \eta) = (-1, -1)$, *and under the hypotheses of Theorem 8.1, the number of eigenvalues of A on the imaginary axis is not less than* $n - 2\{(d - 1)a + h\}$.

9. **Examples.** The following four examples demonstrate that there are significant problems of applied analysis in which the classes of matrix pairs $L_n(\xi, \eta)$ have a role to play. Significant new results are not obtained, although fresh light may be cast on familiar problems. The first three examples concern quadratic eigenvalue problems and admit extensions to higher order systems. However, for the sake of brevity and clarity only the second order systems are presented.

1). Pairs $(A, H) \in L_n(1, 1)$ arise naturally in the theory of vibrations. The time-invariant matrix equation

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = 0$$

in which $M, B, K \in \mathbb{R}^{n\times n}$ and $M > 0$, $B \geq 0$ and $K \geq 0$ is a classical model for externally damped vibrating systems. The corresponding eigenvalue problem

$$L(\lambda)x := (\lambda^2 M + \lambda B + K)x = 0 \tag{9.1}$$

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has the linearization

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}B \end{bmatrix}$$

and if $H := \begin{bmatrix} B & M \\ M & 0 \end{bmatrix}$ then it is easily verified that $(A, H) \in L_{2n}(1, 1)$.

Systems of this kind are known to be 2-definitizable if $B > kM + k^{-1}K$ for some $k > 0$. They are treated in reference [4] in the context of complex hermitian matrices M, B, K. For the perturbation problems of Section 8, for example, the real formulation yields no sharper results. For example, if M, B and K in (9.1) are obtained by rank m, b, and k perturbations of a 2-definitizable system $L_0(\lambda) = \lambda^2 M_0 + \lambda B_0 + K_0$ then the number of real eigenvalues of (9.1), say r, satisfies

$$r \geq 2n - 2(m + b + k).$$

This follows from Theorem (8.1) and is consistent with a result of the paper [4].

2). Gyroscopic systems of the form

$$M\ddot{y}(t) + G\dot{y}(t) - Ky(t) = 0 \tag{9.2}$$

where $M, G, K \in \mathbb{R}^{n\times n}$ and $M > 0$, $G^T = -G \neq 0$, $K > 0$ are discussed in the paper [1]. Defining

$$H = \begin{bmatrix} G & M \\ -M & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ M^{-1}K & -M^{-1}G \end{bmatrix},$$

it is found that A provides a linearization of (9.2) and $(A, H) \in L_{2n}(-1, -1)$.

Stability of the system (9.2) is ensured when $|G| > kM + k^{-1}K$ for some $k > 0$ and the pair (A, H) is then 4-definitizable. As in the first example the conclusions obtained from Corollary 8.2 (for low rank perturbations of G and K) do not improve on those obtained by analysis over $\mathbb{C}$ (and reported in the paper [4]).

3). Suppose now that R, B, $S \in \mathbb{R}^{n\times n}$ with $R^T = -R$ and nonsingular, $B^T = B$, $S^T = -S$. Consider the system

$$R\ddot{y}(t) + B\dot{y}(t) + Sy(t) = 0. \tag{9.3}$$

Defining

$$H = \begin{bmatrix} B & R \\ -R & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -R^{-1}S & -R^{-1}B \end{bmatrix}$$

it is found that A is a linearization for (9.3) and the pair $(A, H) \in L_{2n}(1, -1)$.

4). Let A, D, $C \in \mathbb{R}^{n\times n}$ with $D \geq 0$ and $C^T = C$. Consider the Riccati equation for the unknown $X \in \mathbb{R}^{n\times n}$:

$$XDX - XA - A^TX - C = 0. \tag{9.4}$$

The pair $(\hat{A}, H)$ defined by

$$H = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} -A & D \\ C & A^T \end{bmatrix}$$

is found to be in $L_{2n}(-1,-1)$. It is known from analysis over $\mathbb{C}$ (see [3], for example) that (9.4) has a hermitian solution if and only if $\gamma(\hat{A}, H) = n$. Our Corollary 6.2 confirms that this is the case when the nonzero pure imaginary eigenvalues of $\hat{A}$ have only even partial multiplicities (*cf.* Theorems I.3.21 and II.4.3 of [3]).

Notice also that (from the latter theorem), when (A, D) is a controllable pair and $\gamma(A; H) < n$ (in particular, when $(\hat{A}, H)$ is a definitizable pair, as in Theorem 7.3 above) then (9.4) has *no* real symmetric solutions.

5). Consider equation (9.4) once more. Define $\hat{A}$ as above and let

$$\hat{H} = \begin{bmatrix} C & A^T \\ A & -D \end{bmatrix}.$$

Then $\hat{H}^T = \hat{H}$ and $\hat{H}\hat{A} = -\hat{A}^T\hat{H}$. Thus, there is also a pair from $L_{2n}(1,-1)$ associated with (9.4). Note also that $\hat{H} = H\hat{A}$.

Appendix. In this appendix we list the canonical forms for pairs of matrices in $L_n(\xi, \eta)$ (see Section 2 for the definition) under the transformations $(A, H) \longrightarrow (S^{-1}AS, S^THS)$ for invertible real matrices S. The derivation of these forms can be found in a variety of sources and are collected here for easy reference. See [2], [9], [6], [8] and [3].

The following notations are used:

$$Z = Z_1 \oplus \cdots \oplus Z_q = \bigoplus_{i=1}^{q} Z_1$$

denotes the block-diagonal matrix with blocks $Z_1, \ldots, Z_q$ on the main diagonal. Jordan blocks are defined as follows:

$$J_n(a) = \begin{pmatrix} a & & & \\ 1 & \ddots & 0 & \\ & \ddots & & \\ & 0 & a & \\ & & 1 & a \end{pmatrix}; \quad J_n\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} Z & & \\ I_2 & \ddots & 0 \\ & \ddots & \\ 0 & I_2 & Z \end{pmatrix},$$

where Z stands for the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Here a, b are real numbers with $b > 0$. The size of $J_n(a)$ is $n \times n$ and the size of $J_n\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is $2n \times 2n$.

Several special matrices will be used. Define

$$F_j = \begin{pmatrix} & & & 1 \\ & & 0 & -1 \\ & & & 0 \\ (-1)^{j-1} & & & \end{pmatrix}.$$

Thus, F_j is a $j \times j$ matrix which is symmetric if j is odd and skew-symmetric if j is even. Then write

$$G_j = \begin{pmatrix} & & & F_2^{j-1} \\ & 0 & -F_2^{j-1} & \\ & & 0 & \\ (-1)^{j-1}F_2^{j-1} & & & \end{pmatrix},$$

so that G_j is a $2j \times 2j$ matrix which is symmetric for all j. Also, define real symmetric $j \times j$ matrices

$$K_j = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ & & & \vdots \\ 0 & 1 & & \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Finally, recall that $F_2^j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^j$ and let

$$L_j = \begin{bmatrix} 0 & \cdots & 0 & F_2^j \\ \vdots & & -F_2^j & 0 \\ 0 & & & \vdots \\ (-1)^{j-1}F_2^j & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, L_j is a $2j \times 2j$ skew-symmetric matrix for each j.

The canonical forms are now presented in the order in which they are used in the main text (Section 3–6, respectively).

THEOREM A.1. *Let $(A, H) \in L_n(\xi, \eta)$. Then there exists a real invertible matrix S such that*

$$S^{-1}AS = \bigoplus_i A_i, \quad S^T HS = \bigoplus_i H_i. \tag{A.1}$$

The matrices of a canonical pair (A_i, H_i) have the same size and take the following forms (depending on the four possible choices of (ξ, η)):

CASE 1: $\xi = -1, \eta = 1$. *There are two types, either*

$$A_i = \bigoplus_{j=1}^{p} \big(J_{n_j}(a_i) \oplus J_{n_j}(a_i)^T\big), \tag{A.2}$$

$$H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{n_j} \\ -I_{n_j} & 0 \end{bmatrix}$$

where a_i is real, or

$$A_i = \bigoplus_{j=1}^{p} \left(J_{n_j}\begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \oplus J_{n_j}\begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}^T\right), \quad b_i > 0, \tag{A.3}$$

$$H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{2n_j} \\ -I_{2n_j} & 0 \end{bmatrix}$$

CASE 2: $\xi = 1, \eta = 1$. *There are two types, either*

$$A_i = \bigoplus_{j=1}^{p} J_{n_j}(a_i), \quad H_i = \bigoplus_{j=1}^{p} \epsilon_j K_{n_j} \tag{A.4}$$

where a_i is real and $\epsilon_i = +1$ or -1; or

$$A_i = \bigoplus_{j=1}^{p} J_{n_j} \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}, \quad H_i = \bigoplus_{j=1}^{p} K_{2n_j} \tag{A.5}$$

where a_i, b_i are real and $b_i > 0$.

CASE 3: $\xi = 1, \eta = -1$. *There are four types, either*

$$A_i = \bigoplus_{j=1}^{p} J_{2n_j+1}(0) \oplus \bigoplus_{j=1}^{q} \left(J_{n_{p+j}}(0) \oplus -J_{n_{p+j}}(0)^T\right), \tag{A.6}$$

$$H_i = \bigoplus_{j=1}^{p} \epsilon_j F_{2n_j+1} \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0 & I_{n_{p+j}} \\ I_{n_{p+j}} & 0 \end{bmatrix}.$$

where $n_{p+1}, \ldots, n_{p+q}$ are even integers and $\epsilon_1, \ldots, \epsilon_p$ take the values $+1$, or -1, or

$$A_i = \bigoplus_{j=1}^{p} \left(J_{n_j}(a_i) \oplus -J_{n_j}(a_i)^T\right) \tag{A.7}$$

$$H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{n_j} \\ I_{n_j} & 0 \end{bmatrix}$$

where $a_i > 0$, or

$$A_i = \oplus_{j=1}^{p} J_{n_j} \begin{pmatrix} 0 & b_i \\ -b_i & 0 \end{pmatrix}, \quad H_i = \bigoplus_{j=1}^{p} \epsilon_j G_{n_j} \tag{A.8}$$

where $b_i > 0$ and $\epsilon_1, \ldots, \epsilon_p$ take the values $+1$ or -1, or

$$A_i = \bigoplus_{j=1}^{p} J_{n_j} \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \oplus -J_{n_j} \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}^T \tag{A.9}$$

$$H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{2n_j} \\ I_{2n_j} & 0 \end{bmatrix}.$$

CASE 4: $\xi = -1, \eta = -1$. *There are four types, either*

$$A_i = \bigoplus_{j=1}^{p} J_{2n_j}(0) \oplus \bigoplus_{j=1}^{r} \left\{ \left(J_{2n_{p+j}+1}(0)\right) \oplus \left(-J_{2n_{p+j}+1}(0)\right)^T \right\}, \tag{A.10}$$

$$H_i = \bigoplus_{j=1}^{p} \epsilon_j F_{2n_j} \oplus \bigoplus_{j=1}^{r} \begin{bmatrix} 0 & I_{2n_{p+j}+1} \\ -I_{2n_{p+j}+1} & 0 \end{bmatrix},$$

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where ϵ_j is 1 or -1 (for $j = 1, \ldots, p$), or

$$(\text{A.11}) \qquad A_i = \bigoplus_{j=1}^{p} \left\{ \left(J_{n_j}(a_i)\right) \oplus \left(-J_{n_j}(a_i)\right)^T \right\}, \quad H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{n_j} \\ -I_{n_j} & 0 \end{bmatrix},$$

where $a_i > 0$, or

$$(A.12) \qquad A_i = \bigoplus_{j=1}^{p} J_{n_j} \begin{pmatrix} 0 & b_i \\ -b_i & 0 \end{pmatrix}, \quad H_i = \bigoplus_{j=1}^{p} \epsilon_j L_{n_j},$$

where $b_i > 0$ and ϵ_j is $+1$ or -1, or

$$(\text{A.13}) \qquad A_i = \bigoplus_{j=1}^{p} \left\{ J_{n_j} \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \oplus \left(-J_{n_j} \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \right)^T \right\}, \quad H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{2n_j} \\ -I_{2n_j} & 0 \end{bmatrix},$$

where $a_i > 0$ and $b_i > 0$.

In each of the cases 1, 2, 3 and 4, the canonical form (A.1) is uniquely determined by A and H up to simultaneous permutations of pairs of blocks A_i and H_i.

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