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Poisson Brackets and Structure of Nongraded Hamiltonian Lie Algebras Related to Locally-Finite Derivations

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Abstract. Xu introduced a class of nongraded Hamiltonian Lie algebras. These Lie algebras have a Poisson bracket structure. In this paper, the isomorphism classes of these Lie algebras are determined by employing a "sandwich" method and by studying some features of these Lie algebras. It is obtained that two Hamiltonian Lie algebras are isomorphic if and only if their corresponding Poisson algebras are isomorphic. Furthermore, the derivation algebras and the second cohomology groups are determined.

1 Introduction

A Lie algebra $(\mathcal{A},[\cdot,\cdot])$ is called to have a *Poisson bracket structure* if there exists a commutative associative algebra structure $(\mathcal{A},\cdot)$ such that the compatibility condition holds:

$$[u,v\cdot w]=[u,v]\cdot w+v\cdot[u,w] \quad \text{for } u,v,w\in\mathcal{A}. \tag{1.1}$$

The algebra $(\mathcal{A},\cdot,[\cdot,\cdot])$ with two algebraic structures is also called a *Poisson algebra*. Poisson bracket structures have many applications in areas of mathematics and physics; they are fundamental algebraic structures on phase spaces in classical mechanics; they are also the main objects in symplectic geometry (*cf.* [Z]).

Let $\mathbb{F}$ be a field of characteristic zero. A Lie algebra $\mathcal{A}$ is called *graded* if $\mathcal{A}=\bigoplus_{\alpha\in\Gamma}\mathcal{A}_\alpha$ is a Γ-graded $\mathbb{F}$-vector space for some abelian group Γ such that

$$\dim\mathcal{A}_\alpha<\infty,\quad [\mathcal{A}_\alpha,\mathcal{A}_\beta]\subset\mathcal{A}_{\alpha+\beta} \quad \text{for } \alpha,\beta\in\Gamma. \tag{1.2}$$

A classical Poisson algebra $\mathcal{P}(\ell)$ is a polynomial algebra $\mathcal{A}=\mathbb{F}[t_1,t_2,\dots,t_{2\ell}]$ in 2ℓ variables with the Lie bracket

$$[f,g]=\sum_{i=1}^{\ell}\big(\partial_{t_i}(f)\partial_{t_{\ell+i}}(g)-\partial_{t_{\ell+i}}(f)\partial_{t_i}(g)\big) \quad \text{for } f,g\in\mathcal{A}, \tag{1.3}$$

where ∂_{t_i} stands for partial derivative $\frac{\partial}{\partial t_i}$. Define

$$\mathcal{P}(\ell)_n=\Big\{t_1^{n_1}t_2^{n_2}\cdots t_{2\ell}^{n_{2\ell}}\ \Big|\ n_i\in\mathbb{N},\ \sum_{i=1}^{2\ell}n_i=n+2\Big\} \quad \text{for } -2\le n\in\mathbb{Z}, \tag{1.4}$$

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then $\mathcal{P}(\ell)$ is a $\mathbb{Z}$-graded algebra $\mathcal{P}(\ell) = \bigoplus_{n\in\mathbb{Z}} \mathcal{P}(\ell)_n$. When we consider only its Lie algebra structure, this Lie algebra is denoted by $\mathcal{H}(\ell)$. Then $\mathcal{H}(\ell)$ (or the simple Lie algebra $[\mathcal{H}(\ell), H(\ell)]/\mathbb{F}$) is a classical *Lie algebra of Cartan type H* (also called a *Hamiltonian Lie algebra*) [K1], [K2]. Generalizations of graded Hamiltonian Lie algebras have been studied in [O], [OZ].

Nongraded Lie algebras appear naturally in the theory of vertex algebras and their multi-variable analogues, and they play important roles in mathematical physics. Xu [X2] constructed a family of in general nongraded Hamiltonian Lie algebras based on certain derivation-simple algebras and locally finite derivations (we refer to [SXZ] for the classification of derivation-simple algebras). In [SX], Xu and the author of this paper determined the isomorphism classes of Poisson algebras constructed in [X2] (two Poisson algebras are called isomorphic if there exists an isomorphism which preserves both associative algebra structure and Lie algebra structure). However, the structure theory of the Hamiltonian Lie algebras in general does not seem to be well-developed. Since the Poisson algebras have two compatible algebraic structures while the Hamiltonian Lie algebras only have a Lie algebraic structure, the problem of determination of the isomorphism classes of Hamiltonian Lie algebras is thus more complicated, and one can see that some special treatments are needed in order to determine their isomorphism classes.

In [OZ], Osborn and Zhao determined the isomorphism classes of the graded Hamiltonian Lie algebras under certain finiteness condition on the skew-symmetric $\mathbb{Z}$-bilinear forms ϕ_0. They used the "derivation method" to determine the isomorphism classes of the Hamiltonian Lie algebras, mainly, they first determined the derivation algebras of the Lie algebras in order to obtain their isomorphism theorem. In this paper, we shall determine the isomorphism classes of in general nongraded Hamiltonian Lie algebras $\mathcal{H}(\underline{\ell}, \Gamma)$, where $\underline{\ell}$ is a 7-tuple of nonnegative integers and Γ is some free abelian group, which correspond to the Lie algebras in [SX] with the skew-symmetric $\mathbb{Z}$-bilinear form ϕ being zero and $\ell_4 = 0$. The reason we choose $\phi = \ell_4 = 0$ is that the Hamiltonian Lie algebras look more natural and more explicit, and are therefore easier for application, and also they are general enough to cover already most interesting cases (see Section 2). The Hamiltonian Lie algebras considered in [OZ] in case $\phi_0 = 0$ are the cases of the Hamiltonian Lie algebras $[\mathcal{H}(\underline{\ell}, \Gamma), \mathcal{H}(\underline{\ell}, \Gamma)]/\mathbb{F}$ with $\underline{\ell} = (\ell, 0, \dots, 0)$.

Unlike the graded case, where the sets of ad-locally finite elements and ad-locally nilpotent elements can be determined, in the nongraded case, the determination of the sets of ad-locally finite elements and ad-locally nilpotent elements seems to be un-achievable. Here, we use a "sandwich" method to estimate them (see Lemma 3.1). By studying some important features of the Hamiltonian Lie algebras (Lemma 3.4), we are able to obtain the isomorphism theorem without the need to know the structure of their derivation algebras. We obtain:

Main Theorem *Two Hamiltonian Lie algebras are isomorphic if and only if their corresponding Poisson algebras are isomorphic.*

In Section 2, we shall rewrite the presentations of the above-mentioned Hamiltonian Lie algebras up to certain obvious isomorphisms, which we call *normalized forms*. Then we shall prove the main theorem in Section 3. In Section 4, we shall use a different method from those in [F], [OZ] to determine the derivation algebras of the Hamiltonian Lie algebras. The reason we determine the derivation algebras after the determination of the isomorphism classes is that we want to emphasize that the determination of the isomorphism classes does not depend on the determination of the derivation algebras. Then in the final section, we shall determine the second cohomology groups of the Hamiltonian Lie algebras (the second cohomology groups of the Hamiltonian Lie algebras considered in [OZ] was determined by Jia [J]).

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2 Normalized Forms

Before we present the normalized forms of the Hamiltonian Lie algebras, to better understand general Hamiltonian Lie algebras, we first explain how one can generalize the classical Hamiltonian Lie algebras $\mathcal{H}(\ell)$ defined in (1.3).

For convenience, we denote

$$\bar{i} = i + \ell \quad \text{for } 1 \le i \le \ell. \tag{2.1}$$

The constructional ingredients of the classical Hamiltonian Lie algebra $\mathcal{H}(\ell)$ are the pairs $(\mathcal{A}, \mathcal{D})$ consisting of the polynomial algebra

$$\mathcal{A} = \mathbb{F}[t_1, t_{\bar{1}}, \dots, t_\ell, t_{\bar{\ell}}], \tag{2.2}$$

and a finite dimensional space $\mathcal{D} = \text{span}\{\partial_{t_i}, \partial_{t_{\bar{i}}} \mid 1 \le i \le \ell\}$ of commuting locally finite derivations. The derivations $\partial_{t_i} = \frac{\partial}{\partial t_i}$ are called *down-grading operators* by its obvious meaning for $1 \le i \le 2\ell$. Then the type of derivation pairs $\{(\partial_{t_i}, \partial_{t_{\bar{i}}}) \mid 1 \le i \le \ell\}$ for $\mathcal{H}(\ell)$ is

$$(d, d), \tag{2.3}$$

where d stands for down-grading operators.

If we replace the polynomial algebra by the Laurant polynomial algebra

$$\mathcal{A} = \mathbb{F}[x_1^{\pm 1}, x_{\bar{1}}^{\pm 1}, \dots, x_\ell^{\pm 1}, x_{\bar{\ell}}^{\pm 1}], \tag{2.4}$$

and rewrite (1.3) as

$$[f, g] = \sum_{p=1}^{\ell} (x_p x_{\bar{p}})^{-1} \big(\partial_p^*(f) \partial_{\bar{p}}^*(g) - \partial_{\bar{p}}^*(f) \partial_p^*(g) \big) \quad \text{for } f, g \in \mathcal{A}, \tag{2.5}$$

where ∂_p^* stands for $x_p \frac{\partial}{\partial x_p}$ for $1 \le p \le 2\ell$, then we obtain a Hamiltonian Lie algebra, denoted by $\overline{\mathcal{H}}(\ell)$. Now the derivations ∂_p^* are called *grading operators* by its obvious meaning, and the type of derivation pairs $\{(\partial_p^*, \partial_{\overline{p}}^*) \mid 1 \le p \le \ell\}$ for $\overline{\mathcal{H}}(\ell)$ is then

$$(g, g), \tag{2.6}$$

where g stands for grading operators.

Furthermore, we can replace $\mathcal{A}$ by a semigroup algebra which is the tensor product of a Laurant polynomial algebra (2.4) and a polynomial algebra (2.2):

$$\mathcal{A} = \mathbb{F}[x_1^{\pm 1}, t_1, x_{\overline{1}}^{\pm 1}, t_{\overline{1}}, \dots, x_\ell^{\pm 1}, t_\ell, x_{\overline{\ell}}^{\pm 1}, t_{\overline{\ell}}], \tag{2.7}$$

and replace ∂_p^* by $\partial_p = \partial_p^* + \partial_{t_p}$ for $1 \le p \le 2\ell$, then (2.5) defines a Hamiltonian Lie algebra, denoted by $\widehat{\mathcal{H}}(\ell)$. The derivations ∂_p are called *mixed operators*, and the type of derivation pairs $\{(\partial_p, \partial_{\overline{p}}) \mid 1 \le p \le \ell\}$ for $\widehat{\mathcal{H}}(\ell)$ is now

$$(m, m), \tag{2.8}$$

where m stands for mixed operators.

In the examples above, we can generally denote a monomial as

$$x^{\alpha, \underline{i}} = x_1^{\alpha_1} x_{\overline{1}}^{\alpha_{\overline{1}}} \cdots x_\ell^{\alpha_\ell} x_{\overline{\ell}}^{\alpha_{\overline{\ell}}} t_1^{i_1} t_{\overline{1}}^{i_{\overline{1}}} \cdots t_\ell^{i_\ell} t_{\overline{\ell}}^{i_{\overline{\ell}}}, \tag{2.9}$$

for

$$\alpha = (\alpha_1, \alpha_{\overline{1}}, \dots, \alpha_\ell, \alpha_{\overline{\ell}}) \in \Gamma, \quad \underline{i} = (i_1, i_{\overline{1}}, \dots, i_\ell, i_{\overline{\ell}}) \in \mathcal{J}, \tag{2.10}$$

where Γ is an additive subgroup of $\mathbb{F}^{2\ell}$ such that $\Gamma = \{0\}$ in the case of $\mathcal{H}(\ell)$ (where there are no nonzero grading operators), and $\Gamma = \mathbb{Z}^{2\ell}$ in the cases of $\overline{\mathcal{H}}(\ell)$ and $\widehat{\mathcal{H}}(\ell)$ (where there are nonzero grading operators), and where $\mathcal{J}$ is some semi-subgroup of $\mathbb{N}^{2\ell}$ such that $\mathcal{J} = \mathbb{N}^{2\ell}$ in the cases of $\mathcal{H}(\ell)$ and $\widehat{\mathcal{H}}(\ell)$ (where there are nonzero down-grading operators), and $\mathcal{J} = \{0\}$ in the case of $\overline{\mathcal{H}}(\ell)$ (where there are no nonzero down-grading operators). In all three cases, we can define operators $\partial_p^* = x_p \frac{\partial}{\partial x_p}$, $\partial_{t_p} = \frac{\partial}{\partial t_p}$ and $\partial_p = \partial_p^* + \partial_{t_p}$ such that $\partial_p^* = 0$ in the case of $\mathcal{H}(\ell)$ and $\partial_{t_p} = 0$ in the case of $\overline{\mathcal{H}}(\ell)$.

With the above examples in mind, we can now give generalizations of the Hamiltonian Lie algebras as follows.

First for convenience, for $m, n \in \mathbb{Z}$, we denote

$$\overline{m, n} = \begin{cases} \{m, m+1, \dots, n\} & \text{if } m \le n \\ \varnothing & \text{otherwise.} \end{cases} \tag{2.11}$$

We shall construct a semigroup algebra $\mathbb{F}[\Gamma \times \mathcal{J}]$ (*cf.* (2.7)), where Γ is some free abelian subgroup of an $\mathbb{F}$-vector space $\mathbb{F}^n$ and $\mathcal{J}$ is some semi-subgroup of $\mathbb{N}^n$, and construct 7 groups of derivation pairs $\{(\partial_p, \partial_{\overline{p}}) \mid p \in I_i\}$ for $i \in \overline{1,7}$, where I_i are some indexing sets such that if we denote each type of derivation pairs $\{(\partial_p, \partial_{\overline{p}}) \mid p \in I_i\}$ by $(T_i, T_{\overline{i}})$ for $i \in \overline{1,7}$, then the types of derivation pairs in the order of the groups $\{(\partial_p, \partial_{\overline{p}}) \mid p \in I_i\}$ for $i \in \overline{1,7}$ are

$$(T_1, T_{\overline{1}}) = (g, g), \quad (T_2, T_{\overline{2}}) = (m, g), \quad (T_3, T_{\overline{3}}) = (m, g), \tag{2.12}$$
$$(T_4, T_{\overline{4}}) = (m, m), \quad (T_5, T_{\overline{5}}) = (g, d), \quad (T_6, T_{\overline{6}}) = (m, d), \quad (T_7, T_{\overline{7}}) = (d, d).$$

Then we shall see that (2.3), (2.6) and (2.8) correspond respectively to the three special cases:

(i) $I_7 = \overline{1, \ell}$ and $I_i = \varnothing$ if $i \neq 7$,
(ii) $I_1 = \overline{1, \ell}$ and $I_i = \varnothing$ if $i \neq 1$, and
(iii) $I_4 = \overline{1, \ell}$ and $I_i = \varnothing$ if $i \neq 4$.

To construct, we let

$$\underline{\ell} = (\ell_1, \ldots, \ell_7) \in \mathbb{N}^7 \setminus \{0\}. \tag{2.13}$$

Set

$$\iota_0 = 0, \quad \iota_i = \ell_1 + \ell_2 + \cdots + \ell_i, \quad i \in \overline{1,7}, \tag{2.14}$$

$$I_{i,j} = \overline{\iota_{i-1} + 1, \iota_j} \quad \text{for } i, j \in \overline{1,7},\ i \leq j. \tag{2.15}$$

Denote

$$I_i = I_{i,i}, \quad I = I_{1,7}, \quad J = \overline{1, 2\iota_7}. \tag{2.16}$$

Define the map $\overline{}\colon J \to J$ by

$$\overline{p} = \begin{cases} p + \iota_7 & \text{if } p \in \overline{1, \iota_7}, \\ p - \iota_7 & \text{if } p \in \overline{\iota_7 + 1, 2\iota_7}, \end{cases} \tag{2.17}$$

(*cf.* (2.1)). For any subset K of $\overline{1, 2\iota_7}$, we denote

$$\overline{K} = \{\overline{p} \mid p \in K\}. \tag{2.18}$$

In particular, we have $J = I \cup \overline{I}$. Set

$$J_i = I_i \cup \overline{I}_i, \quad J_{i,j} = I_{i,j} \cup \overline{I}_{i,j} \quad \text{for } i, j \in \overline{1,7}, i \leq j. \tag{2.19}$$

Let $\mathbb{F}$ be a field of characteristic zero. We write an element α of $\mathbb{F}^{2\iota_7}$ in the form

$$\alpha = (\alpha_1, \alpha_{\overline{1}}, \ldots, \alpha_{\iota_7}, \alpha_{\overline{\iota_7}}) \quad \text{with} \quad \alpha_p \in \mathbb{F}, \tag{2.20}$$

(*cf.* (2.10)). Set

$$\varepsilon_p = (\delta_{1,p}, \delta_{\bar{1},p}, \ldots, \delta_{\iota_7,p}, \delta_{\bar{\iota_7},p}) \in \mathbb{F}^{2\iota_7} \quad \text{for } p \in J. \tag{2.21}$$

For $\alpha \in \mathbb{F}^{2\iota_7}$ and $K \subset J$, we use $\alpha_{_K}$ to denote the vector in $\mathbb{F}^{|K|}$ (where $|K|$ is the size of K), obtained from α by deleting all the coordinates α_p with $p \in J \setminus K$; for instance,

$$\alpha_{\{1,3\}} = (\alpha_1, \alpha_3) \in \mathbb{F}^2, \quad \alpha_{\{1,\bar{2},\bar{3}\}} = (\alpha_1, \alpha_{\bar{2}}, \alpha_{\bar{3}}) \in \mathbb{F}^3. \tag{2.22}$$

Sometimes, when the context is clear, we also use $\alpha_{_K}$ to denote the vector in $\mathbb{F}^{2\iota_7}$ by putting its p-th coordinate to be zero for $p \in J \setminus K$.

We fix a set $\{\sigma_p \mid p \in J\}$ of elements in $\mathbb{F}^{2\iota_7}$ as follows:

$$\sigma_p = \begin{cases} \varepsilon_p + \varepsilon_{\bar{p}} & \text{if } p \in I_1 \cup I_{3,4}, \\ \varepsilon_p & \text{if } p \in I_2, \\ 0 & \text{if } p \in I_{5,7}, \end{cases} \tag{2.23}$$

and $\sigma_{\bar{p}} = \sigma_p$. Using the notations (2.9) and (2.23), the factor $(x_p x_{\bar{p}})^{-1}$ that appears in (2.5) is simply $x^{-\sigma_p}$ if $p \in I_1$. If we re-denote x_p^{-1} by x_p (and x_p by x_p^{-1}), then the factor $(x_p x_{\bar{p}})^{-1}$ in (2.5) can be written as

$$(x_p x_{\bar{p}})^{-1} = x^{\sigma_p}. \tag{2.24}$$

Now we take an additive subgroup Γ of $\mathbb{F}^{2\iota_7}$ such that

$$\alpha_{\bar{I}_{5,6} \cup J_7} = 0 \quad \text{for } \alpha \in \Gamma, \tag{2.25}$$

(this condition is necessary since we require that $T_{\bar{5}} = T_{\bar{6}} = T_7 = T_{\bar{7}} = d$ by (2.12), which means that $\partial_p^* = 0$, *i.e.*, we shall have $\alpha_p = 0$ if $p \in \bar{I}_{5,6} \cup J_7$ for $\alpha \in \Gamma$ (*cf.* (2.2) and (2.3))), and we shall also require that

$$\sigma_p \in \Gamma, \quad \varepsilon_q \in \Gamma, \quad \mathbb{F}\varepsilon_r \cap \Gamma \neq \{0\} \quad \text{for } p \in I_{1,4}, q \in I_{5,6}, r \in J_{1,4}, \tag{2.26}$$

where the first condition is necessary since we require that x^{σ_p} will appear as a factor in the Lie bracket (*cf.* (2.5) and (2.24), also see (2.36)), and where the last two conditions are called the *distinguishable conditions* among the derivations ∂_p defined later in (2.33), which are necessary in order to guarantee the simplicity of the Hamiltonian Lie algebras (*cf.* [X2]).

Note that $\mathbb{N}^{2\iota_7}$ is an additive sub-semigroup of $\mathbb{F}^{2\iota_7}$. We take

$$\mathcal{J} = \{\underline{i} = (i_1, i_{\bar{1}}, \ldots, i_{\iota_7}, i_{\bar{\iota_7}}) \in \mathbb{N}^{2\iota_7} \mid \underline{i}_{J_1 \cup \bar{I}_{2,3} \cup I_5} = 0\}, \tag{2.27}$$

(*cf.* (2.10)), where the condition $\underline{i}_{J_1 \cup \bar{I}_{2,3} \cup I_5} = 0$ is necessary since $T_1 = T_{\bar{1}} = T_{\bar{2}} = T_{\bar{3}} = T_5 = g$ by (2.12), which means that $\partial_{t_p} = 0$, *i.e.*, we shall have $i_p = 0$ if $p \in J_1 \cup \bar{I}_{2,3} \cup I_5$ for $\underline{i} \in \mathcal{J}$ (*cf.* (2.4) and (2.6)).

Now we let $\mathcal{A} = \mathbb{F}[\Gamma \times \mathcal{J}]$ be the semigroup algebra with basis

$$\{x^{\alpha,\underline{i}} \mid (\alpha, \underline{i}) \in \Gamma \times \mathcal{J}\}, \tag{2.28}$$

(*cf.* (2.9)), and the multiplication

$$x^{\alpha,\underline{i}} \cdot x^{\beta,\underline{j}} = x^{\alpha+\beta,\underline{i}+\underline{j}} \quad \text{for } (\alpha, \underline{i}), (\beta, \underline{j}) \in \Gamma \times \mathcal{J}. \tag{2.29}$$

Then $\mathcal{A}$ forms a commutative associative algebra with $1 = x^{0,0}$ as the identity element. Set

$$\mathcal{A}_\alpha = \operatorname{span}\{x^{\alpha,\underline{i}} \mid \underline{i} \in \mathcal{J}\} \quad \text{for } \alpha \in \Gamma. \tag{2.30}$$

Then $\mathcal{A}$ is Γ-graded: $\mathcal{A} = \bigoplus_{\alpha\in\Gamma} \mathcal{A}_\alpha$ (but in general $\mathcal{A}_\alpha$ is infinite dimensional). For convenience, we denote

$$x^\alpha = x^{\alpha,0}, \quad t^{\underline{i}} = x^{0,\underline{i}}, \quad t_p = t^{\varepsilon_p}, \quad \text{for } \alpha \in \Gamma, \underline{i} \in \mathcal{J}, p \in J. \tag{2.31}$$

In particular,

$$t^{\underline{i}} = \prod_{p\in J} t_p^{i_p}, \quad x^{\alpha,\underline{i}} = x^\alpha t^{\underline{i}}, \quad \text{for } \alpha \in \Gamma, \underline{i} \in \mathcal{J}, \tag{2.32}$$

(*cf.* (2.9)). Define the derivations $\{\partial_p, \partial_p^*, \partial_{t_p} \mid p \in J\}$ of $\mathcal{A}$ by

$$\partial_p = \partial_p^* + \partial_{t_p} \quad \text{and} \quad \partial_p^*(x^{\alpha,\underline{i}}) = \alpha_p x^{\alpha,\underline{i}}, \quad \partial_{t_p}(x^{\alpha,\underline{i}}) = i_p x^{\alpha,\underline{i}-\varepsilon_p}, \tag{2.33}$$

for $p \in J$, $(\alpha, \underline{i}) \in \Gamma \times \mathcal{J}$, where we treat

$$x^{\alpha,\underline{i}} = 0 \quad \text{if } (\alpha, \underline{i}) \notin \Gamma \times \mathcal{J}. \tag{2.34}$$

In particular,

$$\partial_p^* = 0, \quad \partial_{t_q} = 0 \quad \text{for } p \in \overline{I}_{5,6} \cup J_7, q \in J_1 \cup \overline{I}_{2,3} \cup I_5, \tag{2.35}$$

by (2.25) and (2.27) (*cf.* (2.12)). We call the nonzero derivations ∂_p^* *grading operators*, the nonzero derivations ∂_{t_q} *down-grading operators*, and the derivations $\partial_r^* + \partial_{t_r}$ *mixed operators* if both ∂_r^* and ∂_{t_r} are not zero. Then the types of derivation pairs in the order of the groups $\{(\partial_p, \partial_{\overline{p}}) \mid p \in I_i\}$ for $i \in \overline{1,7}$ are as shown in (2.12).

Now we define the following Lie bracket on $\mathcal{A}$:

$$[u, v] = \sum_{p\in I} x^{\sigma_p} \left(\partial_p(u)\partial_{\overline{p}}(v) - \partial_{\overline{p}}(u)\partial_p(v) \right), \tag{2.36}$$

for $u \in \mathcal{A}_\alpha$, $v \in \mathcal{A}_\beta$ (*cf.* (2.30), (2.5) and (2.24)), where x^{σ_p} appears just as in (2.5) and (2.24). Then $(\mathcal{A}, [\cdot,\cdot])$ forms a Hamiltonian Lie algebra, denoted by $\mathcal{H}(\underline{\ell}, \Gamma)$, and $(\mathcal{A}, \cdot, [\cdot,\cdot])$ forms a Poisson algebra. Then $\mathcal{H}(\underline{\ell}, \Gamma)$ is the normalized form of

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a class of in general nongraded Hamiltonian Lie algebra constructed in [X2]. From this definition, one sees that the classical Hamiltonian Lie algebra $\mathcal{H}(\ell)$ is simply the Lie algebra $\mathcal{H}(\underline{\ell'}, 0)$ with $\underline{\ell'} = (0, \ldots, 0, \ell)$, and the Hamiltonian Lie algebras $\overline{\mathcal{H}}(\ell)$ and $\widehat{\mathcal{H}}(\ell)$ are respectively $\mathcal{H}(\underline{\ell''}, \mathbb{Z}^\ell)$ and $\mathcal{H}(\underline{\ell'''}, \mathbb{Z}^\ell)$, where $\underline{\ell''} = (\ell, 0, \ldots, 0)$, and $\underline{\ell'''} = (0, 0, 0, \ell, 0, 0, 0)$ (*cf.* (2.12) and the statement after it). The Hamiltonian Lie algebras considered in [OZ] in case $\phi_0 = 0$ are the cases of the Hamiltonian Lie algebras $[\mathcal{H}(\underline{\ell}, \Gamma), \mathcal{H}(\underline{\ell}, \Gamma)]/\mathbb{F}$ with $\underline{\ell} = (\ell, 0, \ldots, 0)$.

The Hamiltonian Lie algebras $\mathcal{H}(\underline{\ell}, \Gamma)$ can also be viewed as generalizations of the Lie algebras in [DZ], [X1], [Zh] in the sense that they have some common features stated in Lemma 3.4.

The following theorem was proved in [X2].

Theorem 2.1 *The Lie algebra $\mathcal{H}(\underline{\ell}, \Gamma)$ is central simple, i.e., $[\mathcal{H}(\underline{\ell}, \Gamma), \mathcal{H}(\underline{\ell}, \Gamma)]/\mathbb{F}$ (the derived algebra modulo its center) is simple.*

3 Isomorphism Classes

In this section, we shall determine the isomorphism classes of the Hamiltonian Lie algebras of the form $\mathcal{H} = \mathcal{H}(\underline{\ell}, \Gamma)$. We assume that $\mathbb{F}$ is an algebraically closed field.

By (2.25), (2.27) and (2.35), we can rewrite (2.36) in the following more explicit form:

$$
\begin{aligned}
(3.1) \qquad [x^{\alpha,\underline{i}}, x^{\beta,\underline{j}}] = &\sum_{p \in I_{1,4}} (\alpha_p \beta_{\overline{p}} - \alpha_{\overline{p}} \beta_p) x^{\sigma_p + \alpha + \beta, \underline{i} + \underline{j}} \\
&+ \sum_{p \in I_{4,6}} (\alpha_p j_{\overline{p}} - i_{\overline{p}} \beta_p) x^{\sigma_p + \alpha + \beta, \underline{i} + \underline{j} - \varepsilon_{\overline{p}}} \\
&+ \sum_{p \in I_{2,4}} (i_p \beta_{\overline{p}} - j_p \alpha_{\overline{p}}) x^{\sigma_p + \alpha + \beta, \underline{i} + \underline{j} - \varepsilon_p} \\
&+ \sum_{p \in I_4 \cup I_{6,7}} (i_p j_{\overline{p}} - i_{\overline{p}} j_p) x^{\sigma_p + \alpha + \beta, \underline{i} + \underline{j} - \varepsilon_p - \varepsilon_{\overline{p}}},
\end{aligned}
$$

for $(\alpha, \underline{i}), (\beta, \underline{j}) \in \Gamma \times \mathcal{J}$, where the first summand over $p \in I_{1,4}$ corresponds to the fact that $T_i \neq d \neq T_{\overline{i}}$ for $i = 1, 2, 3, 4$ (*cf.* (2.12)). As for other summands in (3.1), they are also obvious by (2.12). In particular, we have

(3.2)
$$
[x^\alpha, x^\beta] = \sum_{p \in I_{1,4}} (\alpha_p \beta_{\overline{p}} - \alpha_{\overline{p}} \beta_p) x^{\sigma_p + \alpha + \beta} = \sum_{p \in I_{1,4}} \begin{vmatrix} \alpha_{\{p,\overline{p}\}} \\ \beta_{\{p,\overline{p}\}} \end{vmatrix} x^{\sigma_p + \alpha + \beta} \quad \text{for } \alpha, \beta \in \Gamma,
$$

(*cf.* (2.31) and (2.22)), where $\begin{vmatrix} \alpha_{\{p,\overline{p}\}} \\ \beta_{\{p,\overline{p}\}} \end{vmatrix} = \begin{vmatrix} \alpha_p & \alpha_{\overline{p}} \\ \beta_p & \beta_{\overline{p}} \end{vmatrix}$ is a 2×2 determinant, and

$$
(3.3) \qquad [x^{-\sigma_p}, x^{\beta,\underline{j}}] = \begin{cases} (\beta_p - \beta_{\overline{p}}) x^{\beta,\underline{j}} & \text{if } p \in I_1, \\ -\beta_{\overline{p}} x^{\beta,\underline{j}} & \text{if } p \in I_2, \\ (\beta_p - \beta_{\overline{p}}) x^{\beta,\underline{j}} + j_p x^{\beta,\underline{j} - \varepsilon_p} & \text{if } p \in I_3, \\ (\beta_p - \beta_{\overline{p}}) x^{\beta,\underline{j}} + j_p x^{\beta,\underline{j} - \varepsilon_p} - j_{\overline{p}} x^{\beta,\underline{j} - \varepsilon_{\overline{p}}} & \text{if } p \in I_4, \end{cases}
$$

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and

$$[t_{\overline{q}}, x^{\beta,\underline{j}}] = \begin{cases} -\beta_q x^{\beta,\underline{j}} & \text{if } q \in I_5, \\ -\beta_q x^{\beta,\underline{j}} - j_q x^{\beta,\underline{j}-\varepsilon_q} & \text{if } q \in I_6. \end{cases} \tag{3.4}$$

For any $\underline{i} \in \mathcal{J}$, we define the *level* of $\underline{i}$ to be

$$|\underline{i}| = \sum_{p \in J} i_p. \tag{3.5}$$

For any $(\alpha, \underline{i}) \in \Gamma \times \mathcal{J}$, we define the *support* of $(\alpha, \underline{i})$ to be

$$\operatorname{supp}(\alpha, \underline{i}) = \{p \in J \mid \alpha_p \neq 0 \text{ or } i_p \neq 0\}. \tag{3.6}$$

For any Lie algebra $\mathcal{L}$, we denote by $\mathcal{L}^F$ and by $\mathcal{L}^N$ the sets of ad-locally finite elements and of ad-locally nilpotent elements, of $\mathcal{L}$ respectively. Generally, to obtain the isomorphism theorem, the ordinary way is first to find the sets $\mathcal{H}^F$ and $\mathcal{H}^N$. However, in our case here, the determinations of the sets $\mathcal{H}^F$ and $\mathcal{H}^N$ seem to be un-achievable. Thus, we use a "sandwich" method to estimate them. To do this, we introduce the following three subsets of $\mathcal{H}$. Denote

$$H_1 = \{x^{-\sigma_p}, t_{\overline{q}} \mid p \in I_{1,4}, q \in I_{5,6}\}, \tag{3.7}$$

$$H_2 = \{x^{\alpha,\underline{i}} \mid \alpha_{J_{1,4}} = \underline{i}_{J_{1,4} \cup \overline{I}_{5,6}} = 0,\ i_p i_{\overline{p}} = 0 \text{ for } p \in J_7\}, \tag{3.8}$$

$$H_3 = \operatorname{span}\{x^{\alpha,\underline{i}} \mid \alpha_{J_{1,4}} = \underline{i}_{J_{1,4} \cup \overline{I}_{5,6}} = 0\}, \tag{3.9}$$

(*cf.* (2.22). Then our first result is the following "sandwich" lemma.

Lemma 3.1

$$H_1 \cup H_2 \subset \mathcal{H}^F \subset \operatorname{span}(H_1 \cup H_3), \tag{3.10}$$

$$H_2 \subset \mathcal{H}^N \subset H_3. \tag{3.11}$$

Proof By (3.3) and (3.4), we have $H_1 \subset \mathcal{H}^F$. Suppose $x^{\alpha,\underline{i}} \in H_2$. Then by (3.8),

$$\operatorname{supp}(\alpha, \underline{i}) \subset I_{5,6} \cup J_7, \quad \text{and} \quad \overline{p} \notin \operatorname{supp}(\alpha, \underline{i}) \quad \text{if } p \in \operatorname{supp}(\alpha, \underline{i}). \tag{3.12}$$

Let $x^{\beta,\underline{j}} \in \mathcal{H}$. By (3.1) and (3.12), we see

$$[x^{\alpha,\underline{i}}, x^{\beta,\underline{j}}] = \begin{cases} 0, \quad \text{or} \\ \text{a linear combination of the elements} \\ x^{\gamma,\underline{k}} \text{ such that there exists at least a} \\ p \in (\overline{I}_{5,6} \cup J_7) \setminus \operatorname{supp}(\alpha, \underline{i}) \text{ with } k_p < j_p. \end{cases} \tag{3.13}$$

Thus if we set

$$m = 1 + \sum_{p \in (\overline{I}_{5,6} \cup J_7) \setminus \operatorname{supp}(\alpha, \underline{i})} j_p, \tag{3.14}$$

then $\mathrm{ad}^m_{x^{\alpha,\underline{i}}}(x^{\beta,\underline{j}}) = 0$. This proves $H_2 \subset \mathcal{H}^N \subset \mathcal{H}^F$.

Suppose $u \notin \mathrm{span}(H_1 \cup H_3)$. Write

$$u = \sum_{(\alpha,\underline{i})\in S_0} c_{\alpha,\underline{i}} x^{\alpha,\underline{i}}, \quad \text{where} \tag{3.15}$$

$$S_0 = \{(\alpha,\underline{i}) \in \Gamma \times \mathcal{J} \mid c_{\alpha,\underline{i}} \neq 0\} \quad \text{is a finite set.} \tag{3.16}$$

Then by (3.7) and (3.9), there exist $(\gamma,\underline{k}) \in S_0$ and $p \in I_{1,6}$ such that at least one of p and $\overline{p}$ is in $\mathrm{supp}(\gamma,\underline{k})$, mainly,

$$(\gamma_p, \gamma_{\overline{p}}, k_p, k_{\overline{p}}) \neq 0, \tag{3.17}$$

and such that

$$(\gamma,\underline{k}) \neq (-\sigma_p, 0) \quad \text{if } p \in I_{1,4}, \quad \text{and} \tag{3.18}$$

$$(\gamma,\underline{k}) \neq (0, \varepsilon_{\overline{p}}), \quad k_{\overline{p}} \neq 0 \quad \text{if } p \in I_{5,6}. \tag{3.19}$$

We prove that u is not ad-locally finite. To do this, we choose a total order on Γ compatible with group structure of Γ and define the total order on $\Gamma \times \mathcal{J}$ by the lexicographical order, such that the maximal element $(\gamma,\underline{k})$ of S_0 satisfies (3.17)–(3.19) for some $p \in I_{1,6}$, and that $\sigma_p > \sigma_q$ for all $q \neq p$. This is possible because the set of all nonzero σ_q is $\mathbb{F}$-linear independent. To see how it works, say, $p \in I_1$ and $(\gamma_p, \gamma_{\overline{p}}) \neq 0$ (the proof for other cases is similar). Choose $\beta = b\varepsilon_{\overline{p}} \in \Gamma$ for some $b \in \mathbb{F} \setminus \{0\}$ (*cf.* (2.26)) such that

$$\gamma_p b + m(\gamma_{\overline{p}} - \gamma_p) \neq 0 \quad \text{for all } m \in \mathbb{N}. \tag{3.20}$$

Then for $n \in \mathbb{N}$, the "highest" term of $\mathrm{ad}^n_u(x^\beta)$ is $x^{\beta+n\gamma+n\sigma_p, n\underline{k}}$ with the coefficient

$$\prod_{m=0}^{n-1} \big(\gamma_p(\beta_{\overline{p}} + m\gamma_{\overline{p}} - m) - \gamma_{\overline{p}}(m\gamma_p - m)\big) = \prod_{m=0}^{n-1} \big(\gamma_p b + m(\gamma_{\overline{p}} - \gamma_p)\big) \neq 0. \tag{3.21}$$

Thus by (3.18), the set $\{\mathrm{ad}^n_u(x^\beta) \mid n \in \mathbb{N}\}$ is linearly independent, which implies

$$\dim\big(\mathrm{span}\{\mathrm{ad}^n_u(x^\beta) \mid n \in \mathbb{N}\}\big) = \infty. \tag{3.22}$$

Thus $u \notin \mathcal{H}^F$. This proves $\mathcal{H}^F \subset \mathrm{span}(H_1 \cup H_3)$. Similarly, $\mathcal{H}^N \subset H_3$. ∎

For any subset $X \subset \mathcal{H}$, we denote by $E(X)$ the set of the zero vector and the common eigenvectors in $\mathcal{H}$ for ad_X, mainly

$$E(X) = \{u \in \mathcal{H} \mid [X, u] \subset \mathbb{F}u\}. \tag{3.23}$$

Next, we shall determine $E(\mathcal{H}^F)$. To this end, we need to find the eigenvalues for elements of ad_{H_1}. So we define a map $\pi\colon \Gamma \to \mathbb{F}^{\iota_6}$ by

$$\pi(\alpha) = \mu = (\mu_1, \ldots, \mu_{\iota_6}), \quad \text{with} \tag{3.24}$$

$$\mu_p = \begin{cases} \alpha_p - \alpha_{\overline{p}} & \text{if } p \in I_1 \cup I_{3,4}, \\ -\alpha_{\overline{p}} & \text{if } p \in I_2, \\ -\alpha_p & \text{if } p \in I_{5,6}, \end{cases} \tag{3.25}$$

(*cf.* (3.3) and (3.4)). We define

$$\mathcal{M} = \mathrm{span}\{x^\alpha \in \mathcal{H} \mid \alpha \in \Gamma\}, \tag{3.26}$$

$$\mathcal{M}_\mu = \mathrm{span}\{x^\alpha \mid \pi(\alpha) = \mu\} \quad \text{for } \mu \in \pi(\Gamma). \tag{3.27}$$

Then we have:

Lemma 3.2

$$E(\mathcal{H}^F) = \bigcup_{\mu\in\pi(\Gamma)} \mathcal{M}_\mu, \tag{3.28}$$

thus $\mathcal{M} = \mathrm{span}\big(E(\mathcal{H}^F)\big)$.

Proof By (3.10) and the definition (3.23), we have

$$E(H_1 \cup H_2) \supset E(\mathcal{H}^F) \supset E\big(\mathrm{span}(H_1 \cup H_3)\big). \tag{3.29}$$

We want to prove

$$E(H_1 \cup H_2) \subset \bigcup_{\mu\in\pi(\Gamma)} \mathcal{M}_\mu \subset E\big(\mathrm{span}(H_1 \cup H_3)\big). \tag{3.30}$$

Let $\mu \in \pi(\Gamma)$. By (3.3), (3.4), (3.7)–(3.9) and (3.24)–(3.27), elements in $\mathcal{M}_\mu$ are common eigenvectors for ad_{H_1}, and ad_{H_3} acts trivially on $\mathcal{M}_\mu$. Since elements in H_1 commute with each other, elements in $\mathcal{M}_\mu$ are common eigenvectors for $\mathrm{ad}_{\mathrm{span}(H_1\cup H_3)}$. That is,

$$\bigcup_{\mu\in\pi(\Gamma)} \mathcal{M}_\mu \subset E\big(\mathrm{span}(H_1 \cup H_3)\big). \tag{3.31}$$

Suppose

$$u = \sum_{(\alpha,\underline{i})\in S_0} c_{\alpha,\underline{i}} x^{\alpha,\underline{i}} \in \mathcal{H}, \quad \text{where} \quad S_0 = \{(\alpha,\underline{i}) \in \Gamma \times \mathcal{J} \mid c_{\alpha,\underline{i}} \neq 0\}, \tag{3.32}$$

is a common eigenvector for $\mathrm{ad}_{H_1\cup H_2}$. Since ad_{H_2} is locally nilpotent, ad_{H_2} must act trivially on u. If $(\alpha,\underline{i})\in S_0$ with $i_p\neq 0$ for some $p\in\bar{I}_{5,6}\cup J_7$, then we can choose $v\in H_2$:

$$v=\begin{cases}x^{\varepsilon_{\bar{p}}} & \text{if } p\in\bar{I}_{5,6},\\ t_{\bar{p}} & \text{if } p\in J_7,\end{cases}\tag{3.33}$$

such that $[v,x^{\alpha,\underline{i}}]\neq 0$ by (3.1) and thus $[v,u]\neq 0$, contradicting the fact that ad_{H_2} acts trivially on u. Thus $\underline{i}_{\bar{I}_{5,6}\cup J_7}=0$. Similarly, since u is a common eigenvector for ad_{H_1}, we must have $\underline{i}_{I_{2,3}\cup J_4}=0$ (and thus $\underline{i}=0$) and $\pi(\alpha)=\mu$ for some μ if $(\alpha,\underline{i})\in S_0$. This shows that $u\in\mathcal{M}_\mu$. This together with (3.31) proves (3.30). Now (3.29) and (3.30) show that all these sets are equal, *i.e.*, we have (3.28). ■

Next we shall determine the sets $\mathcal{M}^F$ and $\mathcal{M}^N$. Recall that the Lie bracket in $\mathcal{M}$ has the simple form (3.2).

Lemma 3.3

$$\mathcal{M}^F=\mathrm{span}\{x^{-\sigma_p},x^\alpha\mid p\in I_{1,4},\alpha_{J_{1,4}}=0\},\tag{3.34}$$

$$\mathcal{M}^N=\mathrm{span}\{x^\alpha\mid\alpha_{J_{1,4}}=0\}.\tag{3.35}$$

Proof We shall prove (3.34) as the proof (3.35) is similar. It is straightforward to verify that by (3.2) elements in the right-hand side of (3.34) commute with each other and they are ad-locally finite on $\mathcal{M}$. Thus the right-hand side of (3.34) is contained in $\mathcal{M}^F$. Conversely, suppose $u\in\mathcal{M}$ is not in the right-hand side of (3.34). Then we can write u as in (3.15), where now

$$S_0=\{(\alpha,\underline{i})\in\Gamma\times\mathcal{J}\mid\underline{i}=0,\ c_{\alpha,\underline{i}}\neq 0\}\quad\text{is a finite set.}\tag{3.36}$$

Thus we still have (3.17)–(3.19), and the same arguments after (3.19) show that u is not ad-locally finite on $\mathcal{M}$. ■

Now we shall study some important features of the Lie algebra $\mathcal{M}$, which is crucial in the proof of the isomorphism theorem.

Lemma 3.4

(1) *Assume that $\iota_4\neq 0$. For $\mu\in\pi(\Gamma)$, regarding $\mathcal{M}_\mu$ as an $\mathcal{M}_0$-module via the adjoint action, we have*

  (i) *if $\mu_{I_{1,4}}=0$, then the action of $\mathcal{M}_0$ on $\mathcal{M}_\mu$ is trivial and*

  (ii) *if $\mu_{I_{1,4}}\neq 0$, then $\mathcal{M}_\mu$ is a cyclic $\mathcal{M}_0$-module, the nonzero multiplicative scalars of x^α for all $\alpha\in\Gamma$ with $\pi(\alpha)=\mu$, are the only generators.*

(2) *Assume that $\iota_4=0$ and $\iota_6\neq 0$. Then $(\bigcup_{\alpha\in\Gamma}\mathbb{F}x^\alpha)\setminus\{0\}$ are the set of the common eigenvectors of $\mathcal{H}^F$ in $\mathcal{M}$.*

Proof (1) Assume that $\iota_4 \neq 0$. From (3.2) and the definition of π in (3.24), we see that x^α commutes with x^β if $\pi(\alpha) = 0$ and $\big(\pi(\beta)\big)_{I_{1,4}} = 0$. Thus if $\mu_{I_{1,4}} = 0$, the adjoint action of $\mathcal{M}_0$ on $\mathcal{M}_\mu$ is trivial. Assume

$$u = \sum_{\beta\in S_0} c_\beta x^\beta \in \mathcal{M}_\mu \quad \text{with} \quad \mu_{I_{1,4}} \neq 0, \quad \text{where} \tag{3.37}$$

$$S_0 = \{\beta \in \Gamma \mid \pi(\beta) = \mu, c_\beta \neq 0\} \quad \text{is a finite set.} \tag{3.38}$$

By (3.2), one has

$$[x^\alpha, u] = -\sum_{p\in I_{1,4}} \alpha_p \mu_p x^{\sigma_p+\alpha} \cdot u \quad \text{if } \pi(\alpha) = 0. \tag{3.39}$$

Thus the subspace

$$U = \operatorname{span}\Big\{ x^{\sigma_p+\alpha} \cdot u = \sum_{\beta\in S_0} c_\beta x^{\sigma_p+\alpha+\beta} \;\Big|\; \alpha \in \ker_\pi, p \in I_{1,4} \Big\}, \tag{3.40}$$

is a $\mathcal{M}_0$-submodule of $\mathcal{M}_\mu$. Let $\langle u\rangle$ denote the cyclic submodule of $\mathcal{M}_\mu$ generated by u. Then $\langle u\rangle \subset U$. If the size $|S_0|$ of S_0 is ≥ 2, then U in (3.40) is a proper submodule of $\mathcal{M}_\mu$ and so u is not a generator of $\mathcal{M}_\mu$.

Now assume that S_0 is a singleton $\{\beta\}$ with $\pi(\beta) = \mu$. Suppose $\mu_p \neq 0$ for some $p \in I_{1,4}$. For any $k \neq 1$, by (3.25), $k\sigma_p \in \ker_\pi$, thus

$$x^{\beta+k\sigma_p} = -\big((k-1)\mu_p\big)^{-1}[x^{(k-1)\sigma_p}, x^\beta] \in \langle u\rangle. \tag{3.41}$$

For any $\alpha \in \ker_\pi$, by (3.25), $\alpha - (k+1)\sigma_p \in \ker_\pi$. Thus by (3.2), (3.25) and (3.41), noting that $\beta_{\overline{q}} = \beta_q - \mu_q$ for $q \in I_{1,4}$, it is straightforward to compute that

$$k\mu_p x^{\alpha+\beta} + \sum_{q\in I_{1,4}} (\delta_{p,q} - \alpha_q)\mu_q x^{\alpha+\beta-\sigma_p+\sigma_q} = [x^{\alpha-(k+1)\sigma_p}, x^{\beta+k\sigma_p}] \in \langle u\rangle. \tag{3.42}$$

This shows that $x^{\alpha+\beta} \in \langle u\rangle$ for all $\alpha \in \ker_{\pi_1}$, but $\mathcal{M}_\mu$ is spanned by such elements. Thus u is a generator of $\mathcal{M}_\mu$.

(2) is obtained directly from (3.28). ■

Let $\mathcal{H}(\underline{\ell}', \Gamma')$ be another Hamiltonian Lie algebra defined in last section. We shall add a prime on all the constructional ingredients related to $\mathcal{H}(\underline{\ell}', \Gamma')$; for instance, $\mathcal{H}', \mathcal{J}', \sigma_i', \ell_i', \iota_i'$, *etc.*

To state our isomorphism theorem, denote by $M_{m\times n}$ the space of $m \times n$ matrices with entries in $\mathbb{F}$ and by GL_m the group of $m \times m$ invertible matrices with entries in $\mathbb{F}$.

Definition 3.5 Let Γ, Γ' be two additive subgroups of $\mathbb{F}^{2\iota_7}$ satisfying (2.25) and (2.26). A group isomorphism $\tau\colon \alpha \mapsto \alpha^*$ from $\Gamma \to \Gamma'$ is called *preserving* if τ

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has the following form: there exists a permutation $\nu\colon p \mapsto p^*$ on the index set $I_{1,4}$, which maps $I_k \to I_k$ for $k = 1, 2, 3, 4$, such that

$$\alpha^*_{\{p^*,\overline{p^*}\}} = \alpha_{\{p,\overline{p}\}} A_p \quad \text{for } p \in I_{1,4}, \tag{3.43}$$

(*cf.* (2.22)), where $A_p \in \mathrm{GL}_2$; the multiplication in the right-hand side of (3.43) is the vector-matrix multiplication;

$$A_p = \begin{pmatrix} a_p + b_p & a_p \\ 1 - a_p - b_p & 1 - a_p \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ a_p & b_p \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} b_p & 0 \\ 1 - b_p & 1 \end{pmatrix}, \tag{3.44}$$

if $p \in I_1 \cup I_4$ or I_2 or I_3 respectively, for some $a_p, b_p \in \mathbb{F}$ with $b_p \neq 0$;

$$\alpha^*_{I_5} = (\alpha_{I_1} - \alpha_{\overline{I}_1}) B_{1,5} - \alpha_{\overline{I}_2} B_{2,5} + \alpha_{I_5} B_{5,5}, \quad \text{where} \tag{3.45}$$

$$B_{1,5} \in M_{\ell_1 \times \ell_5}, \quad B_{2,5} \in M_{\ell_2 \times \ell_5}, \quad B_{5,5} \in \mathrm{GL}_{\ell_5}; \tag{3.46}$$

and

$$\alpha^*_{I_6} = (\alpha_{I_1} - \alpha_{\overline{I}_1}) B_{1,6} - \alpha_{\overline{I}_2} B_{2,6} + (\alpha_{I_{3,4}} - \alpha_{\overline{I}_{3,4}}) B_{3,6} + \alpha_{I_5} B_{5,6} + \alpha_{I_6} B_{6,6}, \quad \text{where} \tag{3.47}$$

$$B_{1,6} \in M_{\ell_1 \times \ell_6}, \quad B_{2,6} \in M_{\ell_2 \times \ell_6}, \quad B_{3,6} \in M_{(\ell_3+\ell_4) \times \ell_6}, \quad B_{5,6} \in M_{\ell_5 \times \ell_6}, \quad B_{6,6} \in \mathrm{GL}_{\ell_6}. \tag{3.48}$$

Note that the above uniquely determine the isomorphism by (2.25). Let us explain the above definition. First we introduce the following notations. For any $m \times n$ matrix $A = (a_{p,q})$, we denote by $\widetilde{A} = (\widetilde{a}_{p,q})$ (resp. $\widehat{A} = (\widehat{a}_{p,q})$) the $2m \times n$ matrix such that the odd rows of $\widetilde{A}$ (resp. $\widehat{A}$) form the matrix A (resp. the $m \times n$ zero matrix) and the even rows of $\widetilde{A}$ (resp. $\widehat{A}$) form the matrix $-A$, *i.e.*,

$$\widetilde{a}_{2p-1,q} = -\widetilde{a}_{2p,q} = a_{p,q}, \quad \widehat{a}_{2p-1,q} = 0, \quad \widehat{a}_{2p,q} = -a_{p,q} \tag{3.49}$$

for $p \in \overline{1,m}$, $q \in \overline{1,n}$. A preserving isomorphism τ can be decomposed into the composition of two isomorphisms $\tau = \tau_\nu \cdot \tau_0$ such that τ_ν only involves the permutation ν, *i.e.*, in (3.43)–(3.48), all A_p and $B_{i,i}$ are identity matrices and all $B_{i,j}$ are zero matrices for $i \neq j$; and τ_0 only involves matrices, *i.e.*, $\nu = \mathbf{1}_{I_{1,4}}$ in (3.43). Furthermore, τ_0 can be decomposed into $\tau_0 = \tau_1 \cdot \tau_2$ such that τ_1, τ_2 have the following forms:

$$\tau_1\colon (\alpha^*_{J_{1,4}}, \alpha^*_{I_{5,6}}) = (\alpha_{J_{1,4}}, \alpha_{I_{5,6}}) A, \quad \text{where} \tag{3.50}$$

$$A = \mathrm{diag}(A_1, \ldots, A_{\iota_4}, B_{5,5}, B_{6,6}), \tag{3.51}$$

and

$$\tau_2\colon (\alpha^*_{J_{1,4}}, \alpha^*_{I_{5,6}}) = (\alpha_{J_{1,4}}, \alpha_{I_{5,6}}) C, \quad C = \mathbf{1}_{2\iota_4+\ell_5+\ell_6} + D, \tag{3.52}$$

where in general $\mathbf{1}_m$ denotes the $m \times m$ identity matrix, and where D has the form

$$D = (0, D_5, D_6), \quad D_5 = \begin{pmatrix} \widetilde{B}_{1,5} \\ \widehat{B}_{2,5} \\ 0 \end{pmatrix}, \quad D_6 = \begin{pmatrix} \widetilde{B}_{1,6} \\ \widehat{B}_{2,6} \\ B_{3,6} \\ B_{5,6} \\ 0 \end{pmatrix}, \tag{3.53}$$

where 0 denotes some proper zero matrices whose orders are clear from the context.

Now we can state the main result of this paper.

Theorem 3.6 *$\theta\colon \mathcal{H}(\underline{\ell}, \Gamma) \cong \mathcal{H}(\underline{\ell}', \Gamma')$ if and only if $\underline{\ell} = \underline{\ell}'$ and there exists a preserving isomorphism $\tau\colon \Gamma \cong \Gamma'$.*

Theorem 3.7 **(Main Theorem)** *Two Hamiltonian Lie algebras are isomorphic if and only if their corresponding Poisson algebras are isomorphic.*

Proof By Theorem 3.6 and by [SX], the condition for two Hamiltonian Lie algebras being isomorphic is the same as the condition for the corresponding two Poisson algebras being isomorphic. ■

Proof of Theorem 3.6 "$\Leftarrow$": Suppose $\underline{\ell} = \underline{\ell}'$ and $\tau\colon \Gamma \to \Gamma'$ is a preserving isomorphism. By the explanation above, τ can be written as $\tau = \tau_\nu \cdot \tau_1 \cdot \tau_2$, thus it suffices to consider the following 3 cases.

Case A First assume that $\tau = \tau_\nu$ is determined by permutation ν.

For any $\underline{i} \in \mathcal{J}$, we define $\underline{i}^* \in \mathcal{J}$ which is obtained from $\underline{i}$ by permutation ν. Then it is straightforward to verify that the linear map

$$\theta_\nu\colon \mathcal{H} \to \mathcal{H}' \quad \text{such that } \theta_\nu(x^{\alpha,\underline{i}}) = x^{\alpha^*,\underline{i}^*}, \tag{3.54}$$

is a Lie algebra isomorphism.

Case B Next assume that $\tau = \tau_1$ as in (3.50).

We shall define an isomorphism $\theta\colon \mathcal{H} \to \mathcal{H}'$ as Poisson algebra isomorphism (then θ is clearly a Lie algebra isomorphism). By (1.1), it suffices to find the images of the generators x^α, t_p for $\alpha \in \Gamma$, $p \in I_{2,4} \cup I_{6,7} \cup \overline{I}_{4,7}$ (*cf.* (3.58) and (3.62)–(3.64) below) such that the following conditions hold (*cf.* [SX]):

$$\theta([x^\alpha, x^\beta]) = [\theta(x^\alpha), \theta(x^\beta)], \quad \theta([t_p, x^\beta]) = [\theta(t_p), \theta(x^\beta)], \\ \theta([t_p, t_q]) = [\theta(t_p), \theta(t_q)], \tag{3.55}$$

for $\alpha, \beta \in \Gamma$ and $p, q \in I_{2,4} \cup I_{6,7} \cup \overline{I}_{4,7}$.

Let $\Delta = \sum_{p \in I_{1,4}} \mathbb{Z}\sigma_p$ be the subgroup of Γ generated by $\{\sigma_p \mid p \in I_{1,4}\}$ and define $\chi\colon \Delta \to \mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ to be the *character* of Δ (*i.e.*, the group homomorphism $\Delta \to \mathbb{F}^\times$) determined by

$$\chi(\sigma_p) = b_p \quad \text{for } p \in I_{1,4}, \tag{3.56}$$

where b_p are elements in $\mathbb{F}$ appearing as entries of matrices A_p in (3.44). We prove that χ can be extended to a character $\chi\colon \Gamma \to \mathbb{F}^\times$ as follows: Assume that $\Delta_1 \supset \Delta$ is a maximal subgroup of Γ such that χ can be extended to a character $\chi\colon \Delta_1 \to \mathbb{F}^\times$. If $\Delta_1 \neq \Gamma$, then we choose $\alpha \in \Gamma \setminus \Delta_1$ and extend χ to $\Delta_2 = \mathbb{Z}\alpha + \Delta_1 \to \mathbb{F}^\times$ by defining

$$\chi(m\alpha + \beta) = \begin{cases} \chi(\beta) & \text{if } \mathbb{Z}\alpha \cap \Delta_1 = \{0\}, \\ a^m \chi(\beta) & \text{if } \mathbb{Z}\alpha \cap \Delta_1 = \mathbb{Z}n\alpha, \end{cases} \tag{3.57}$$

for $m \in \mathbb{Z}$, $\beta \in \Delta_1$, where a is an n-th root of $\chi(n\alpha)$ in the second case (recall that $\mathbb{F}$ is algebraically closed). This leads to a contradiction with the maximality of Δ_1. Thus χ can be extended to a character $\chi\colon \Gamma \to \mathbb{F}^\times$.

Now we define the images of x^α to be

$$\theta(x^\alpha) = \chi(\alpha) x'^{\alpha^*} \quad \text{for } \alpha \in \Gamma, \tag{3.58}$$

(recall that we add prime on the constructional ingredients related to $\mathcal{H}'$). Then by (3.2) we see that the first equation of (3.55) holds because (3.44) and (3.50) guarantees that $\sigma_p^* = \sigma_p'$ and that the determinant of A_p is $|A_p| = b_p = \chi(\sigma_p)$ and

$$\chi(\alpha)\chi(\beta) \begin{vmatrix} \alpha^*_{\{p,\overline{p}\}} \\ \beta^*_{\{p,\overline{p}\}} \end{vmatrix} = \chi(\alpha+\beta) \begin{vmatrix} \alpha_{\{p,\overline{p}\}} \\ \beta_{\{p,\overline{p}\}} \end{vmatrix} \cdot |A_p| = \chi(\sigma_p + \alpha + \beta) \begin{vmatrix} \alpha_{\{p,\overline{p}\}} \\ \beta_{\{p,\overline{p}\}} \end{vmatrix}. \tag{3.59}$$

Next we shall find the image of t_p. To do this, we introduce a new notation: For any vector $s = (s_1, s_{\overline{1}}, s_2, s_{\overline{2}}, \ldots, s_{\iota_7}, s_{\overline{\iota}_7})$ (with entries in $\mathbb{F}$, $\mathcal{H}$ or in $\mathcal{H}'$), we denote

$$\overline{s} = (-s_{\overline{1}}, s_1, -s_{\overline{2}}, s_2, \ldots, -s_{\overline{\iota}_7}, s_{\iota_7}). \tag{3.60}$$

For a subset $K \subset J$, we denote by $\overline{s}_K$ the vector obtained from $\overline{s}$ by deleting $-s_{\overline{p}}, s_q$ for $\overline{p}, q \in J \setminus K$; for instance,

$$\overline{s}_{\{\overline{1},\overline{2},3,4,\overline{4}\}} = (-s_{\overline{1}}, -s_{\overline{2}}, s_3, -s_{\overline{4}}, s_4), \tag{3.61}$$

(*cf.* (2.22)). We define

$$\theta(t_p) = s_p \quad \text{for } p \in I_{2,4} \cup I_{6,7} \cup \overline{I}_{4,7}, \text{ where} \tag{3.62}$$

$$s_p = t'_p, \quad s_q = b_q t'_q, \quad (-s_{\overline{r}}, s_r) = b_r(-t'_{\overline{r}}, t'_r)A_r^{-1} \quad \text{for } p \in I_2, q \in I_3, r \in I_4, \tag{3.63}$$

$$\overline{s}_{\overline{I}_{5,6}} = \overline{t}'_{\overline{I}_{5,6}} \operatorname{diag}(B_{5,5}, B_{6,6})^{-1}, \quad \overline{s}_{I_6} = \overline{t}'_{I_6} B_{6,6}^T, \quad s_{J_7} = t'_{J_7}, \tag{3.64}$$

where the up-index "T" stands for the transpose of a matrix. Then if $p \in I_{2,3}$, we have

$$[\theta(t_p), \theta(x^\alpha)] = \chi(\alpha) b_p \alpha_{\overline{p}} x'^{\alpha^* + \sigma'_p} = \theta(\alpha_{\overline{p}} x^{\alpha+\sigma_p}) = \theta([t_p, x^\alpha]), \tag{3.65}$$

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because by (3.44) and (3.50), $\alpha^*_{\overline{p}} = b_p \alpha_{\overline{p}}$ if $p \in I_2$ and $\alpha^*_{\overline{p}} = \alpha_{\overline{p}}$ if $p \in I_3$. If $p \in I_4$, as 1×2 matrices with entries in $\mathcal{H}$, we have

$$(3.66)\qquad [\theta(\bar{t}_{\{p,\overline{p}\}}), \theta(x^\alpha)] = \chi(\alpha) b_p \alpha^*_{\{p,\overline{p}\}} A_p^{-1} x'^{\alpha^*+\sigma'_p} = \chi(\alpha+\sigma_p)\alpha_{\{p,\overline{p}\}} x'^{\alpha^*+\sigma'_p} = \theta([\bar{t}_{\{p,\overline{p}\}}, x^\alpha]).$$

Furthermore, we have $[(\bar{t}_{\bar{I}_{5,6}}, \bar{t}_{I_6\cup J_7}), x^\alpha] = (\alpha_{I_{5,6}}, 0)x^\alpha$, and

$$(3.67)\qquad \alpha^*_{I_{5,6}} = \alpha_{I_{5,6}} \operatorname{diag}(B_{5,5}, B_{6,6}),$$

by (3.50). From this and (3.64), we obtain

$$(3.68)\qquad \big[\big(\theta(\bar{t}_{\bar{I}_{5,6}}), \theta(\bar{t}_{I_6\cup J_7})\big), \theta(x^\alpha)\big] = \theta\big([(\bar{t}_{\bar{I}_{5,6}}, \bar{t}_{I_6\cup J_7}), x^\alpha]\big).$$

From this and (3.66), we obtain the second equation of (3.55).

To verify the last equation of (3.55), note that

$$(3.69)\qquad \big[\bar{t}^T_{I_{2,3}\cup J_4\cup I_5\cup J_{6,7}}, \bar{t}_{I_{2,3}\cup J_4\cup I_5\cup J_{6,7}}\big] = \operatorname{diag}(0, S^\sigma_{\ell_4}, 0, S_{\ell_6+\ell_7}),$$

where

$$(3.70)\qquad S^\sigma_{\ell_4} = \operatorname{diag}\left(\begin{pmatrix} 0 & x^{\sigma_{\iota_3+1}} \\ -x^{\sigma_{\iota_3+1}} & 0\end{pmatrix}, \dots, \begin{pmatrix} 0 & x^{\sigma_{\iota_4}} \\ -x^{\sigma_{\iota_4}} & 0\end{pmatrix}\right),$$

is a $2\ell_4 \times 2\ell_4$ matrix with entries in $\mathcal{H}$, and where, in general

$$(3.71)\qquad S_m = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right) \in \mathrm{GL}_{2m}.$$

Using (3.69), (3.63) and (3.64), we can obtain

$$(3.72)\qquad \big[\theta(\bar{t}_{I_{2,3}\cup J_4\cup I_5\cup J_{6,7}})^T, \theta(\bar{t}_{I_{2,3}\cup J_4\cup I_5\cup J_{6,7}})\big] = \theta\big(\big[\bar{t}^T_{I_{2,3}\cup J_4\cup I_5\cup J_{6,7}}, \bar{t}_{I_{2,3}\cup J_4\cup I_5\cup J_{6,7}}\big]\big).$$

For example, if $p \in I_4$, by (3.56), (3.58) and (3.63), we have

$$(3.73)\qquad [\theta(\bar{t}_{\{p,\overline{p}\}})^T, \theta(\bar{t}_{\{p,\overline{p}\}})] = b_p (A_p^{-1})^T [\bar{t}'^T_{\{p,\overline{p}\}}, \bar{t}'_{\{p,\overline{p}\}}] b_p A_p^{-1} = b_p \begin{pmatrix} 0 & x'^{\sigma'_p} \\ -x'^{\sigma'_p} & 0 \end{pmatrix} = \theta([\bar{t}^T_{\{p,\overline{p}\}}, \bar{t}_{\{p,\overline{p}\}}]).$$

This proves the last equation of (3.55).

Case C Assume that $\tau = \tau_2$ as in (3.52).

We define (3.58) with $\chi(\alpha) = 1$ and we define (3.62) with

$$(3.74)\qquad \bar{s}_{I_{2,3}\cup J_4} = \bar{t}'_{I_{2,3}\cup J_4} + \bar{t}'_{I_6} E_1,$$

$$(3.75)\qquad \bar{s}_{\bar{I}_{5,6}} = \bar{t}'_{\bar{I}_{5,6}} E_2 + \bar{t}'_{I_6} E_3 + x'^{-\sigma'} E_4, \quad \bar{s}_{I_6\cup J_7} = \bar{t}'_{I_6\cup J_7},$$

where $E_1, \dots, E_4$ are some matrices to be determined in order that (3.55) holds and where $x'^{-\sigma'}$ denotes the vector

$$x'^{-\sigma'} = (x'^{-\sigma'_1}, \dots, x'^{-\sigma'_{\iota_4}}). \tag{3.76}$$

We shall not give the explicit forms of $E_1, \dots, E_4$ here, but an interested reader can find the solutions by considering two special cases of (3.53):

(1) $D_5 = 0$,
(2) $D_6 = 0$ (the general case is the composition of the two special cases), or refer to [SX] (also, *cf.* the proof of necessity).

"$\Rightarrow$": Assume that there exists a Hamiltonian Lie algebra isomorphism $\theta\colon \mathcal{H}(\underline{\ell}, \Gamma) \to \mathcal{H}(\underline{\ell}', \Gamma')$.

First, we make the following conventions: If a subset of $\mathcal{H}$ is defined, then we take the definition of the corresponding subset of $\mathcal{H}'$ for granted. If a property about $\mathcal{H}$ is given, the same property also holds for $\mathcal{H}'$, without description.

Clearly, θ maps $\mathcal{H}^F, \mathcal{H}^N$ to $\mathcal{H}'^F, \mathcal{H}'^N$ respectively, thus also maps $\mathcal{M} \to \mathcal{M}'$ by Lemma 3.2. By Lemma 3.3, we have $\dim(\mathcal{M}^F/\mathcal{M}^N) = \iota_4$. This shows that

$$\iota_4 = \iota'_4. \tag{3.77}$$

For simplicity, we assume that $\iota_4 \neq 0$ (if $\iota_4 = 0$, using Lemma 3.4 (2), one sees that all statements or arguments below either work or do not apply to the case; if $\iota_6 = 0$, then one can go directly to Claim 8 below). Denote

$$\Gamma_{1,4} = \left\{ \alpha \in \Gamma \mid \left(\pi(\alpha)\right)_{I_{1,4}} = 0 \right\}, \tag{3.78}$$

(*cf.* (3.24) and (2.22)). By Lemma 3.2, there exists a bijection $\tau_1\colon \pi(\Gamma) \to \pi(\Gamma')$ such that

$$\theta(\mathcal{M}_\mu) = \mathcal{M}'_{\tau_1(\mu)} \quad \text{for } \mu \in \pi(\Gamma) \text{ and } \tau_1(0) = 0. \tag{3.79}$$

From this and Lemma 3.4, there exists a bijection $\Gamma \setminus \Gamma_{1,4} \to \Gamma' \setminus \Gamma'_{1,4}$ which shall be denoted by $\tau\colon \alpha \mapsto \alpha^*$ such that

$$\theta(x^\alpha) = c_\alpha x'^{\alpha^*} \quad \text{for } \alpha \in \Gamma \setminus \Gamma_{1,4} \text{ and some } c_\alpha \in \mathbb{F}^\times. \tag{3.80}$$

We shall prove the necessity by establishing several claims.

Claim 1 There exists a bijection $I_{1,4} \to I'_{1,4}$ denoted by $\nu\colon p \mapsto p^*$ such that

$$\theta(x^{-\sigma_p}) = d_p x'^{-\sigma'_{p^*}} \quad \text{for } p \in I_{1,4} \text{ and some } d_p \in \mathbb{F}^\times. \tag{3.81}$$

By (3.7)–(3.9) and Lemma 3.3, we have

$$\begin{aligned} \{u \in \mathcal{M}^F \mid [u, H_1 \cup H_2] = 0\} &= \operatorname{span}\{x^{-\sigma_p} \mid p \in I_{1,4}\} \\ &= \{u \in \mathcal{B}^F \mid [u, H_1 \cup H_3] = 0\}. \end{aligned} \tag{3.82}$$

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Thus by Lemma 3.1,

$$\{u \in \mathcal{M}^F \mid [u, \mathcal{H}^F] = 0\} = \operatorname{span}\{x^{-\sigma_p} \mid p \in I_{1,4}\}. \tag{3.83}$$

Let $p \in I_{1,4}$. Then by (3.83),

$$\theta(x^{-\sigma_p}) \in \sum_{q \in I'_{1,4}} \mathbb{F} x'^{-\sigma'_q}. \tag{3.84}$$

Suppose

$$\theta(x^{-\sigma_p}) \notin \bigcup_{q \in I'_{1,4}} \mathbb{F} x'^{-\sigma'_q}. \tag{3.85}$$

By (2.26), there exists $a \in \mathbb{F}^\times$ such that $a\varepsilon_{\overline{p}} \in \Gamma$. By (3.2), we have

$$[x^{a\varepsilon_{\overline{p}}-\sigma_p}, x^{-a\varepsilon_{\overline{p}}-\sigma_p}] = 2ax^{-\sigma_p}. \tag{3.86}$$

Note that $a\varepsilon_{\overline{p}} - \sigma_p, -a\varepsilon_{\overline{p}} - \sigma_p \notin \Gamma_{1,4}$, by (3.81),

$$\theta(x^{a\varepsilon_{\overline{p}}-\sigma_p}) \in \mathbb{F}x'^{\alpha} \setminus \{0\}, \quad \theta(x^{-a\varepsilon_{\overline{p}}-\sigma_p}) \in \mathbb{F}x'^{\beta} \setminus \{0\} \quad \text{for some } \alpha, \beta \in \Gamma' \setminus \Gamma'_{1,4}. \tag{3.87}$$

By (3.2), we have

$$[x'^{\alpha}, x'^{\beta}] = \sum_{q \in I'_{1,4}} (\alpha_q \beta_{\overline{q}} - \alpha_{\overline{q}} \beta_q) x'^{\sigma'_q + \alpha + \beta}. \tag{3.88}$$

By (3.84)–(3.86) and (3.88), there exist $q, r \in I'_{1,4}$ with $q \neq r$ such that $\sigma'_q + \alpha + \beta = -\sigma'_r$. Thus

$$\beta = -\alpha - \sigma'_q - \sigma'_r, \tag{3.89}$$

and (3.88) becomes

$$[x'^{\alpha}, x'^{\beta}] = (\alpha_q \eta'_{\overline{q}} + \alpha_{\overline{q}}) x'^{-\sigma'_r} + (\alpha_r \eta'_{\overline{r}} + \alpha_{\overline{r}}) x'^{-\sigma'_q}, \tag{3.90}$$

where in general, for $q \in J_{1,4}$, we denote

$$\eta_q = \begin{cases} 1 & \text{if } q \in I_{1,4}, \\ -1 & \text{if } q \in \overline{I}_1 \cup I_{3,4}, \\ 0 & \text{if } q \in \overline{I}_2, \end{cases} \tag{3.91}$$

and we define η'_q similarly (then $\sigma'_q = \varepsilon'_q - \eta'_{\overline{q}} \varepsilon'_{\overline{q}}$, *cf.* (2.23)). By (3.85), both coefficients in (3.90) are nonzero. Since $2a\varepsilon_{\overline{p}} - \sigma_p \in \Gamma \setminus \Gamma_{1,4}$, we have

$$\theta(x^{2a\varepsilon_{\overline{p}}-\sigma_p}) \in \mathbb{F}x'^{\gamma} \setminus \{0\} \quad \text{for some } \gamma \in \Gamma' \setminus \Gamma'_{1,4}. \tag{3.92}$$

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From $[x^{2a\varepsilon_{\overline{p}}-\sigma_p}, x^{-a\varepsilon_{\overline{p}}-\sigma_p}] \in \mathbb{F}x^{a\varepsilon_{\overline{p}}-\sigma_p}$, it follows from (3.87) that

$$[x'^{\gamma}, x'^{\beta}] \in \mathbb{F}x'^{\alpha}. \tag{3.93}$$

Thus there exists $q' \in I'_{1,4}$ such that

$$\gamma_{q'}\beta_{\overline{q}'} - \gamma_{\overline{q}'}\beta_{q'} \neq 0 \quad \text{and} \quad \sigma'_{q'} + \gamma + \beta = \alpha. \tag{3.94}$$

Hence

$$\gamma = \alpha - \beta - \sigma'_{q'} = 2\alpha + \sigma'_q + \sigma'_r - \sigma'_{q'}, \tag{3.95}$$

by (3.89). If $q \neq q' \neq r$, we deduce from (3.89) and (3.95) that

$$\begin{aligned}[x'^{\gamma}, x'^{\beta}] = (\alpha_q\eta'_{\overline{q}} + \alpha_{\overline{q}})x'^{\sigma'_q+\gamma+\beta} + (\alpha_r\eta'_{\overline{r}} + \alpha_{\overline{r}})x'^{\sigma'_r+\gamma+\beta} \\ + (\gamma_{q'}\beta_{\overline{q}'} - \gamma_{\overline{q}'}\beta_{q'})x'^{\sigma'_{q'}+\gamma+\beta} \notin \mathbb{F}x'^{\alpha},\end{aligned} \tag{3.96}$$

a contradiction with (3.93). Similarly, if $q' = q$ or $q' = r$, we can still deduce a contradiction from (3.89), (3.93) and (3.95). This proves the claim.

We extend ν to $\nu\colon J_{1,4} \to J'_{1,4}$ such that $\nu(\overline{p}) = \overline{p}^*$ for $p \in I_{1,4}$. For $p \in I_{1,4}$, by (2.26), we fix $e_p \in \mathbb{F}^{\times}$ such that

$$\lambda_p = e_p\varepsilon_{\overline{p}} \in \Gamma \setminus \{0\}. \tag{3.97}$$

Then $\lambda_p \notin \Gamma_{1,4}$. Denote $\lambda_p^* = \tau(\lambda_p)$ (*cf.* (3.80)). Write

$$\lambda_p^* = (\lambda^*_{p,1}, \lambda^*_{p,\overline{1}}, \ldots, \lambda^*_{p,\iota'_7}, \lambda^*_{p,\overline{\iota}'_7}) \in \Gamma' \subset \mathbb{F}^{2\iota'_7}, \tag{3.98}$$

(*cf.* (2.20)). For $p, q \in I_{1,4}$, applying θ to $[x^{\lambda_p}, x^{-\sigma_q}] = \delta_{p,q}e_p x^{\lambda_p}$, by (3.80) and (3.81), we obtain

$$d_q(\eta'_{\overline{q}^*}\lambda^*_{p,q^*} + \lambda^*_{p,\overline{q}^*}) = \delta_{p,q}e_p \quad \text{for } p, q \in I_{1,4}. \tag{3.99}$$

Let $p \neq q$. Applying θ to $0 = [x^{\lambda_p}, x^{\lambda_q}]$ and using (3.99), we obtain

$$0 = \lambda^*_{p,q^*}\lambda^*_{q,\overline{q}^*} - \lambda^*_{p,\overline{q}^*}\lambda^*_{q,q^*} = \lambda^*_{p,q^*}(\lambda^*_{q,\overline{q}^*} + \eta'_{\overline{q}^*}\lambda^*_{q,q^*}) = \lambda^*_{p,q^*}d_q^{-1}e_q. \tag{3.100}$$

The above two equations imply

$$\lambda^*_{p,q^*} = 0 \quad \text{for } p \in I_{1,4}, q \in J_{1,4}, q \neq p, \overline{p}. \tag{3.101}$$

Denote

$$\Gamma_p = (\mathbb{F}\varepsilon_p + \mathbb{F}\varepsilon_{\overline{p}}) \cap \Gamma. \tag{3.102}$$

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Exactly as in the proof of (3.101), we have

$$\alpha^*_{q^*} = 0 \quad \text{for } \alpha \in \Gamma_p \setminus \Gamma_{1,4},\ p, q \in J_{1,4},\ q \neq p, \overline{p}. \tag{3.103}$$

Claim 2 $\tau\colon \alpha \mapsto \alpha^*$ can be uniquely extended to a group isomorphism $\tau\colon \Gamma \to \Gamma'$ such that $\sigma^*_p = \sigma'_{p^*}$ for $p \in I_{1,4}$.

Noting that by (3.24), (3.25) and (3.78), $\alpha \notin \Gamma_{1,4}$ implies $\alpha + k\sigma_1 \notin \Gamma_{1,4}$ for $k \in \mathbb{Z}$. For any $\alpha \in \Gamma$, $\beta \in \Gamma_1$ with $\alpha, \beta, \alpha + \beta \notin \Gamma_{1,4}$, we have (recall (3.91))

$$\begin{aligned}\big(\alpha_1(\beta_{\overline{1}} + \eta_{\overline{1}}) - \alpha_{\overline{1}}(\beta_1 - 1)\big)\, c_{\alpha+\beta-\sigma_1} x'^{(\alpha+\beta)^*} \\ = c_\alpha c_{\beta-\sigma_1}\big(\alpha^*_{1^*}(\beta - \sigma_1)^*_{\overline{1}^*} - \alpha^*_{\overline{1}^*}(\beta - \sigma_1)^*_{1^*}\big)\, x'^{\sigma'_{1^*} + \alpha^* + (\beta-\sigma_1)^*},\end{aligned} \tag{3.104}$$

by applying θ to (3.2) and by (3.103). By comparing the power of x', this implies

$$(\alpha + \beta)^* = \sigma'_{1^*} + \alpha^* + (\beta - \sigma_1)^* \tag{3.105}$$

if α, β satisfy

$$\beta \in \Gamma_1, \quad \alpha, \beta, \alpha + \beta \in \Gamma \setminus \Gamma_{1,4}, \quad \text{and} \quad \alpha_1(\beta_{\overline{1}} + \eta_{\overline{1}}) - \alpha_{\overline{1}}(\beta_1 - 1) \neq 0. \tag{3.106}$$

Let $\alpha \in \Gamma \setminus \Gamma_{1,4}$. We prove by induction on $|k|$ that

$$(k\alpha)^* - k\alpha^* \in \widetilde{\Gamma}'_1, \quad \text{where} \quad \widetilde{\Gamma}'_1 = \{\beta \in \Gamma' \mid \beta_q = 0 \text{ for } q \in J_{1,4},\ q \neq 1^*, \overline{1}^*\}. \tag{3.107}$$

Let $\gamma \in \Gamma$ such that $\gamma, \alpha + \gamma \notin \Gamma_{1,4}$. We have

$$\sum_{p \in I_{1,4}} (\alpha_p \gamma_{\overline{p}} - \alpha_{\overline{p}} \gamma_p) c_{\sigma_p+\alpha+\gamma} x'^{(\sigma_p+\alpha+\gamma)^*} = c_\alpha c_\gamma \sum_{p \in I'_{1,4}} (\alpha^*_{p^*}\gamma^*_{\overline{p}^*} - \alpha^*_{\overline{p}^*}\gamma^*_{p^*}) x'^{\sigma'_{p^*} + \alpha^* + \gamma^*}. \tag{3.108}$$

We inductively assume that (3.107) holds for k (for instance, $k = 1$). Let $\gamma = k\alpha + \beta$ for some suitable $\beta \in \Gamma_1$ such that condition (3.106) holds for all the involved pairs for which we need to make use of (3.105) in the following proof (when α, k are fixed, by (2.26), such β exists), by (3.107) (note that we assume (3.107) holds for k), (3.105) and (3.103), we see that all terms in (3.108) vanish except the terms corresponding to $p = 1$ in both sides. Thus we obtain

$$\begin{aligned}\sigma'_{1^*} + \big((k+1)\alpha\big)^* + \beta^* &= \big(\sigma_1 + (k+1)\alpha + \beta\big)^* \\ &= \sigma'_{1^*} + \alpha^* + (k\alpha + \beta)^* \\ &= 2\sigma'_{1^*} + \alpha^* + (k\alpha)^* + (\beta - \sigma_1)^*,\end{aligned} \tag{3.109}$$

where the first and last equalities follow from (3.105) and the second follows from (3.108). From this we see that (3.107) holds for $k + 1$. This proves (3.107). Now replacing α by $j\alpha$ (with $j \neq 0$) and β by $k\alpha + \beta - \sigma_1$ in (3.108) (with suitable $\beta \in \Gamma_1$), since (3.107) holds, we have again that all terms in (3.108) vanish except

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the terms corresponding to $p = 1$ in both sides. Thus we have a similar formula to (3.109):

$$\big((j+k)\alpha+\beta\big)^* = 2\sigma'_{1^*} + (j\alpha)^* + (k\alpha)^* + (\beta - 2\sigma_1)^*. \tag{3.110}$$

From this we obtain

$$(j\alpha)^* + (k\alpha)^* = (j'\alpha)^* + (k'\alpha)^* \quad \text{if } j+k = j'+k',\ j,k,j',k' \neq 0. \tag{3.111}$$

From this we obtain

$$(j\alpha)^* = j\alpha^* \quad \text{for } \alpha \in \Gamma \setminus \Gamma_{1,4},\ j \in \mathbb{Z} \setminus \{0\}. \tag{3.112}$$

For some suitable $\beta \in \Gamma_1$, by (3.105), (3.110) and (3.112), we have

$$\begin{aligned}\sigma'_{1^*} + (j\alpha+\sigma_1)^* + (\beta-2\sigma_1)^* &= \big((j\alpha+\sigma_1)+(\beta-\sigma_1)\big)^* \\ &= (j\alpha+\beta)^* = 2\sigma'_{1^*} + j\alpha^* + (\beta-2\sigma_1)^*.\end{aligned} \tag{3.113}$$

From this we obtain

$$(j\alpha+\sigma_1)^* = j\alpha^* + \sigma'_{1^*} \quad \text{for } \alpha \in \Gamma \setminus \Gamma_{1,4},\ j \in \mathbb{Z} \setminus \{0\}. \tag{3.114}$$

Now take any $\alpha, \gamma \in \Gamma$ such that

$$\alpha, \gamma, \alpha+\gamma \in \Gamma \setminus \Gamma_{1,4} \quad \text{and} \quad \alpha_1\gamma_{\bar{1}} - \alpha_{\bar{1}}\gamma_1 \neq 0. \tag{3.115}$$

Using (3.114) in (3.108), by comparing the term $x'^{(\sigma_1+\alpha+\gamma)^*}$ in both sides, we obtain

$$\begin{aligned}&(\alpha+\gamma)^* = \alpha^* + \gamma^* + \sum_{p \in I_{1,4}} k^{(p)}_{\alpha,\gamma}(\sigma'_{p^*} - \sigma'_{1^*}), \quad \text{where} \\ &k^{(p)}_{\alpha,\gamma} = 0,1 \quad \text{such that} \sum_{p \in I_{1,4}} k^{(p)}_{\alpha,\gamma} \leq 1.\end{aligned} \tag{3.116}$$

We claim that $(\alpha+\gamma)^* = \alpha^* + \gamma^*$ if the pairs (α,γ), $(2\alpha, 2\gamma)$ satisfy (3.115). Assume that $k^{(q)}_{\alpha,\gamma} = 1$ for some $q \in I_{1,4}$. Then we obtain

$$\begin{aligned}(2\alpha)^* + (2\gamma)^* + \sum_{p \in I_{1,4}} k^{(p)}_{2\alpha,2\gamma}(\sigma'_{p^*} - \sigma'_{1^*}) &= (2\alpha+2\gamma)^* \\ &= \big(2(\alpha+\gamma)\big)^* = 2(\alpha+\gamma)^* \\ &= 2\Big(\alpha^* + \gamma^* + \sum_{p \in I_{1,4}} k^{(p)}_{\alpha,\gamma}(\sigma'_{p^*} - \sigma'_{1^*})\Big),\end{aligned} \tag{3.117}$$

from this we obtain $k^{(q)}_{2\alpha,2\gamma} = 2k^{(q)}_{\alpha,\gamma} > 1$, which is a contradiction to (3.116).

For any $\alpha, \beta, \alpha+\beta \in \Gamma \setminus \Gamma_{1,4}$, we can always choose $\gamma \in \Gamma \setminus \Gamma_{1,4}$ such that the pairs

$$(\alpha+\beta,\gamma), \quad (2\alpha+2\beta,2\gamma), \quad (\alpha,\beta+\gamma), \quad (2\alpha,2\beta+2\gamma), \quad (\beta,\gamma), \quad (2\beta,2\gamma), \tag{3.118}$$

satisfy (3.115). Hence

$$(\alpha+\beta)^* + \gamma^* = (\alpha+\beta+\gamma)^* = \alpha^* + (\beta+\gamma)^* = \alpha^* + \beta^* + \gamma^*, \tag{3.119}$$

which shows

$$(\alpha+\beta)^* = \alpha^* + \beta^* \quad \text{for } \alpha,\beta,\alpha+\beta \in \Gamma \setminus \Gamma_{1,4}. \tag{3.120}$$

This shows that τ can be uniquely extended to a group isomorphism $\tau \colon \Gamma \to \Gamma'$ such that $\sigma_1^* = \sigma'_{1^*}$ and so similarly $\sigma_p^* = \sigma'_{p^*}$ for $p \in I_{1,4}$. This proves the claim.

Claim 3 We have $\nu(I_i) = I'_i$ for $i = 1,2,3,4$. In particular, $(\ell_1,\ell_2,\ell_3,\ell_4) = (\ell'_1,\ell'_2,\ell'_3,\ell'_4)$, $I_i = I'_i$ for $i = 1,2,3,4$, and $\sigma_p = \sigma'_p$, $\eta_p = \eta'_p$ for $p \in I_{1,4}$ (*cf.* (2.23) and (3.91)).

Note that $\mathrm{ad}_{x^{-\sigma_p}}$ is a semi-simple operator on $\mathcal{H}$ if and only if $p \in I_{1,2}$ (*cf.* (3.3)). Thus

$$\nu(I_{1,2}) = I'_{1,2}, \quad \text{and so} \quad \nu(I_{3,4}) = I'_{3,4}. \tag{3.121}$$

Denote

$$\begin{aligned} \mathcal{N} &= \{u \in \mathcal{H} \mid [u,\mathcal{M}] \subset \mathcal{M}\} \\ &= \mathcal{M} + \mathrm{span}\{x^{\alpha,\underline{i}} \mid \alpha = \alpha_{I_{5,6}},\ |\underline{i}| = 1 \text{ or } \underline{i} = \underline{i}_{I_6\cup J_7}\}, \end{aligned} \tag{3.122}$$

$$\begin{aligned} \mathcal{N}_0 &= \mathcal{M} + \{u \in \mathcal{N} \mid [x^{-\sigma_p},u] = 0 \text{ for } p \in I_{1,4}\} \\ &= \mathcal{M} + \mathrm{span}\{x^{\alpha,\varepsilon_q}, x^{\alpha,\underline{j}} \mid \alpha = \alpha_{I_{5,6}},\ q \in \overline{I}_{5,6},\ \underline{j} = \underline{j}_{I_6\cup J_7}\}, \end{aligned} \tag{3.123}$$

$$\mathcal{N}_p = \mathcal{M} + \mathrm{span}\{u \in \mathcal{N} \mid [x^{-\sigma_p},u] = 0\} \quad \text{for } p \in I_{1,4}. \tag{3.124}$$

Then $\mathcal{N}_0$ is a Lie algebra and $\mathcal{N}$ is an $\mathcal{N}_0$-module such that $\mathcal{N}_p$ is a submodule for $p \in I_{1,4}$. Note that the quotient module $\mathcal{N}/\mathcal{N}_p$ is zero if $p \in I_1$, is a cyclic $\mathcal{N}_0$-module (with generator $t_{\overline{p}}$) if $p \in I_{2,3}$, and is not cyclic (with two generators $t_p, t_{\overline{p}}$) if $p \in I_4$. Applying θ to the above sets and by (3.121), we obtain the claim.

Using Claim 3 and (3.54), by replacing $\mathcal{H}$ by $\theta_\nu(\mathcal{H})$ (*cf.* (3.54)), we can now suppose $\nu = 1$.

Claim 4 . There exists $A = \mathrm{diag}(A_1,\ldots,A_{\iota_4}) \in \mathrm{GL}_{2\iota_4}$, where

$$A_p = \begin{pmatrix} a_p + b_p & a_p \\ 1 - a_p - b_p & 1 - a_p \end{pmatrix}, \quad A_q = \begin{pmatrix} 1 & 0 \\ a_q & b_q \end{pmatrix} \in \mathrm{GL}_2, \tag{3.125}$$

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for $p \in I_1 \cup I_{3,4}$, $q \in I_2$, such that $\alpha^*_{\{p,\overline{p}\}} = \alpha_{\{p,\overline{p}\}} A_p$ for $\alpha \in \Gamma \setminus \Gamma_{1,4}$, $p \in I_{1,4}$.

Using that τ is a group isomorphism and applying θ to

$$[x^{-\sigma_p}, x^\alpha] = (\alpha_{\overline{p}} + \eta_{\overline{p}}\alpha_p)x^\alpha \tag{3.126}$$

(*cf.* (3.3) and (3.91)), by (3.80) and (3.81), we obtain

$$d_p(\alpha^*_{\overline{p}} + \eta_{\overline{p}}\alpha^*_p) = \alpha_{\overline{p}} + \eta_{\overline{p}}\alpha_p \quad \text{if } \alpha_{\overline{p}} + \eta_{\overline{p}}\alpha_p \neq 0, \alpha \in \Gamma \setminus \Gamma_{1,4}, p \in I_{1,4}. \tag{3.127}$$

Comparing the coefficients in (3.108), we obtain

$$\begin{aligned}(\alpha_p\gamma_{\overline{p}} - \alpha_{\overline{p}}\gamma_p)c_{\sigma_p+\alpha+\gamma} = c_\alpha c_\gamma(\alpha^*_p\gamma^*_{\overline{p}} - \alpha^*_{\overline{p}}\gamma^*_p) \quad \text{if} \\ \alpha_p\gamma_{\overline{p}} - \alpha_{\overline{p}}\gamma_p \neq 0,\ \alpha, \gamma, \alpha+\gamma \in \Gamma \setminus \Gamma_{1,4}.\end{aligned} \tag{3.128}$$

Suppose $\alpha \pm \gamma \notin \Gamma_{1,4}$. Replacing γ by $-\gamma$ in (3.128), and dividing by the result from (3.128), we obtain

$$c_{-\gamma}c_\gamma^{-1} = c_{\sigma_p+\alpha-\gamma}c^{-1}_{\sigma_p+\alpha+\gamma} = c_{\alpha-\gamma}c^{-1}_{\alpha+\gamma}. \tag{3.129}$$

In particular, by taking $\gamma = \sigma_p + \lambda_p$ (recall (3.97)) and replacing α by $\alpha + \sigma_p + \lambda_p$, we obtain that

$$c_{-\sigma_p-\lambda_p}c_{\alpha+2\sigma_p+2\lambda_p} = c_\alpha c_{\sigma_p+\lambda_p}, \tag{3.130}$$

holds under some conditions on α (these conditions are linear inequalities on α_p, $\alpha_{\overline{p}}$). Setting $\gamma = \sigma_p + 2\lambda_p$ in (3.128) and using (3.130), we obtain

(3.131)

$$\big(\alpha_p(-\eta_{\overline{p}}+2e_p) - 2\alpha_{\overline{p}}\big)\, c^{-1}_{-\sigma_p-\lambda_p} = c^{-1}_{\sigma_p+\lambda_p}c_{\sigma_p+2\lambda_p}\big(\alpha^*_p(-\eta_{\overline{p}}+2\lambda^*_{p,\overline{p}}) - \alpha^*_{\overline{p}}(1+2\lambda^*_{p,p})\big),$$

holds under some conditions on α. Recall from (3.91) that $\eta_{\overline{p}} = 0$ if $p \in I_2$ and $\eta_{\overline{p}} = -1$ otherwise. Noting that when p is fixed, all coefficients (such as $\lambda^*_{p,p}$) of α_p, $\alpha_{\overline{p}}$, α^*_p, $\alpha^*_{\overline{p}}$ appearing in (3.127) and (3.131) are constant. From (3.127) and (3.131), using (3.99), we can solve $\alpha^*_p, \alpha^*_{\overline{p}}$ as linear combinations of $\alpha_p, \alpha_{\overline{p}}$ with the coefficient matrices as required in the claim (*i.e.*, as shown in (3.125)); furthermore, we have $b_p = d_p^{-1}$. Since τ is a group isomorphism, the condition on α can be removed, *i.e.*, the claim holds for all $\alpha \in \Gamma$.

Claim 5 In (3.125), $a_p = 0$ if $p \in I_3$.

Let $p \in I_3$. We write $\theta(t_p) = bt'_p + \sum_{(0,\varepsilon_p)\neq(\beta,\underline{j})\in\Gamma'\times\mathcal{J}'} b_{\beta,\underline{j}}x^{\beta,\underline{j}}$ for some $b, b_{\beta,\underline{j}} \in \mathbb{F}$. Then we have

$$\alpha_{\overline{p}}c_{\alpha+\sigma_p}x'^{\alpha^*+\sigma_p} = \theta([t_p, x^\alpha]) = bc_\alpha\alpha^*_{\overline{p}}x'^{\alpha^*+\sigma_p} + \cdots \tag{3.132}$$

for $\alpha \in \Gamma \setminus \Gamma_{1,4}$, where the missed terms do not contain $x'^{\alpha^*+\sigma_p}$. Thus by (3.125), we have

$$\alpha_{\overline{p}}c_{\alpha+\sigma_p} = bc_\alpha\big(a_p\alpha_p + (1-a_p)\alpha_{\overline{p}}\big). \tag{3.133}$$

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Hence $b \neq 0$. Take $0 \neq \alpha \in \mathbb{F}\varepsilon_p \cap \Gamma$ (then $\alpha \notin \Gamma_{1,4}$), we obtain $a_p = 0$. This also proves (3.43) and (3.44).

Claim 6 Denote $\sigma = \sum_{p \in I_{1,4}} \sigma_p$. For any $\alpha \in \Gamma$ with $\alpha_{J_{1,4}} \neq \sigma$, we have $\theta(x^\alpha) = c_\alpha x'^{\alpha^*}$ for some $c_\alpha \in \mathbb{F}^\times$.

Assume that $\alpha \in \Gamma_{1,4}$ with $\alpha_{J_{1,4}} \neq \sigma$. Then by (2.26), we can always choose $\beta = \beta_p \varepsilon_p + \beta_{\overline{p}} \varepsilon_{\overline{p}} \in \Gamma \setminus \Gamma_{1,4}$ for some $p \in I_{1,4}$, such that

$$a = \beta_p(\alpha_{\overline{p}} - \beta_{\overline{p}} + \eta_p) - \beta_{\overline{p}}(\alpha_p - \beta_p - 1) \neq 0. \tag{3.134}$$

Then $\beta, \alpha - \beta - \sigma_p \notin \Gamma_{1,4}$ and $\beta^* \in \widetilde{\Gamma}'_p$ (where $\widetilde{\Gamma}'_p$ is a similar notation as in (3.107)). We have

$$\theta(x^\alpha) = a^{-1}\theta([x^\beta, x^{\alpha-\beta-\sigma_p}]) = a^{-1} c_\beta c_{\alpha-\beta-\sigma_p} [x'^{\beta^*}, x'^{\alpha^*-\beta^*-\sigma_p}] \in \mathbb{F}x'^{\alpha^*}. \tag{3.135}$$

By (3.7) and Lemma 3.1, we have

$$\theta(t_p) \in \mathcal{H}'^F \subset \operatorname{span}(H'_1 \cup H'_3) = \sum_{q \in I'_{1,4}} \mathbb{F}x'^{-\sigma_q} + \sum_{r \in \overline{I}'_{5,6}} \mathbb{F}t'_r + H'_3, \tag{3.136}$$

for $p \in \overline{I}_{5,6}$. Thus, using notations (2.22) and (3.76), we have (also recall notations (3.60) and (3.61))

$$\theta^{-1}(\overline{t}'_{\overline{I}'_{5,6}}) \equiv (x^{-\sigma})_{I_{1,4}} F_1 + \overline{t}_{\overline{I}_{5,6}} F_2 \pmod{H_3}, \tag{3.137}$$

for some

$$F_1 = (a_{p,q})_{p \in I_{1,4}, q \in \overline{I}'_{5,6}} \in M_{\iota_4 \times (\ell'_5 + \ell'_6)}, \tag{3.138}$$

$$F_2 = (b_{p,q})_{p \in \overline{I}_{5,6}, q \in \overline{I}'_{5,6}} \in \mathrm{GL}_{\ell_5+\ell_6}, \tag{3.139}$$

(in particular $\ell_5 + \ell_6 = \ell'_5 + \ell'_6$).

Claim 7 We have

$$a_{p,q} = 0 \quad \text{if } p \in I_{3,4}, q \in \overline{I}'_5, \tag{3.140}$$

$$b_{p,q} = 0 \quad \text{if } p \notin I_5, q \in \overline{I}'_5, \tag{3.141}$$

which implies $(\ell_5, \ell_6) = (\ell'_5, \ell'_6)$ and $I_i = I'_i$ for $i = 5, 6$.

Note that the center of $\mathcal{M}$ is $\mathcal{C} = \{x^\alpha \in \Gamma \mid \alpha = \alpha_{I_{5,6}}\}$. Denote the centralizer $C_{\mathcal{H}}(\mathcal{C}) = \{u \in \mathcal{H} \mid [u, \mathcal{C}] = 0\}$. It is straightforward to check that

$$\{t_p \mid p \in I_{2,3} \cup J_4 \cup I_6\} \subset C_{\mathcal{H}}(\mathcal{C}) \subset \operatorname{span}\{x^{\alpha,\underline{i}} \mid \underline{i}_{\overline{I}_{5,6}} = 0\}. \tag{3.142}$$

For $p \in I_{1,4}$, (3.142) implies that $\operatorname{ad}_{x^{-\sigma_p}}|_{C_{\mathcal{H}}(\mathcal{C})}$ is semi-simple if and only if $p \in I_{1,2}$, and $\operatorname{ad}_{t_q}|_{C_{\mathcal{H}}(C)}$ is semi-simple for $q \in \overline{I}_5$ by (3.142) and is not semi-simple for $q \in \overline{I}_6$.

Moreover, by (3.1), for $p \in \overline{I}_{5,6}$, ad_{t_p} is semi-simple if and only if $p \in \overline{I}_5$. We obtain the claim.

By (3.140) and (3.141), we can write F_1 and F_2 is the forms

$$F_1 = \begin{pmatrix} B_{1,5} & B_{1,6} \\ B_{2,5} & B_{2,6} \\ 0 & B_{3,6} \end{pmatrix}, \quad F_2 = \begin{pmatrix} B_{5,5} & B_{5,6} \\ 0 & B_{6,6} \end{pmatrix}, \tag{3.143}$$

such that all $B_{i,j}$ have the forms in (3.46) and (3.48).

For any $\alpha \in \Gamma$, we denote

$$\widehat{\alpha} = (\alpha_{\overline{1}} + \eta_{\overline{1}}\alpha_1, \dots, \alpha_{\overline{\iota_4}} + \eta_{\overline{\iota_4}}\alpha_{\iota_4}) \in \mathbb{F}^{\iota_4}, \tag{3.144}$$

(*cf.* (3.91)). For $\alpha \in \Gamma \setminus \Gamma_{1,4}$, applying θ^{-1} to

$$\alpha^*_{I_{5,6}} c_\alpha^{-1} x'^{\alpha^*} = [\overline{t}'_{I_{5,6}}, c_\alpha^{-1} x'^{\alpha^*}], \tag{3.145}$$

(*cf.* (3.4)), using (3.137), and noting that $[H_3, \mathcal{M}] = 0$, we obtain

$$\alpha^*_{I_{5,6}} x^\alpha = [(x^{-\sigma})_{I_{1,4}} F_1 + \overline{t}_{\overline{I}_{5,6}} F_2, x^\alpha] = (\widehat{\alpha} F_1 + \alpha_{I_{5,6}} F_2) x^\alpha, \tag{3.146}$$

that is

$$\alpha^*_{I_{5,6}} = \widehat{\alpha} F_1 + \alpha_{I_{5,6}} F_2, \tag{3.147}$$

holds for all $\alpha \in \Gamma \setminus \Gamma_{1,4}$ and so holds for all $\alpha \in \Gamma$ since $\tau \colon \alpha \mapsto \alpha^*$ is an isomorphism. From this and (3.143), we obtain formulas (3.43)–(3.48) (*cf.* (3.49)).

Claim 8 $\ell_7 = \ell'_7$.

Observe from (3.9) that

$$H_3 = C_{\mathcal{H}}(\mathcal{M}) \quad \text{(the centralizer of } \mathcal{M}), \tag{3.148}$$

$$\mathrm{span}\{x^{\alpha,\underline{i}} \in H_3 \mid \underline{i}_{J_7} = 0\} = C(H_3) \quad \text{(the center of } H_3). \tag{3.149}$$

By exchanging $\mathcal{H}$ with $\mathcal{H}'$ if necessary, we can suppose $\ell_7 \leq \ell'_7$. As in the proof of sufficiency, we can construct an embedding $\overline{\theta} \colon \mathcal{H} \to \mathcal{H}'$ such that

$$\overline{\theta}(x^\alpha) = \theta(x^\alpha), \quad \overline{\theta}(\overline{t}_{\overline{I}_{5,6}}) \equiv \theta(\overline{t}_{\overline{I}_{5,6}}) \pmod{H'_3}, \tag{3.150}$$

(*cf.* Claim 6 and (3.137). Note that using (3.137), we can now obtain that Claim 6 holds for all $\alpha \in \Gamma$ if $\ell_5 + \ell_6 \neq 0$). Thus by identifying $\mathcal{H}$ with $\overline{\theta}(\mathcal{H})$, we can assume that $\mathcal{H}$ is a subalgebra of $\mathcal{H}'$ such that there exists an isomorphism θ satisfying

$$\theta(x^\alpha) = x^\alpha, \quad \theta(t_{\overline{p}}) \equiv t_{\overline{p}} \pmod{H'_3} \quad \text{for } \alpha \in \Gamma,\ p \in I_{5,6}. \tag{3.151}$$

By restricting θ to H_3, we want to prove

$$\theta(t_p) = t_p + c_p \quad \text{for } p \in I_6 \text{ and some } c_p \in \mathbb{F}, \tag{3.152}$$

$$\theta(x^{\alpha,\underline{i}}t^{\underline{j}}) = x^\alpha \prod_{p\in I_6}\big(\theta(t_p)\big)^{i_p} \prod_{q\in J_7}\big(\theta(t_q)\big)^{j_q} \quad \text{for } \alpha = \alpha_{I_{5,6}},\ \underline{i} = \underline{i}_{I_6},\ \underline{j} = \underline{j}_{J_7}. \tag{3.153}$$

To prove (3.152), first by (3.149), we have $c_p = \theta(t_p) - t_p \in C(H_3')$. Then by (3.151), we have

$$[t_{\overline{q}}, c_p] = \theta\big([\theta^{-1}(t_{\overline{q}}), t_p]\big) - [t_{\overline{q}}, t_p] = 0, \tag{3.154}$$

where the second equality follows from the fact that $\theta^{-1}(t_{\overline{q}}) = t_{\overline{q}} \pmod{H_3}$ and $[H_3, t_p] = 0$. From (3.154), we obtain that $c_p \in \mathbb{F}$. Thus we have (3.152). Similarly, we have

$$\theta(x^{\alpha,\varepsilon_p}) = x^\alpha(t_p + c_{\alpha,p}) \quad \text{for } p \in I_6 \text{ and some } c_{\alpha,p} \in \mathbb{F}. \tag{3.155}$$

By considering $\theta([x^\alpha, t_p t_{\overline{p}}]) = [\theta(x^\alpha), \theta(t_p t_{\overline{p}})]$, we see that $c_{\alpha,p} = c_p$, and we obtain

$$\theta(t_p t_{\overline{p}}) = (t_p + c_p)t_{\overline{p}} + u_p \quad \text{for } p \in I_6 \text{ and some } u_p \in C_{\mathcal{H}'}\big(C(H_3')\big). \tag{3.156}$$

From this and (3.152), we can deduce

$$\theta(x^{\alpha,\varepsilon_p}) = x^\alpha(t_p + c_p) \quad \text{for } p \in I_6. \tag{3.157}$$

Similar to (3.156), we have

$$\theta(x^{-\sigma_p,\varepsilon_p+\varepsilon_{\overline{p}}}) = x^{-\sigma_p,\varepsilon_{\overline{p}}}(t_p + c_p) + u_p' \quad \text{for } p \in I_6 \text{ and some } u_p' \in C_{\mathcal{H}'}\big(C(H_3')\big). \tag{3.158}$$

Now from (3.152), (3.155)–(3.158), we can obtain (3.153) by induction on $|\underline{i}|$ in case $\underline{j} = 0$.

Assume that (3.153) holds for all $\underline{j}$ with $|\underline{j}| < n$, where $n \geq 1$. We denote by $A_{\alpha,\underline{i},\underline{j}}$ the difference between the left-hand side and the right-hand side of (3.153). Then the inductive assumption says that $A_{\alpha,\underline{i},\underline{j}} = 0$ if $|\underline{j}| < n$. Now suppose $|\underline{j}| = n$. Say $j_r \geq 1$ for some $r \in I_7$ (the proof is similar if $r \in \overline{I}_7$). Let $\underline{k} = \underline{j} - \varepsilon_r + \varepsilon_{\overline{r}}$. Then we have

$$\begin{aligned}
[\theta(t_r), A_{\alpha,\underline{i},\underline{k}}] &= \theta([t_r, x^{\alpha,\underline{i}}t^{\underline{k}}]) - \theta\big([t_r, \theta^{-1}(x^\alpha)]\big) \prod_{p\in I_6}\big(\theta(t_p)\big)^{i_p} \prod_{q\in J_7}\big(\theta(t_q)\big)^{k_q} \\
&\quad - x^\alpha\Big[\theta(t_r), \prod_{p\in I_6}\big(\theta(t_p)\big)^{i_p} \prod_{q\in J_7}\big(\theta(t_q)\big)^{k_q}\Big] \\
&= (j_{\overline{r}} + 1)\Big(\theta(x^{\alpha,\underline{i}}t^{\underline{j}-\varepsilon_r}) - x^\alpha \prod_{p\in I_6}\big(\theta(t_p)\big)^{i_p} \prod_{q\in J_7}\big(\theta(t_q)\big)^{j_q-\delta_{q,r}}\Big) \\
&= (j_{\overline{r}} + 1)A_{\alpha,\underline{i},\underline{j}-\varepsilon_r} = 0,
\end{aligned} \tag{3.159}$$

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where the first equality follows from (1.1), the second equality follows from (1.1) and (3.151). By (1.1) and (3.159), we obtain

$$[\theta(t_r^2), A_{\alpha,\underline{i},\underline{k}}] = \theta\big([t_r^2, \theta^{-1}(A_{\alpha,\underline{i},\underline{k}})]\big) = 2\theta\big(t_r[t_r, \theta^{-1}(A_{\alpha,\underline{i},\underline{k}})]\big) = 0. \tag{3.160}$$

On the other hand, exactly similar to (3.159), we have

$$[\theta(t_r^2), A_{\alpha,\underline{i},\underline{k}}] = 2(j_{\bar{r}} + 1)A_{\alpha,\underline{i},\underline{j}}. \tag{3.161}$$

Now (3.160) and (3.161) show that $A_{\alpha,\underline{i},\underline{j}} = 0$. This proves (3.153). By (3.152), (3.153) and by identifying $C(H_3)$ with $C(H_3')$ using the isomorphism, we see that θ is an associative algebra isomorphism $H_3 \to H_3'$ over the domain ring $C(H_3)$. From this we obtain $\ell_7 = \ell_7'$ since $2\ell_7$ is the transcendental degree of H_3 over the domain ring $C(H_3)$. This completes the proof of Theorem 3.6. ■

4 Derivations

In this section, we shall determine the structure of the derivation algebra of the Hamiltonian Lie algebra $\mathcal{H} = \mathcal{H}(\underline{\ell}, \Gamma)$. As pointed out in [F], the significance of derivations for Lie theory primarily resides in their affinity to low dimensional cohomology groups; their determination therefore frequently affords insight into structural features of Lie algebras which do not figure prominently in the defining properties. Some general results concerning derivations of graded Lie algebras were established in [F]. However in our case the algebras are in general nongraded, the results in [F] can not be applied to our case here. Thus we try a different method to determine derivations of the Hamiltonian Lie algebras $\mathcal{H}$. Our method is also different from that used in [OZ].

Recall that a *derivation* d of the Lie algebra $\mathcal{H}$ is a linear transformation on $\mathcal{H}$ such that

$$d([u_1, u_2]) = [d(u_1), u_2] + [u_1, d(u_2)] \quad \text{for } u_1, u_2 \in \mathcal{H}. \tag{4.1}$$

Denote by $\operatorname{Der}\mathcal{H}$ the space of the derivations of $\mathcal{H}$, which is a Lie algebra. Moreover, $\operatorname{ad}_{\mathcal{H}}$ is an ideal. Elements in $\operatorname{ad}_{\mathcal{H}}$ are called *inner derivations*, while elements in $\operatorname{Der}\mathcal{H} \setminus \operatorname{ad}_{\mathcal{H}}$ are called *outer derivations*.

We can embed $\mathcal{H}$ into a larger Lie algebra $\widetilde{\mathcal{H}}$ such that $\widetilde{\mathcal{H}}$ has a basis $\{x^{\alpha,\underline{i}} \mid (\alpha, \underline{i}) \in \Gamma \times \mathbb{N}^{2\iota_7}\}$ (*i.e.*, in $\widetilde{\mathcal{H}}$, we replace $\mathcal{J}$ by $\mathbb{N}^{2\iota_7}$, *cf.* (2.27), and we have (3.1) with the last three summands running over $p \in I_{1,6}$, $p \in I_{1,4}$ and $p \in I$ respectively). Then for $p \in J_1 \cup \bar{I}_{2,3} \cup I_5$, clearly, $t_p \notin \mathcal{H}$, but $[t_p, \mathcal{H}] \subset \mathcal{H}$. Thus

$$d_p = \operatorname{ad}_{t_p}|_{\mathcal{H}} \quad \text{for } p \in J_1 \cup \bar{I}_{2,3} \cup I_5, \tag{4.2}$$

defines an outer derivation of $\mathcal{H}$. For $p \in I_{2,3} \cup J_4 \cup \bar{I}_5 \cup J_{6,7}$, obviously, ∂_{t_p} is a derivation of $\mathcal{H}$ (*cf.* (2.27), (2.35) and (3.1)). For $p \in J$, we define $\operatorname{sgn}(p) = 1$ if $p \in I$ and $\operatorname{sgn}(p) = -1$ if $p \in \bar{I}$. Then

$$\partial_{t_p} = \operatorname{sgn}(p)\operatorname{ad}_{t_{\bar{p}}} \quad \text{for } p \in \bar{I}_5 \cup J_{6,7}. \tag{4.3}$$

Define $d_0(x^{\alpha,\underline{i}}) = (\sum_{p\in I_{1,4}} \alpha_p + 1)x^{\alpha,\underline{i}}$ for $(\alpha,\underline{i}) \in \Gamma \times \mathcal{J}$. It is straightforward to verify that d_0 is an outer derivation of $\mathcal{H}$. Denote $\sigma = \sum_{p\in I_{1,4}} \sigma_p$. If $\iota_7 = \ell_1$, then $\mathcal{H} = [\mathcal{H},\mathcal{H}] + \mathbb{F}x^{\sigma}$, and we can define an outer derivation d_0' by setting

$$d_0'([\mathcal{H},\mathcal{H}]) = 0, \quad d_0'(x^{\sigma}) = 1_{\mathcal{H}}. \tag{4.4}$$

If $\iota_7 \neq \ell_1$, we set $d_0' = 0$.

We denote by $\mathrm{Hom}^+_{\mathbb{Z}}(\Gamma,\mathbb{F})$ the set of group homomorphisms $\mu\colon \Gamma \to \mathbb{F}$ such that $\mu(\sigma_p) = 0$ for $p \in I_{1,4}$. For $\mu \in \mathrm{Hom}^+_{\mathbb{Z}}(\Gamma,\mathbb{F})$, we define a linear transformation d_μ on $\mathcal{H}$ by

$$d_\mu(x^{\alpha,\underline{i}}) = \mu(\alpha)x^{\alpha,\underline{i}} \quad \text{for } (\alpha,\underline{i}) \in \Gamma \times \mathcal{J}. \tag{4.5}$$

Clearly, by (3.1), d_μ is a derivation of $\mathcal{H}$. We identify $\mathrm{Hom}^+_{\mathbb{Z}}(\Gamma,\mathbb{F})$ with a subspace of $\mathrm{Der}\,\mathcal{H}$ by $\mu \mapsto d_\mu$. For $p \in I_{1,6}$, we define $\mu_p \in \mathrm{Hom}^+_{\mathbb{Z}}(\Gamma,\mathbb{F})$ by

$$\mu_p(\alpha) = \begin{cases} \alpha_{\overline{p}} + \eta_{\overline{p}}\alpha_p & \text{if } p \in I_{1,4}, \\ \alpha_p & \text{if } p \in I_{5,6}, \end{cases} \tag{4.6}$$

for $\alpha \in \Gamma$ (*cf.* (3.91)). By (3.3) and (3.4), we have

$$\mathrm{ad}_{x^{-\sigma_p}} = \begin{cases} d_{\mu_p} & \text{if } p \in I_{1,2}, \\ d_{\mu_p} + \partial_{t_p} & \text{if } p \in I_3, \\ d_{\mu_p} + \partial_{t_p} - \partial_{t_{\overline{p}}} & \text{if } p \in I_4, \end{cases} \tag{4.7}$$

$$\mathrm{ad}_{t_{\overline{q}}} = \begin{cases} -d_{\mu_q} & \text{if } q \in I_5, \\ -d_{\mu_q} - \partial_{t_q} & \text{if } q \in I_6. \end{cases} \tag{4.8}$$

We fix a subspace $\mathrm{Hom}^*_{\mathbb{Z}}(\Gamma,\mathbb{F})$ of $\mathrm{Hom}^+_{\mathbb{Z}}(\Gamma,\mathbb{F})$ such that

$$\mathrm{Hom}^+_{\mathbb{Z}}(\Gamma,\mathbb{F}) = \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma,\mathbb{F}) \oplus \mathrm{span}\{\mu_p \mid p \in I_{1,6}\}, \tag{4.9}$$

is a direct sum as vector spaces. Since $\mathrm{ad}_{\mathbb{F}} = 0$, we set $\mathcal{H}^* = \mathrm{span}\{x^{\alpha,\underline{i}} \mid (0,0) \neq (\alpha,\underline{i}) \in \Gamma \times \mathcal{J}\}$.

Theorem 4.1 . *The derivation algebra* $\mathrm{Der}\,\mathcal{H}$ *is spanned by*

$$d_0',\ d_p,\ \partial_{t_q},\ d_\mu,\ \mathrm{ad}_{\mathcal{H}^*} \quad \textit{for } p \in \{0\} \cup J_1 \cup \overline{I}_{2,3} \cup I_5,\ q \in I_{2,3} \cup J_4,\ \mu \in \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma,\mathbb{F}). \tag{4.10}$$

Furthermore, we have the following vector space decomposition as a direct sum of subspaces:

$$\mathrm{Der}\,\mathcal{H} = \Big(\Big(\mathbb{F}d_0' + \sum_{p\in\{0\}\cup J_1\cup\overline{I}_{2,3}\cup I_5} \mathbb{F}d_p\Big) \oplus \sum_{q\in I_{2,3}\cup J_4} \mathbb{F}\partial_{t_q} \oplus \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma,\mathbb{F})\Big) \oplus \mathrm{ad}_{\mathcal{H}^*}. \tag{4.11}$$

In particular, all derivations of the classical Hamiltonian Lie algebras $\mathcal{H}(\ell)$ (cf. (1.3)) are inner.

Proof First note that in [OZ], d_0 was written as a derivation of the form d_μ with μ satisfying $\mu(\sigma_p) = \mu(\sigma_1)$ for $p \in I_{1,4}$. Let $d \in \operatorname{Der}\mathcal{H}$ and let D be the subspace of $\operatorname{Der}\mathcal{H}$ spanned by the elements in (4.10). Note that $D \supset \operatorname{Hom}^+_{\mathbb{Z}}(\Gamma, \mathbb{F})$ by (4.7) and (4.8). We shall prove that after a number of steps in each of which d is replaced by $d - d'$ for some $d' \in D$ the 0 derivation is obtained and thus proving that $d \in D$. This will be done by a number of claims.

Claim 1 We can suppose

(i) $d(1) = 0$,
(ii) $d(x^{-\sigma_p}) = 0$ for $p \in I_{3,4}$,
(iii) $d(x^{\varepsilon_q}) = d(t_r) = 0$ for $q \in I_{5,6}$, $r \in \overline{I}_6 \cup J_7$.

By replacing d by $d - d(1)d_0$, we can suppose $d(1) = 0$. For any $(\alpha, \underline{i}) \in \Gamma \times \mathcal{J}$, we write

$$(4.12)\qquad d(x^{\alpha,\underline{i}}) = \sum_{(\beta,\underline{j})\in M_{\alpha,\underline{i}}} c^{(\beta,\underline{j})}_{\alpha,\underline{i}} x^{\alpha+\beta,\underline{j}} \quad \text{for some } c^{(\beta,\underline{j})}_{\alpha,\underline{i}} \in \mathbb{F}, \quad \text{where}$$

$$(4.13)\qquad M_{\alpha,\underline{i}} = \{(\beta, \underline{j}) \in \Gamma \times \mathcal{J} \mid c^{(\beta,\underline{j})}_{\alpha,\underline{i}} \neq 0\},$$

is a finite set. We set $c^{(\beta,\underline{j})}_{\alpha,\underline{i}} = 0$ if $(\beta, \underline{j}) \notin M_{\alpha,\underline{i}}$. We shall denote $M_{\alpha,0}$ simply by M_α. Using the inductive assumption, suppose we have proved that $d(x^{-\sigma_r}) = 0$ for $r \in I_{3,4}$ and $r < p$. Let $(\beta, \underline{j}) \in M_{-\sigma_p}$. Using (3.3), one can deduce by induction on $|\underline{j}|$ that

$$(4.14)\qquad x^{-\sigma_p+\beta,\underline{j}} = [u_{\beta,\underline{j}}, x^{-\sigma_p}] \quad \text{for some } u_{\beta,\underline{j}} \in \mathcal{H},$$

such that $u_{\beta,\underline{j}}$ has the following form

$$(4.15)\qquad u_{\beta,\underline{j}} = \sum_{k,\ell\in\mathbb{Z}} b_{k,\ell} x^{-\sigma_p+\beta,\underline{j}+k\varepsilon_p+\ell\varepsilon_{\overline{p}}} \quad \text{for some } b_{k,\ell} \in \mathbb{F},$$

(recall convention (2.34)). Thus we can take

$$(4.16)\qquad u = \sum_{(\beta,\underline{j})\in M_{-\sigma_p}} c^{(\beta,\underline{j})}_{-\sigma_p,0} u_{\beta,\underline{j}} \in \mathcal{H} \quad \text{such that } (d - \operatorname{ad}_u)(x^{-\sigma_p}) = 0,$$

Applying d to $[x^{-\sigma_r}, x^{-\sigma_p}] = 0$, we obtain

$$(4.17)\qquad \sum_{(\beta,\underline{j})\in M_{-\sigma_p}} c^{(\beta,\underline{j})}_{-\sigma_p,0} [x^{-\sigma_r}, x^{-\sigma_p+\beta,\underline{j}}] = 0 \quad \text{for } r \in I_{3,4}, r < p,$$

i.e.,

$$(4.18)\qquad c^{(\beta,\underline{j})}_{-\sigma_p,0}(\beta_r - \beta_{\overline{r}}) - c^{(\beta,\underline{j}+\varepsilon_{\overline{r}})}_{-\sigma_p,0}(j_{\overline{r}} + 1) + c^{(\beta,\underline{j}+\varepsilon_r)}_{-\sigma_p,0}(j_r + 1) = 0,$$

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for $r \in I_{3,4}$, $r < p$, from this and by induction on $j_r + j_{\overline{r}}$ ranging from $\max\{k_r + k_{\overline{r}} \mid (\beta, \underline{k}) \in M_{-\sigma_p}\}$ down to zero, we obtain

$$\beta_r = \beta_{\overline{r}}, \quad j_r = j_{\overline{r}} = 0 \quad \text{for } (\beta, \underline{j}) \in M_{-\sigma_p}, r \in I_{3,4}, r < p. \tag{4.19}$$

Then (4.15), (4.16) and (4.19) show that $\mathrm{ad}_u(x^{-\sigma_r}) = 0$ for $r \in I_{3,4}$, $r < p$. Thus if we replace d by $d - \mathrm{ad}_u$, we have $d(x^{-\sigma_r}) = 0$ for $r \in I_{3,4}$, $r \leq p$. This proves Claim 1 (ii). Note that for $v = x^{\varepsilon_q}$, $q \in I_{5,6}$, or $v = t_r$, $r \in \overline{I}_6 \cup J_7$, we have $\mathrm{ad}_v(\mathcal{H}) = \mathcal{H}$. Thus similar to the above proof, we have Claim 1 (iii).

Note that for $(\beta, \underline{j}) \in \Gamma \times \mathcal{J}$, by (3.3) and (3.4), we have

$$(\beta_{\overline{p}} + \eta_{\overline{p}}\beta_p)x^{-\sigma_p+\beta,\underline{j}} = [x^{-\sigma_p+\beta,\underline{j}}, x^{-\sigma_p}] \quad \text{for } p \in I_{1,2}, \tag{4.20}$$

$$(-1 + \beta_p)e_p x^{\lambda_p+\beta,\underline{j}} = [x^{-\sigma_p+\beta,\underline{j}}, x^{\lambda_p}] + j_p x^{\lambda_p+\beta,\underline{j}-\varepsilon_p} \quad \text{for } p \in I_{1,4}, \tag{4.21}$$

(recall notations λ_p, $p \in I_{1,4}$ in (3.97)), and

$$\beta_p x^{\beta,\underline{j}} = [x^{\beta,\underline{j}}, t_{\overline{p}}] \quad \text{for } p \in I_5. \tag{4.22}$$

Claim 2 By replacing d by $d - d'$ for some $d' \in D$, we can suppose

$$\beta_{\overline{p}} + \eta_{\overline{p}}\beta_p = 0 \quad \text{for } (\beta, \underline{j}) \in M_{-\sigma_p}, p \in I_{1,2}, \tag{4.23}$$

$$\beta_p = 1 \quad \text{for } (\beta, \underline{j}) \in M_{\lambda_p}, p \in I_{1,4}, \tag{4.24}$$

$$\beta_p = 0 \quad \text{for } (\beta, \underline{j}) \in M_{0,\varepsilon_{\overline{p}}}, p \in I_5. \tag{4.25}$$

The proof of (4.23) is similar to that of Claim 1. To prove (4.24), suppose we have proved

$$\beta_r = 1 \quad \text{for } (\beta, \underline{j}) \in M_{\lambda_r}, i \in I_{1,4}, r < p. \tag{4.26}$$

To see how the proof works, for simplicity, we assume that $p \in I_1$ (the proof for $p \in I_{2,4}$ is exactly similar). Then the second term on the right-hand side of (4.21) vanishes. Let

$$u = \sum_{(\beta,\underline{j}) \in M_{\lambda_p},\, \beta_p \neq 1} c_{\lambda_p,0}^{(\beta,\underline{j})} \big((-1 + \beta_p)e_p\big)^{-1} x^{-\sigma_p+\beta}. \tag{4.27}$$

Then by replacing d by $d - \mathrm{ad}_u$, from (4.21), we see that (4.24) holds for p. We want to prove that after this replacement, Claim 1, (4.23) and (4.26) still hold. It suffices to prove

$$[u, x^{-\sigma_q}] = [u, x^{\varepsilon_{q'}}] = [u, t_{q''}] = [u, x^{\lambda_r}] = 0, \tag{4.28}$$

for $q \in I_{1,4}$, $q' \in I_{5,6}$, $q'' \in \overline{I}_6 \cup J_7$, $r \in I_{1,4}$, $r < p$.

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We have

$$(4.29)\qquad -e_p \sum_{(\beta,\underline{j})\in M_{\lambda_p}} c_{\lambda_p,0}^{(\beta,\underline{j})} x^{\lambda_p+\beta,\underline{j}} = -e_p d(x^{\lambda_p}) = d([x^{-\sigma_p}, x^{\lambda_p}]$$
$$= \sum_{(\beta,\underline{j})\in M_{-\sigma_p}} c_{-\sigma_p,0}^{(\beta,\underline{j})}(-1+\beta_p)x^{\lambda_p+\beta,\underline{j}} + \sum_{(\beta,\underline{j})\in M_{\lambda_p}} c_{\lambda_p,0}^{(\beta,\underline{j})}(\beta_p - \beta_{\overline{p}} - e_p)x^{\lambda_p+\beta,\underline{j}}.$$

This gives

$$(4.30)\qquad \beta_p - \beta_{\overline{p}} = (c_{\lambda_p,0}^{(\beta,\underline{j})})^{-1}(\beta_p - 1)c_{-\sigma_p,0}^{(\beta,\underline{j})} \quad \text{for } (\beta,\underline{j}) \in M_{\lambda_p}.$$

If $(\beta, \underline{j}) \notin M_{-\sigma_p}$, then the right-hand side of (4.30) is zero; on the other hand, if $(\beta, \underline{j}) \in M_{-\sigma_p}$, then (4.23) gives $\beta_p - \beta_{\overline{p}} = 0$. In any case, we have $\beta_p - \beta_{\overline{p}} = 0$ for $(\beta, \underline{j}) \in M_{\lambda_p}$. Thus by (4.27),

$$(4.31)\qquad [u, x^{-\sigma_p}] = \sum_{(\beta,\underline{j})\in M_{\lambda_p},\beta_p\neq 1} c_{\lambda_p,0}^{(\beta,\underline{j})}\big((-1+\beta_p)e_p\big)^{-1}(\beta_p - \beta_{\overline{p}})x^{-\sigma_p+\beta} = 0.$$

Similarly, we can prove other equations in (4.28). This proves (4.24). Similarly, we have (4.25).

Claim 3 By replacing d by $d - \sum_{p\in J_1\cup \overline{I}_{2,3}\cup I_5} a_p d_p - d_\mu$ for some $a_p \in \mathbb{F}$ and some $\mu \in \mathrm{Hom}^{+}_{\mathbb{Z}}(\Gamma, \mathbb{F})$, we can suppose $d(x^{-\sigma_p}) = d(x^{\lambda_q}) = d(t_r) = 0$ for $p \in I_{1,2}$, $q \in I_{1,4}$, $r \in I_5$.

Again for simplicity, we prove that after some replacement, $d(x^{-\sigma_p}) = d(x^{\lambda_p}) = 0$ for $p \in I_1$. Defining $\mu \in \mathrm{Hom}^{+}_{\mathbb{Z}}(\Gamma, \mathbb{F})$ by $\mu(\alpha) = c_{\lambda_p,0}^{(0,0)} e_p^{-1}(\alpha_{\overline{p}} - \alpha_p)$, and by replacing d by $d - d_\mu$, we obtain $c_{\lambda_p,0}^{(0,0)} = 0$ (recall (4.12) that $c_{\alpha,\underline{i}}^{(\beta,\underline{j})}$ is the coefficient of $x^{\alpha+\beta,\underline{j}}$, not that of $x^{\beta,\underline{j}}$). Obviously, this replacement does not affect the result we have obtained so far. Recalling the definition of d_p in (4.2), we have

(4.32)
$$d_p(x^{-\sigma_p}) = [t_p, x^{-\sigma_p}] = -1, \quad d_{\overline{p}}(x^{-\sigma_p}) = 1, \quad d_p(x^{\lambda_p}) = e_p x^{\sigma_p+\lambda_p}, \quad d_{\overline{p}}(x^{\lambda_p}) = 0.$$

Thus by replacing d by $d - a_p d_p - a_{\overline{p}} d_{\overline{p}}$ for some $a_p, a_{\overline{p}} \in \mathbb{F}$, we can suppose

$$(4.33)\qquad c_{-\sigma_p,0}^{(\sigma_p,0)} = c_{\lambda_p,0}^{(\sigma_p,\lambda_p)} = c_{\lambda_p,0}^{(0,0)} = 0.$$

Note again that the replacement does not affect the results we have obtained so far.

Let $q \in I_1$, $q \neq p$. We have

(4.34)
$$0 = d([x^{-\sigma_p}, x^{-\sigma_q}]) = \sum \Big(c_{-\sigma_p,0}^{(\beta,\underline{j})}(\beta_{\overline{q}} - \beta_q) + c_{-\sigma_q,0}^{(\beta+\sigma_q-\sigma_p,\underline{j})}(\beta_p - \beta_{\overline{p}})\Big)\, x^{-\sigma_p+\beta,\underline{j}},$$

(4.35)
$$0 = d([x^{-\sigma_p}, x^{\lambda_q}]) = \sum \Big(c_{-\sigma_p,0}^{(\beta,\underline{j})}\beta_q e_q + c_{\lambda_q,0}^{(\beta+\sigma_q-\sigma_p,\underline{j})}(\beta_p - \beta_{\overline{p}})\Big)\, x^{-\sigma_p+\sigma_q+\beta+\lambda_q,\underline{j}}.$$

Now (4.23), (4.34) and (4.35) show that $\beta_{\overline{q}} = \beta_q = 0$ if $(\beta, \underline{j}) \in M_{-\sigma_p}$. Similarly, we can prove $\beta_r = j_r = 0$ for all $r \in J, r \neq p, \overline{p}$ if $(\beta, \underline{j}) \in M_{-\sigma_p}$. This and (4.30) show that $(\beta, \underline{j}) = (\sigma_p, 0)$ if $(\beta, \underline{j}) \in M_{-\sigma_p}$. But $(\sigma_p, 0) \notin M_{-\sigma_p}$ by (4.33), *i.e.*, $M_{-\sigma_p} = \varnothing$. Thus $d(x^{-\sigma_p}) = 0$. Similarly $d(x^{\lambda_p}) = 0$. Analogously, we can obtain other results of Claim 3.

Claim 4 We can suppose $d = 0$.

Note that x^α is a common eigenvector for the elements of the set

$$A = \{x^{-\sigma_p}, x^{\varepsilon_q}, t_r \mid p \in I_{1,4},\ q \in I_{5,6},\ r \in \overline{I}_{5,6} \cup J_7\}. \tag{4.36}$$

Since $d(A) = 0$, $d(x^\alpha)$ is also a common eigenvector for the elements of A. From this and Lemma 3.2, we obtain

$$\eta_{\overline{p}}\beta_p + \beta_{\overline{p}} = \beta_q = 0, \quad \underline{j} = 0 \quad \text{for } p \in I_{1,4}, q \in I_{5,6} \text{ and } (\beta, \underline{j}) \in M_\alpha. \tag{4.37}$$

For simplicity, we denote $c_\alpha^{(\beta)} = c_{\alpha,0}^{(\beta,0)}$. We want to prove

$$d(x^\alpha) = m_\alpha x^\alpha \quad \text{for } \alpha \in \Gamma \text{ and some } m_\alpha \in \mathbb{F}, \tag{4.38}$$

i.e., M_α is either empty or a singleton $\{(0, 0)\}$. Thus assume that

$$\beta_p \neq 0 \quad \text{for some } (\beta, \underline{j}) \in M_\alpha, p \in I_{1,4}, \alpha \in \Gamma. \tag{4.39}$$

For convenience, we again suppose $p \in I_1$. Denote $\Gamma_p = (\mathbb{F}\varepsilon_p + \mathbb{F}\varepsilon_{\overline{p}}) \cap \Gamma$ as in (3.102), and set $\mathcal{H}_p = \operatorname{span}\{x^\alpha \mid \alpha \in \Gamma_p\}$. We have

$$d(x^\alpha) \in \mathcal{H}_p \quad \text{for } \alpha \in \Gamma_p, \tag{4.40}$$

by using the fact that x^α commutes with elements of A except possibly $x^{-\sigma_p}, x^{\lambda_p}$. By (4.37) and by

$$0 = d([x^{-\sigma_p-\lambda_p}, x^{\lambda_p}]) = [d(x^{-\sigma_p-\lambda_p}), x^{\lambda_p}] = \sum c_{-\sigma_p-\lambda_p}^{(\beta)} e_p x^\beta, \tag{4.41}$$

and by (4.40), we obtain

$$d(x^{-\sigma_p-\lambda_p}) = a_p x^{-\lambda_p}, \quad d(x^{\lambda_p-\sigma_p}) = -a_p x^{\lambda_p} \quad \text{for some } a_p \in \mathbb{F}, \tag{4.42}$$

where the second equation is obtained from $d([x^{-\sigma_p-\lambda_p}, x^{\lambda_p-\sigma_p}]) = 0$. Applying d to

$$[x^{-2\sigma_p}, x^{\lambda_p}] = -2e_p x^{-\sigma_p+\lambda_p}, \quad [x^{-2\sigma_p}, x^{-\sigma_p-\lambda_p}] = 2e_p x^{-2\sigma_p-\lambda_p}, \tag{4.43}$$

$$[x^{-2\sigma_p-\lambda_p}, x^{\lambda_p-\sigma_p}] = -3e_p x^{-2\sigma_p}, \tag{4.44}$$

we obtain respectively

$$d(x^{-2\sigma_p}) = -2a_p x^{-\sigma_p}, \quad d(x^{-2\sigma_p-\lambda_p}) = 0, \quad a_p = 0. \tag{4.45}$$

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Thus all equations in (4.42) and (4.45) are zero. Applying d to $[x^{\lambda_p-\sigma_p}, x^{k\lambda_p}] = -kx^{(k+1)\lambda_p}$, using induction on k, we obtain

$$d(x^{k\lambda_p}) = 0 \quad \text{for } k \geq 1. \tag{4.46}$$

Applying d to

(4.47)
$$[x^\alpha, x^{-2\sigma_p}] = 2(\alpha_{\overline{p}} - \alpha_p)x^{\alpha-\sigma_p}, \quad [x^\alpha, x^{-\sigma_p-k\lambda_p}] = (\alpha_{\overline{p}} - \alpha_p - k\alpha_{\overline{p}}e_p)x^{\alpha-k\lambda_p},$$

$$[x^{\alpha-k\lambda_p}, x^{k\lambda_p}] = k\alpha_p e_p x^{\alpha+\sigma_p}, \tag{4.48}$$

for $k \geq 1$, using (4.37), we obtain

$$2(\alpha_{\overline{p}} - \alpha_p)c_\alpha^{(\beta)} = 2(\alpha_{\overline{p}} - \alpha_p)c_{\alpha+\sigma_p}^{(\beta)}, \tag{4.49}$$

$$\big(\alpha_{\overline{p}} - \alpha_p - k(\alpha_p + \beta_p)e_p\big)\, c_\alpha^{(\beta)} = (\alpha_{\overline{p}} - \alpha_p - k\alpha_p e_p)c_{\alpha-k\lambda_p}^{(\beta)}, \tag{4.50}$$

$$k(\alpha_p + \beta_p)e_p c_{\alpha-k\lambda_p}^{(\beta)} = k\alpha_p e_p c_{\alpha+\sigma_p}^{(\beta)}. \tag{4.51}$$

If $\alpha_p \neq \alpha_{\overline{p}}$, then the above three equations gives $\beta_p = 0$, a contradiction with (4.39). Thus we obtain

$$\beta_p \neq 0,\ (\beta, 0) \in M_\alpha \Rightarrow \alpha_p = \alpha_{\overline{p}}. \tag{4.52}$$

Replacing α by $\alpha - \sigma_p$ in (4.51), it gives

$$(\alpha_p - 1)c_\alpha^{(\beta)} = (\alpha_p - 1 + \beta_p)c_{\alpha-\sigma_p-\lambda_p}^{(\beta)}. \tag{4.53}$$

Noting that for $\alpha' = \alpha - \sigma_p - \lambda_p$, we have $\alpha'_p \neq \alpha'_{\overline{p}}$. Assume $(\beta, 0) \in M_\alpha$. If $(\beta, 0) \in M_{\alpha'}$, then (4.52) shows that $\beta_p = 0$, again a contradiction with (4.39). Thus $(\beta, 0) \notin M_{\alpha'}$ and the right-hand side of (4.53) is zero. This and (4.52) show that $\alpha_{\overline{p}} = \alpha_p = 1$. Note that for $\alpha'' = \alpha - k\lambda_p$, $k \geq 1$, the relation $\alpha''_{\overline{p}} = \alpha''_p = 1$ does not hold, thus the right-hand side of (4.50) is zero. We obtain $\alpha_p + \beta_p = 0$. Hence

$$\alpha_{\overline{p}} = \alpha_p = -\beta_p = -\beta_{\overline{p}} = 1 \quad \text{if } \beta_p \neq 0, (\beta, 0) \in M_\alpha. \tag{4.54}$$

If $\alpha_{J_{1,4}} \neq \sigma$ (*cf.* (2.22)), say $(\alpha_q, \alpha_{\overline{q}}) \neq (1, 1)$ for some $q \in I_1$, $q \neq p$. Suppose $\alpha_q \neq 1$ (the proof is similar if $\alpha_{\overline{q}} \neq 1$), then we can write

$$x^\alpha = \big((\alpha_q - 1)e_q\big)^{-1}[x^{\alpha-\sigma_q-\lambda_q+\sigma_p}, x^{\lambda_q-\sigma_p}]. \tag{4.55}$$

Since for $\alpha' = \alpha - \sigma_q - \lambda_q + \sigma_p$ or $\lambda_q - \sigma_p$, the relation $\alpha'_p = \alpha'_{\overline{p}} = 1$ does not hold, we have $\beta_p = 0$ if $(\beta, 0) \in M_{\alpha'}$. Then applying d to (4.55) gives that $\beta_p = 0$ if $(\beta, 0) \in M_\alpha$, which contradicts (4.39) again. Hence

$$\alpha_{J_{1,4}} = \sigma, \quad \text{and} \quad \beta = -\sigma, \tag{4.56}$$

by (4.37) and (4.54). If $\ell_5+\ell_6+\ell_7\neq 0$, we can write

$$(4.57)\quad x^{\alpha}=\begin{cases}(\alpha_q+k)^{-1}[x^{\alpha+\sigma_p+k\varepsilon_q},x^{-\sigma_p-k\varepsilon_q,\varepsilon_{\overline{q}}}] & \text{for } q\in I_{5,6}, k\in\mathbb{Z}, \alpha_q+k\neq 0,\\ [x^{\alpha+\sigma_p,\varepsilon_r},x^{-\sigma_p,\varepsilon_{\overline{r}}}] & \text{for } r\in I_7.\end{cases}$$

Note that for

$$(4.58)\quad (\alpha',\underline{i}')=(\alpha+\sigma_p+k\varepsilon_q,0),\quad (-\sigma_p-k\varepsilon_q,\varepsilon_{\overline{q}}),\quad (\alpha+\sigma_p,\varepsilon_r)\quad\text{or}\quad(-\sigma_p,\varepsilon_{\overline{r}}),$$

the relation $\alpha'_p=\alpha'_{\overline{p}}=1$ does not hold; one can prove as above that $\beta_p=0$ if $(\beta,0)\in M_{\alpha',\underline{i}'}$. Then applying d to (4.57) gives that $\beta_p=0$ if $(\beta,0)\in M_\alpha$, which again contradicts (4.39). Hence $\ell_5+\ell_6+\ell_7=0$. Similarly, one can prove $\ell_2+\ell_3+\ell_4=0$. But then $\iota_7=\ell_1$, and we can replace d by $d-c_\alpha^{(\beta)}d'_0$ (*cf.* (4.4) and (4.56)), so that $c_\alpha^{(\beta)}$ becomes zero. This proves that the assumption (4.39) does not hold. Thus we have (4.38).

Now we prove

$$(4.59)\qquad d(t^{\underline{i}})=0\quad\text{if } \underline{i}=\underline{i}_{J_7}.$$

By Claim 1, we can suppose $|\underline{i}|=n\geq 2$. Assume that we have proved (4.59) for $|\underline{i}|<n$. Then $d([\nu,t^{\underline{i}}])=0$ for $\nu\in A$. From this, we obtain $d(t^{\underline{i}})\in\mathbb{F}1_{\mathcal{H}}$. Suppose $i_p>0$ for some $p\in I_7$. Then $t^{\underline{i}}=(i_{\overline{p}}+1)^{-1}[t_p^2,t^{\underline{i}-\varepsilon_p+\varepsilon_{\overline{p}}}]$, and $|\underline{i}-\varepsilon_p+\varepsilon_{\overline{p}}|=n$, thus $d(t^{\underline{i}})\in[\mathbb{F},t^{\underline{i}-\varepsilon_p+\varepsilon_{\overline{p}}}]+[t_p^2,\mathbb{F}]=0$. Similarly, by replacing d by $d-d'$ for some $d'\in\sum_{q\in I_{2,3}\cup J_4}\mathbb{F}\partial_{t_q}$ (which does not affect the results we have obtained so far), we can suppose

$$(4.60)\qquad d(x^{\alpha,\underline{i}})=m_\alpha x^{\alpha,\underline{i}}\quad\text{for } \alpha\in\Gamma, \underline{i}=\underline{i}_{J_7},$$

and

$$(4.61)\qquad d(t_p)=d(t_q^2)=0\quad\text{for } p\in I_{2,3}\cup J_4, q\in\overline{I}_5\cup J_6.$$

Note that $\mathcal{H}$ is generated by elements in (4.60) and (4.61), thus we obtain that (4.60) holds for all $(\alpha,\underline{i})\in\Gamma\times\mathcal{J}$. From this and (3.1), one can easily deduce that

$$(4.62)\qquad \mu\colon \alpha\mapsto m_\alpha\quad\text{is a group homomorphism such that } \mu\in\mathrm{Hom}_{\mathbb{Z}}^{+}(\Gamma,\mathbb{F}) \text{ if } \iota_7\neq\ell_1.$$

Assume that $\iota_7=\ell_1$. Then by (3.2) and (4.38), we have

$$(4.63)\qquad m_\alpha+m_\beta=m_{\alpha+\beta+\sigma_p}\quad\text{if } \alpha_p\beta_{\overline{p}}\neq\alpha_{\overline{p}}\beta_p \text{ and } \alpha,\beta\in\Gamma, p\in I_{1,4}.$$

By (4.42), (4.45), (4.46), and by induction on $|i|+|j|$, one can prove

$$(4.64)\qquad m_{i\sigma_p+j\lambda_p}=0\quad\text{for } i,j\in\mathbb{Z}, p\in I_{1,4}.$$

From this we want to prove

$$(4.65)\qquad m_\alpha=m_{\alpha+i\sigma_p+j\lambda_p}\quad\text{for } \alpha\in\Gamma, i,j\in\mathbb{Z}.$$

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By replacing α by some $\alpha + \sigma_p$ if necessary, we can suppose $(\alpha_p, \alpha_{\overline{p}}) \neq (0,0), (1,1)$. By (4.64) and by $[x^\alpha, x^{-\sigma_p + j\lambda_p}] = (\alpha_p(i + j\lambda_p) - i\alpha_{\overline{p}})x^{\alpha+(i+1)\sigma_p+j\lambda_p}$, we obtain $m_\alpha = m_{\alpha+i\sigma_p+j\lambda_p}$ if $\alpha_p(i-1+je_p) \neq (i-1)\alpha_{\overline{p}}$, from this, one can deduce (4.65). Now from (4.63) and (4.65), we obtain (4.62) again. Thus by replacing d by $d - d_\mu$, we have $d = 0$. This proves Claim 4 and also (4.10).

To prove that (4.11) is a direct sum, suppose

$$(4.66)\quad d = a_0' d_0' + \sum_{p \in \{0\} \cup J_1 \cup \overline{I}_{2,3} \cup I_5} a_p d_p + \sum_{q \in I_{2,3} \cup J_4} b_q \partial_{t_q} + d_\mu + \sum_{(0,0) \neq (\alpha,\underline{i}) \in \Gamma \times \mathcal{J}} c_{\alpha,\underline{i}} \operatorname{ad}_{x^{\alpha,\underline{i}}},$$

is the 0 derivation. Applying d to $A \cup \{1, t_p \mid p \in I_{2,3} \cup J_4 \cup I_6\}$ (*cf.* (4.36)), we obtain that all coefficients are zero except a_0'. Thus $d = a_0' d_0' = 0$. By (4.4), we obtain either $a_0' = 0$ or $d_0' = 0$. Thus (4.11) is a direct sum. ■

5 Second Cohomology Groups

In this section, we shall determine the second cohomology groups of the Hamiltonian Lie algebra $\mathcal{H} = \mathcal{H}(\underline{\ell}, \Gamma)$. It is well known that all one-dimensional central extensions of a Lie algebra are determined by the second cohomology group. Central extensions are often used in the structure theory and the representation theory of Kac-Moody algebras [K3]. Using central extension, we can construct many infinite dimensional Lie algebras, such as affine Lie algebras, infinite dimensional Heisenberg algebras, and generalized Virasoro and super-Virasoro algebras, which have a profound mathematical and physical background (*cf.* [K3], [S1], [SZ]). Since the cohomology groups are closely related to the structures of Lie algebras, the computation of cohomology groups seems to be important and interesting as well (*cf.* [J], [LW], [S1], [S2], [S3], [SZ]).

Recall that a *2-cocycle* on $\mathcal{H}$ is an $\mathbb{F}$-bilinear function $\psi\colon \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ satisfying the following conditions:

$$(5.1)\qquad \psi(v_1, v_2) = -\psi(v_2, v_1) \qquad \text{(skew-symmetry)},$$

$$(5.2)\qquad \psi([v_1, v_2], v_3) + \psi([v_2, v_3], v_1) + \psi([v_3, v_1], v_2) = 0 \qquad \text{(Jacobian identity)},$$

for $v_1, v_2, v_3 \in \mathcal{H}$. Denote by $C^2(\mathcal{H}, \mathbb{F})$ the vector space of 2-cocycles on $\mathcal{H}$. For any $\mathbb{F}$-linear function $f\colon \mathcal{H} \to \mathbb{F}$, one can define a 2-cocycle ψ_f as follows

$$(5.3)\qquad \psi_f(v_1, v_2) = f([v_1, v_2]) \quad \text{for } v_1, v_2 \in \mathcal{H}.$$

Such a 2-cocycle is called a *2-coboundary* or a *trivial 2-cocycle* on $\mathcal{H}$. Denote by $B^2(\mathcal{H}, \mathbb{F})$ the vector space of 2-coboundaries on $\mathcal{H}$. A 2-cocycle ϕ is said to be *equivalent to* a 2-cocycle ψ if $\phi - \psi$ is trivial. For a 2-cocycle ψ, we denote by $[\psi]$ the equivalent class of ψ. The quotient space

$$(5.4)\qquad H^2(\mathcal{H}, \mathbb{F}) = C^2(\mathcal{H}, \mathbb{F}) / B^2(\mathcal{H}, \mathbb{F}) = \{\text{the equivalent classes of 2-cocycles}\},$$

is called the *second cohomology group* of $\mathcal{H}$.

Lemma 5.1 *If $\iota_7 \neq \ell_1$, then $H^2(\mathcal{H}, \mathbb{F}) = 0$.*

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Proof Let ψ be a 2-cocycle. Say $\ell_4 \neq 0$ (the proof is exactly similar if $\ell_i \neq 0$ for $i \neq 1, 4$). We fix $p \in I_4$. Define a linear function f by induction on $i_{\overline{p}}$ as follows:

$$f(x^{\alpha,\underline{i}}) = \begin{cases} \alpha_{\overline{p}}^{-1}\big(\psi(t_p, x^{\alpha,\underline{i}}) - i_{\overline{p}} f(x^{\alpha,\underline{i}-\varepsilon_{\overline{p}}})\big) & \text{if } \alpha_{\overline{p}} \neq 0, \\ (i_{\overline{p}}+1)^{-1}\psi(t_p, x^{\alpha,\underline{i}+\varepsilon_{\overline{p}}}) & \text{if } \alpha_{\overline{p}} = 0, \end{cases} \tag{5.5}$$

for $(\alpha, \underline{i}) \in \Gamma \times \mathcal{J}$. Set $\phi = \psi - \psi_f$. Then (5.5) shows that

$$\phi(t_p, x^{\alpha,\underline{i}}) = 0 \quad \text{for } (\alpha, \underline{i}) \in \Gamma \times \mathcal{J}. \tag{5.6}$$

Using Jacobian identity (5.2), we obtain

$$0 = \phi(t_p, [x^{\alpha,\underline{i}}, x^{\beta,\underline{j}}]) = (\alpha_{\overline{p}}+\beta_{\overline{p}})\phi(x^{\alpha,\underline{i}}, x^{\beta,\underline{j}}) + i_{\overline{p}}\phi(x^{\alpha,\underline{i}-\varepsilon_{\overline{p}}}, x^{\beta,\underline{j}}) + j_{\overline{p}}\phi(x^{\alpha,\underline{i}}, x^{\beta,\underline{j}-\varepsilon_{\overline{p}}}), \tag{5.7}$$

for $(\alpha, \underline{i}), (\beta, \underline{j}) \in \Gamma \times \mathcal{J}$. If $\alpha_{\overline{p}} + \beta_{\overline{p}} \neq 0$, by induction on $i_{\overline{p}} + j_{\overline{p}}$, we obtain $\phi(x^{\alpha,\underline{i}}, x^{\beta,\underline{j}}) = 0$. On the other hand, if $\alpha_{\overline{p}} + \beta_{\overline{p}} = 0$, then (5.7) gives

$$\phi(x^{\alpha,\underline{i}}, x^{\beta,\underline{j}}) = -j_{\overline{p}}(i_{\overline{p}}+1)^{-1}\phi(x^{\alpha,\underline{i}+\varepsilon_{\overline{p}}}, x^{\beta,\underline{j}-\varepsilon_{\overline{p}}}), \tag{5.8}$$

and by induction on $j_{\overline{p}}$, we again have $\phi(x^{\alpha,\underline{i}}, x^{\beta,\underline{j}}) = 0$. Thus $\phi = 0$. ∎

Assume that $\iota_7 = \ell_1$. Denote $\sigma = \sum_{p\in I_1} \sigma_p$, and we use notation $\mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})$ as in (4.9) (*cf.* (4.6)). We construct 2-cocycles ϕ_p, ϕ'_p, ϕ_μ for $p \in I_1$, $\mu \in \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})$ as follows:

$$\phi_p(x^\alpha, x^\beta) = \alpha_p \delta_{\alpha+\beta,\sigma-\sigma_p}, \tag{5.9}$$

$$\phi'_p(x^\alpha, x^\beta) = \alpha_{\overline{p}} \delta_{\alpha+\beta,\sigma-\sigma_p}, \tag{5.10}$$

$$\phi_\mu(x^\alpha, x^\beta) = \mu(\alpha) \delta_{\alpha+\beta,\sigma}, \tag{5.11}$$

for $\alpha, \beta \in \Gamma$. It is straightforward to verify that they are 2-cocycles (*cf.* [J]). From the proof of Theorem 5.2 below, one can see why we construct such 2-cocycles.

Theorem 5.2

(1) $H^2(\mathcal{H}, \mathbb{F}) = 0$ *if* $\iota_7 \neq \ell_1$;
(2) *if* $\iota_7 = \ell_1$, *then* $H^2(\mathcal{H}, \mathbb{F})$ *is the vector space spanned by* $B = \{[\phi_p], [\phi'_p], [\phi_\mu] \mid p \in I_1, \mu \in \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})\}$.

Furthermore, for $a_p, b_p \in \mathbb{F}$, $\mu \in \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})$, *we have*

$$\sum_{p\in I_1}(a_p[\phi_p] + b_p[\phi'_p]) + [\phi_\mu] = 0 \Leftrightarrow a_p = b_p = \mu = 0. \tag{5.12}$$

Proof (1) follows from Lemma 5.1, while (2) follows from [J]. We give a simple proof of (2) as follows.

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First we prove (5.12). Thus suppose

$$\psi = \sum_{p\in I_1}(a_p\phi_p + b_p\phi'_p) + \phi_\mu + \psi_f, \tag{5.13}$$

is the zero 2-cocycle for some $a_p, b_p \in \mathbb{F}$ and some linear function f. Then for $p \in I_1$, $\alpha \in \Gamma$, by applying ψ to $(x^{-\sigma_p}, x^\sigma)$, $(x^{\lambda_p}, x^{\sigma-\lambda_p-\sigma_p})$, $(x^\alpha, x^{\sigma-\alpha})$, we have

$$0 = \psi(x^{-\sigma_p}, x^\sigma) = -a_p - b_p, \tag{5.14}$$

$$0 = \psi(x^{\lambda_p}, x^{\sigma-\lambda_p-\sigma_p}) = e_p b_p, \tag{5.15}$$

$$0 = \psi(x^\alpha, x^{\sigma-\alpha}) = \mu(\alpha) + \sum_{p\in I_1}(\alpha_p - \alpha_{\overline{p}})f(x^{\sigma_p+\sigma}), \tag{5.16}$$

(*cf.* the definition of λ_p in (3.97)). We obtain that $a_p = b_p = 0$ for $p \in I_1$ and by (4.9),

$$\mu = \sum_{p\in I_1} c_p\mu_p \in \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma,\mathbb{F}) \cap \mathrm{span}\{\mu_p \mid p \in I_1\} = \{0\}, \tag{5.17}$$

where $c_p = -f(x^{\sigma_p+\sigma}) \in \mathbb{F}$. This proves (5.12).

Now suppose ψ is a 2-cocycle. We define a linear function f as follows: set $f(x^\sigma) = 0$, and for $\alpha \in \Gamma \setminus \{\sigma\}$, we define

$$p_\alpha = \min\{p \in I_1 \mid (\alpha_p, \alpha_{\overline{p}}) \neq (1,1)\}, \tag{5.18}$$

and set

$$f(x^\alpha) = \begin{cases} (\alpha_p - \alpha_{\overline{p}})^{-1}\psi(x^{-\sigma_p}, x^\alpha) & \text{if } \alpha_{\overline{p}} \neq \alpha_p, \\ e_p^{-1}(1-\alpha_p)^{-1}\psi(x^{\lambda_p}, x^{\alpha-\sigma_p-\lambda_p}) & \text{if } \alpha_{\overline{p}} = \alpha_p \neq 1, \end{cases} \tag{5.19}$$

for $p = p_\alpha$. Set

$$\phi = \psi - \sum_{p\in I_1}(a_p\phi_p + b_p\phi'_p) - \psi_f, \tag{5.20}$$

where

$$a_p = -\psi(x^{-\sigma_p}, x^\sigma) - b_p, \quad b_p = e_p^{-1}\psi(x^{\lambda_p}, x^{\sigma-\lambda_p-\sigma_p}), \tag{5.21}$$

(*cf.* (5.14) and (5.15)). Then one can prove

$$\phi(x^{-\sigma_p}, x^\alpha) = 0 \quad \text{for } p \in I_1, \alpha \in \Gamma. \tag{5.22}$$

In fact, if $\alpha = \sigma$, it follows from (5.9), (5.10), (5.20) and (5.21). Assume $\alpha \neq \sigma$. Let $p = p_\alpha$ and write

$$x^\alpha = \begin{cases} (\alpha_p - \alpha_{\overline{p}})^{-1}[x^{-\sigma_p}, x^\alpha] & \text{if } \alpha_{\overline{p}} \neq \alpha_p, \\ e_p^{-1}(1-\alpha_p)^{-1}[x^{\lambda_p}, x^{\alpha-\sigma_p-\lambda_p}] & \text{if } \alpha_{\overline{p}} = \alpha_p \neq 1, \end{cases} \tag{5.23}$$

(*cf.* (5.19)), we can obtain (5.22) by the Jacobian identity (5.2). From (5.22), by considering $\phi(x^{-\sigma_p}, [x^\alpha, x^\beta])$ and by (5.2), we obtain

$$\phi(x^\alpha, x^\beta) = 0 \quad \text{if } \alpha_p + \beta_p \neq \alpha_{\overline{p}} + \beta_{\overline{p}} \text{ for some } p \in I_1. \tag{5.24}$$

Now we want to prove

$$\phi(x^{\lambda_q}, x^\alpha) = 0 \quad \text{for } \alpha \in \Gamma, q \in I_1. \tag{5.25}$$

By (5.24), we can suppose $\alpha_p = \alpha_{\overline{p}}$ if $p \neq q$, and $\alpha_q = \alpha_{\overline{q}} + e_q$. If $\alpha = \sigma - \lambda_q - \sigma_q$, (5.25) follows from (5.9), (5.10), (5.20) and (5.21). So suppose $\alpha \neq \sigma - \lambda_q - \sigma_q$. Let $\alpha' = \alpha + \lambda_q + \sigma_q \neq \sigma$. If $p = p_{\alpha'} \neq q$, then $\alpha'_p = \alpha'_{\overline{p}} = \alpha_p = \alpha_{\overline{p}} \neq 1$, and by writing $x^\alpha = e_p^{-1}(1-\alpha_p)^{-1}[x^{\lambda_p}, x^{\alpha-\sigma_p-\lambda_p}]$, we obtain

$$\phi(x^{\lambda_q}, x^\alpha) = -e_p^{-1}(1-\alpha_p)^{-1} e_q \alpha_q \phi(x^{\lambda_p}, x^{\alpha'-\sigma_p-\lambda_p}) = 0, \tag{5.26}$$

by (5.19). On the other hand, if $p = q$, we again have $\phi(x^{\lambda_q}, x^\alpha) = \phi(x^{\lambda_q}, x^{\alpha'-\sigma_p-\lambda_p}) = 0$ by (5.19).

Now by (5.25) and by writing $(\alpha_p - 1)x^\alpha = -e_p^{-1}[x^{\lambda_p}, x^{\alpha-\lambda_p-\sigma_p}]$, we obtain

$$(\alpha_p - 1)\phi(x^\alpha, x^\beta) = -\beta_p \phi(x^{\alpha-\lambda_p-\sigma_p}, x^{\beta+\lambda_p+\sigma_p}), \tag{5.27}$$

$$(\alpha_p - 2)\phi(x^{\alpha-\lambda_p-\sigma_p}, x^{\beta+\lambda_p+\sigma_p}) = -(\beta_p+1)\phi(x^{\alpha-2\lambda_p-2\sigma_p}, x^{\beta+2\lambda_p+2\sigma_p}), \tag{5.28}$$

for $p \in I_1$, where (5.28) is obtained from (5.27). Using (5.28) in (5.27) and by writing

$$\big(3(\alpha_{\overline{p}} - \alpha_p) - 2\alpha_p e_p\big) x^{\alpha-2\lambda_p-2\sigma_p} = [x^\alpha, x^{-2\lambda_p-3\sigma_p}], \tag{5.29}$$

we obtain

(5.30)

$$\begin{aligned}&\big(3(\alpha_{\overline{p}} - \alpha_p) - 2\alpha_p e_p\big)(\alpha_p - 1)(\alpha_p - 2)\phi(x^\alpha, x^\beta)\\ &\quad = \beta_p(\beta_p+1)\big(\phi([x^\alpha, x^{\beta+2\lambda_p+2\sigma_p}], x^{-2\lambda_p-3\sigma_p}) + \phi(x^\alpha, [x^{-2\lambda_p-3\sigma_p}, x^{\beta+2\lambda_p+2\sigma_p}])\big).\end{aligned}$$

We prove that

$$\phi(x^\alpha, x^{-2\lambda_p-3\sigma_p}) = 0 \quad \text{for } \alpha \in \Gamma, p \in I_1. \tag{5.31}$$

By (5.24), we can suppose $\alpha_{\overline{p}} = \alpha_p + 2e_p$. If $\alpha_p \neq 1, 2$, by setting $\beta = -2\lambda_p - 3\sigma_p$ in (5.27) and (5,28), then the right-hand side of (5.28) is zero by (5.22), and thus (5.31) holds. Suppose $\alpha_p = 1, 2$. Then $\alpha_{\overline{p}} = 1 + 2e_p$ or $2 + 2e_p$, thus we can write α in the following form

(5.32)

$$\alpha = \alpha' + \sigma_p + 2\lambda_p \text{ or } \alpha' + 2\sigma_p + 2\lambda_p \text{ for some } \alpha' \in \Gamma \text{ such that } (\alpha'_p, \alpha'_{\overline{p}}) = (0, 0).$$

We denote

(5.33)
$$c_i = \phi(x^{\alpha'+i(\sigma_p+\lambda_p)-\sigma_p}, x^{-i(\sigma_p+\lambda_p)-\sigma_p}), \qquad c'_i = \phi(x^{\alpha'+i(\sigma_p+\lambda_p)}, x^{-i(\sigma_p+\lambda_p)-\sigma_p}),$$

(5.34) $$d_i = \phi(x^{\alpha'+i(\sigma_p+\lambda_p)-2\sigma_p}, x^{-i(\sigma_p+\lambda_p)}), \qquad d'_i = \phi(x^{\alpha'+i(\sigma_p+\lambda_p)-\sigma_p}, x^{-i(\sigma_p+\lambda_p)}),$$

for $i \in \mathbb{Z}$. By writing

(5.35) $$(i-2j)e_p x^{-i(\sigma_p+\lambda_p)-\sigma_p} = [x^{-j(\sigma_p+\lambda_p)-\sigma_p}, x^{-(i-j)(\sigma_p+\lambda_p)-\sigma_p}] \quad \text{for } j \in \mathbb{Z},$$

we obtain

(5.36) $$(i-2j)c_i = (i+j)c_{i-j} - (2i-j)c_j, \quad (i-2j)c'_i = i(c'_{i-j} - c'_j) \quad \text{for } i, j \in \mathbb{Z}.$$

By writing

(5.37) $$2(j-i)e_p x^{\alpha'+i(\sigma_p+\lambda_p)-\sigma_p} = [x^{\alpha'+j(\sigma_p+\lambda_p)-2\sigma_p}, x^{(i-j)(\sigma_p+\lambda_p)}],$$

(5.38) $$(j-i)e_p x^{\alpha'+i(\sigma_p+\lambda_p)} = [x^{\alpha'+j(\sigma_p+\lambda_p)-\sigma_p}, x^{(i-j)(\sigma_p+\lambda_p)}],$$

we obtain

(5.39)
$$2(j-i)c_i = (2i+j)d_{j-i} - (i-j)d_j, \quad (j-i)c'_i = (i+j)d'_{j-i} + (i-j)d'_j \quad \text{for } i, j \in \mathbb{Z}.$$

Note that the system (5.36) has up to multiplicative scalars unique solutions for c_i, c'_i, and we find that

(5.40) $$c_i = (i^3 - i)c, \quad c'_i = i^2 c' \quad \text{for } i \in \mathbb{Z} \text{ and some } c, c' \in \mathbb{F},$$

are the only solutions. If we substitute j by 1 and by $i+1$ in (5.39), we then obtain $c_i = c'_i = d_i = d'_i = 0$ for all $i \in \mathbb{Z}$. This in particular proves (5.31) by (5.32)–(5.34).

Now using (5.31) in (5.30), noting that $\beta_p - \beta_{\overline{p}} = \alpha_{\overline{p}} - \alpha_p$ by (5.24), we deduce that

(5.41)
$$\Big(\beta_p(\beta_p+1)\big(3(\alpha_{\overline{p}} - \alpha_p) + 2e_p(\beta_p - 1)\big) - (\alpha_p - 1)(\alpha_p - 2)(3(\alpha_{\overline{p}} - \alpha_p) - 2\alpha_p e_p)\Big)\phi(x^\alpha, x^\beta) = 0.$$

As in the proof of (5.31), we can prove $\phi(x^\alpha, x^{2\lambda_p}) = 0$. Thus we can replace λ_p by $2\lambda_p$ in the above discussion, *i.e.*, if we replace e_p by $2e_p$, (5.41) still holds. This forces

(5.42) $$\phi(x^\alpha, x^\beta) = 0 \quad \text{or} \quad \beta_p - 1 = -\alpha_p \quad \text{for all } p \in I_1,$$

i.e., if $\alpha + \beta \neq \sigma$, then $\phi(x^\alpha, x^\beta) = 0$. Thus we can suppose

(5.43) $$\phi(x^\alpha, x^\beta) = m_\alpha \delta_{\alpha+\beta,\sigma} \quad \text{for } \alpha, \beta \in \Gamma \text{ and some } m_\alpha \in \mathbb{F}.$$

As in the proof of (5.31), we can prove $m_{i\sigma_p+j\lambda_p} = 0$ for $i, j \in \mathbb{Z}$, $p \in I_1$. Then for any $\alpha, \beta \in \Gamma$, $p \in I_1$, let $v_1 = x^\alpha$, $v_2 = x^\beta$, $v_3 = x^{\sigma-\alpha-\beta-\sigma_p}$ in (5.2), one can easily deduce that $\mu\colon \alpha \mapsto m_\alpha$ is a group homomorphism $\mu\colon \Gamma \to \mathbb{F}$ such that $\mu(\sigma_p) = 0$. Thus $\mu \in \mathrm{Hom}^+_{\mathbb{Z}}(\Gamma, \mathbb{F})$ and $\phi = \psi_\mu$. Furthermore, we can write $\mu = \nu + \lambda$ for $\nu \in \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})$, $\lambda \in \mathrm{span}\{\mu_p \mid p \in I_1\}$ by (4.9). Then $\phi = \psi_\nu + \psi_\lambda$. But from (5.16) and (5.17), one can see that ψ_λ corresponds to a trivial 2-cocycle, thus we can suppose $\phi = \phi_\nu$. This proves Theorem 5.1. ■

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