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PARTIAL COMPLEMENTS IN FINITE GROUPS

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Abstract

Let G be a finite group with normal subgroup N. A subgroup K of G is a partial complement of N in G if N and K intersect trivially. We study the partial complements of N in the following case: G is soluble, N is a product of minimal normal subgroups of G, N has a complement in G, and all such complements are G-conjugate.

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1. Introduction

Let G be a finite soluble group with a normal subgroup N. A subgroup H of G is a complement of N in G if H intersects N trivially and $G = NH$. We define K to be a partial complement of N in G if K is a subgroup of G and K and N intersect trivially. Consider the following question: if G is a finite soluble group, when is each partial complement of N in G contained in a complement of N in G? Hall [4] proved that if $G = NH$, where N is a p-group (p a prime) and p does not divide the order of H, then each partial complement of N in G is contained in a complement of N in G.

Rose considered related problems in [6]. Assume that p is a prime, N is an abelian normal p-subgroup of G and P is a Sylow p-subgroup of G. He proves in [6, Theorem 4] that P splits over N and all the complements of N in P form a single conjugacy class in P if and only if the following conditions hold: G splits over N; all the complements of N in G form a single conjugacy class in G; and each partial complement of N in G is contained in some complement of N in G. If G is a finite primitive soluble group, then it splits over N and all the complements of N form a single conjugacy class in G. Rose gives examples of such groups G where:

(i) each partial complement of N is contained in a complement of N [6, Theorem 11];

(ii) a partial complement of N is not contained in a complement of N [6, Example 16].

The problem was again raised by Doerk and Hawkes in [2] (see the discussion between Propositions 15.9 and 15.10). Like Rose, they provide an example [2, Example VIII, 2, 19] of a finite soluble primitive group G where there is a partial complement of N in G that is not contained in any complement of N in G.

Assume that G is a finite soluble group, N is a product of minimal normal subgroups of G, N has a complement in G and all its complements are conjugate. In this paper we give a necessary and sufficient condition for each partial complement of N in G to be contained in a complement of N in G. Let $\mathrm{GF}(p)$ denote the field with p elements.

THEOREM 1. *Let G be a finite soluble group. Let N be a product of minimal normal subgroups of G where N is complemented in G and all its complements are conjugate in G. Let p be a prime and let N_p be the Sylow p-subgroup of N. Then each partial complement of N in G is contained in some complement of N in G if and only if, for each prime p dividing the order of N, N_p is projective as a $\mathrm{GF}(p)(G/N)$-module.*

2. Preliminaries

In this section we collect a number of results required in the proof of Theorem 1. Notation is mostly standard; in general, we use the same notation as [2]. We assume throughout the paper that p is a prime number.

LEMMA 2 [6, Lemma 9(i)]. *Suppose G is a finite group. Let N be a normal subgroup of G and let H be a complement of N in G. Then the number of conjugates of H in G is $|N|/|C_N(H)|$.*

Let F be a field of characteristic p and let K be a group of order p. We define an FK-module U to be uniserial if the successive quotients of the radical series of U are simple.

The following lemma is a consequence of [5, Theorem VII, 5.3] and its proof.

LEMMA 3. *Let F be a field of characteristic p and let K be a group of order p. Then the following hold.*

(i) *The regular FK-module is uniserial.*
(ii) *An FK-module generated by a single element is indecomposable.*
(iii) $\mathrm{Rad}^{p-1}(FK) \neq 0$ *and* $\mathrm{Rad}^{p}(FK) = 0$.
(iv) $\mathrm{Rad}^{i}(FK)/\mathrm{Rad}^{i+1}(FK)$ *is simple, so* $\mathrm{Rad}^{i}(FK)/\mathrm{Rad}^{i+1}(FK)$ *has order* $|F|$.

LEMMA 4. *Let F be a field of characteristic p where p is a prime. Let K be a group of order p. Let the FK-module V be the direct sum of W_1 and W_2 where W_1 contains no free submodule and W_2 is a free submodule or $W_2 = 0$. Then* $\mathrm{Rad}^{p-1}(V) = 0$ *if and only if* $W_2 = 0$.

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PROOF. If $W_2 = 0$ then $V = W_1$. Let U be an indecomposable submodule of W_1. By [5, Theorem VII, 5.3], there exists $0 < i < p$ such that $U \cong FK/\mathrm{Rad}^i(FK)$. Therefore $\mathrm{Rad}^{p-1}(U) = 0$. Since the radical of the direct sum of submodules is the sum of the radicals of each submodule, $\mathrm{Rad}^{p-1}(V) = 0$.

If $W_2 \neq 0$ then $\mathrm{Rad}^{p-1}(W_2) \neq 0$ since W_2 is free and $\mathrm{Rad}^{p-1}(FK) \neq 0$. Therefore $\mathrm{Rad}^{p-1}(V) \neq 0$. Note that in both cases, $\mathrm{Rad}^p(V) = 0$. □

In the following, we will deal with the semidirect product $G = VK$ where K is a group of order p and V is an elementary abelian p-subgroup of G. Of course V is a $\mathrm{GF}(p)K$-module, and we will denote by $\mathrm{Rad}(V)$ the radical of V as a $\mathrm{GF}(p)K$-module so that we can apply the previous results; observe that V is now viewed as a multiplicative group.

The next result is similar to [6, Lemma 9(ii)].

LEMMA 5. *Let G be the semidirect product of a normal elementary abelian subgroup V and a cyclic subgroup K of order p. Assume that $V = W_1 \times W_2$ is the direct product of two $\mathrm{GF}(p)K$-submodules of V such that W_1 contains no free submodule and W_2 is a free submodule. Then the elements of order p in G but not in V are those of the form kw, where $w \in W_1 \times \mathrm{Rad}(W_2)$ and $1 \neq k \in K$.*

PROOF. Let $k \in K$ and $w \in V$. Define $r_1 = [w, k]$ and $r_{i+1} = [r_i, k]$. By induction,

$$(kw)^p = k^p w^p r_1^{s_1} \cdots r_{p-2}^{s_{p-2}} r_{p-1} \quad \text{where } s_i = \binom{p}{i+1}$$

for every $1 \leq j \leq p-2$. Note that if $w \in W_1 \times \mathrm{Rad}^j(W_2)$ for any $0 \leq j \leq p$ then $r_i \in \mathrm{Rad}(W_1) \times \mathrm{Rad}^{j+i}(W_2)$. Moreover, $k^p = v^p = 1$. On the other hand, since $r_i \in V$ and p divides all the exponents s_i, then $r_1^{s_1} \cdots r_{p-2}^{s_{p-2}} = 1$. Observe that $r_{p-1} = 1$ if and only if $w \in W_1 \times \mathrm{Rad}(W_2)$ (by Lemma 4). The result follows. □

COROLLARY 6. *Let K be a group of order p where p is a prime. Let the $\mathrm{GF}(p)K$-module V be the direct product of W_1 and W_2. Assume that W_1 has order p^s and does not contain any free module and that W_2 is a free module of rank l. Then the number of subgroups of order p in VK that intersect V trivially is $p^{s+l(p-1)}$.*

PROOF. First we apply Lemma 5 to find the number of elements of order p that are in VK but not in V. Since any indecomposable module of V is uniserial,

$$|W_1 \times \mathrm{Rad}(W_2)| = p^s p^{l(p-1)} = p^{l(p-1)+s}.$$

So there are $(p^{l(p-1)+s})(p-1)$ elements of order p in VK but not in V. Then the number of subgroups of order p in VK that intersect V trivially is

$$\frac{(p^{l(p-1)+s})(p-1)}{p-1} = p^{l(p-1)+s}.$$

This concludes the proof. □

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COROLLARY 7. *Let G be the semidirect product NK, where K is a subgroup of order p and N is an elementary abelian p-subgroup of G. Then the number of subgroups of order p in G that intersect N trivially is equal to the number of conjugates of K in G if and only if N is free as a* $\mathrm{GF}(p)K$*-module.*

PROOF. Let the $\mathrm{GF}(p)K$-module N be the direct product of W_1 and W_2, where W_1 has order p^s and does not contain any free module, and W_2 is a free module of rank l. By Corollary 6, the number of subgroups of order p in NK that intersect N trivially is $p^{s+l(p-1)}$. Let $W_1 = U_1 \times \cdots \times U_r$, where U_i are indecomposable submodules. Observe that S_i, the minimal submodule of U_i, is in the centralizer $C_{U_i}(K)$ and if $w \in U_i$ is not contained in S_i, then w is not centralized by K. Therefore $|C_{U_i}(K)| = |S_i| = p$. Hence $|C_{W_1}(K)| = p^r$.

Similarly, $|C_{W_2}(K)| = p^l$. Hence $|C_N(K)| = p^r p^l = p^{l+r}$.

So, by Lemma 2, we obtain that the number of conjugates of K in NK is

$$\frac{|N|}{|C_N(K)|} = \frac{p^{s+pl}}{p^{r+l}} = p^{s+l(p-1)-r}.$$

Comparing this result with that of Corollary 6, we see that the number of subgroups of order p in NK that intersect N trivially is the same as the number of conjugates of K in NK if and only if $r = 0$; that is, if and only if N is free as a $\mathrm{GF}(p)K$-module. □

LEMMA 8. *Let G be a finite soluble group. Let N be a product of minimal normal p-subgroups of G where p is a prime. Assume that H is a complement of N in G and that all the complements of N in G are conjugates of H. Let H_0 be a subgroup of H of order p. Then each subgroup of order p in NH_0 (the semidirect product) but not in N is contained in a complement of N in G if and only if the number of conjugates of H_0 in NH_0 is equal to the number of subgroups of order p that are in NH_0 but not in N.*

PROOF. Let L be a subgroup of order p in NH_0 but not in N. Then L is contained in a complement of N in G if and only if there exists an element n in N such that $L \subseteq NH_0 \cap H^n$. Observe that $NH_0 = (NH_0)^n$ for each $n \in N$. Hence $L \subseteq NH_0 \cap H^n$ if and only if

$$L \subseteq (NH_0)^n \cap H^n = (NH_0 \cap H)^n.$$

However,

$$(NH_0 \cap H)^n \cong N^n(NH_0 \cap H)^n/N^n \cong (NH_0 \cap NH)^n/N^n = (NH_0)^n/N^n \cong H_0^n.$$

Hence $|(NH_0 \cap H)^n| = |H_0^n| = p$. Therefore $L \subseteq (NH_0 \cap H)^n$ if and only if $L = H_0^n$. Hence each subgroup of order p in NH_0 but not in N is contained in some complement of N in G if and only if the number of conjugates of H_0 in NH_0 is equal to the number of subgroups of order p that are in NH_0 but not in N. □

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Before we prove Lemma 9, we first note a result needed in the proof. Let H be a group, let F be a finite field of prime characteristic p and let E be its algebraic closure. Let V_E denote the EH-module $E \otimes_F V$ [5, Definition VII, 1.1]. By [5, Exercise VII. 7. 19], the FH-module V is projective if and only if the EH-module V_E is projective.

LEMMA 9. *Let F be a field of prime characteristic p. Let H be a soluble group and let V be a semisimple FH-module. Then V restricted to each subgroup C of H of order p is projective as a FC-module if and only if V is projective.*

PROOF. If V is projective as a $\mathrm{GF}(p)H$-module, then its restriction to a subgroup of H is also projective [5, Theorem VII. 7.11(a)].

To prove the other direction, we use the main theorem from [1] which shows that, for a soluble group H and an algebraically closed field E, an EH-module is primitive if and only if it is quasi-primitive.

Assume that V is not projective. Since V is not projective, V_E is not projective. On the other hand, since V is semisimple and the semisimplicity of V is retained when changing fields (by [5, Theorem VII, 1.8]), V_E is semisimple. Since V_E is not projective, there exists a simple direct summand U of V_E which is not projective. Furthermore, since H is soluble and U is a simple EH-module, applying [1, main theorem], we deduce that there is a primitive EA-module W where $A \leq H$ and $U \cong W^H$ (A is called a stabilizer limit for U). We have that W is simple as an A-module by definition, because it is primitive.

First we show that if X is a subgroup of A having order p, then W_X is projective. Let X be a subgroup of A of order p. Since $U \cong W^H$, we have $U_X \cong (W^H)_X$. On the other hand, let $\{1, g_2, \ldots, g_m\}$ be a full set of (A, X)-double coset representatives of H. Applying Mackey's theorem [2, Theorem B. 6.20], we see that $(W^H)_X$ is isomorphic to the tensor product

$$((W \otimes 1)_{A\cap X})^X \oplus \left[\bigoplus_{i=2}^{m}((W \otimes g_i)_{A^{g_i}\cap X})^X\right].$$

But observe that

$$((W \otimes 1)_{A\cap X})^X = (W_X)^X = W_X.$$

As $U_X \cong (W^H)_X$ is projective by assumption, W_X is also projective by [2, Proposition B. 2.4]. Since W_X is projective, it has dimension divisible by p. By [7, note after Theorem 12], the dimension of W is coprime to p. Therefore A does not contain any subgroup of order p and so W is projective as an EA-module. By [2, Proposition B, 6.12], W^H is projective. Hence, U is projective since $U \cong W^H$. Therefore V_E is projective and V is also projective, which gives the final contradiction. □

3. Proof of main theorem

Suppose that $G = NH$ where N is a p-group. Suppose first that N is not projective as a $\mathrm{GF}(p)H$-module. By Lemma 9, there exists a subgroup H_0 of H of order p such

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that N is not projective as a $\mathrm{GF}(p)H_0$-module. By [2, Theorem B. 4.12] and since N_{H_0} is not projective, we know that N_{H_0} is not free. Now by Corollary 7, the number of subgroups of order p in NH_0 that intersect N trivially is different from the number of conjugates of H_0 in NH_0. Therefore by Lemma 8, there exists a partial complement of N in G of order p which is not contained in a complement of N in G.

We now suppose that N is projective as a $\mathrm{GF}(p)H$-module. Let K be a partial complement of N in G. We find a subgroup $H_0 \leq H$ such that K and H_0 are both complements of N in NK. By Dedekind's lemma [2, Lemma A. 1.3],

$$N(NK \cap H) \cong (NK \cap NH) = NK.$$

Observe that, by the isomorphism theorems,

$$(NK \cap H) \cong N(NK \cap H)/N \cong NK/N \cong K,$$

since K and N have trivial intersection. Hence $(NK \cap H) = H_0$ for some $H_0 \leq H$ where $K \cong H_0$.

By [5, Theorem VII. 7.11(a)], N as a $\mathrm{GF}(p)H_0$-module is projective. Furthermore, by [3, Section 2.2], all cohomologies (in particular, the first) vanish. On the other hand, by [2, Theorem A. 15.10], the number of conjugacy classes of complements of NH_0 is the order of the first cohomology group and so all complements are conjugate. Now H_0 and K are both partial complements of N in NH_0 so $K = H_0^n$ for some n in N. As a consequence, $K = H_0^n \leq H^n$. That is, K is in a complement of N in G. Thus we have proved the theorem when N is a p-group.

Now let $N = N_{p'}N_p$ and $G = NH$. Observe that H is a complement of N in G if and only if $N_{p'}H$ is a complement of N_p in G. Firstly, all subgroups $N_{p'}H^n$ are complements of N_p in G. Secondly, assume that C is any complement of N_p in G. Let q be a prime which divides the order of $N_{p'}$. Since the index of C is prime to q, there is a Sylow q-subgroup of G in C, and as a consequence $N_q \leq C$. Since this holds for any prime q that divides the order of $N_{p'}$, $N_{p'} \leq C$. Now we have to show that all the complements of N_p are conjugate. First observe that

$$C = C \cap N_{p'}N_pH = N_{p'}(C \cap N_pH) \quad \text{and} \quad N_{p'} \cap (C \cap N_pH) = 1.$$

Therefore $C \cap N_pH$ is a complement of $N_{p'}$ in C and so $C \cap N_pH$ is a complement of $N_pN_{p'}$ in G. Therefore $C \cap N_pH$ is a conjugate of H, and as a consequence all the complements of N_p in G are conjugates of $N_{p'}H$.

By [2, Theorem B, 4.11] if N_p is projective as a $\mathrm{GF}(p)H$-module then N_p is projective as a $\mathrm{GF}(p)N_{p'}H$-module. By [5, Theorem VII. 7.11(a)] if N_p is projective as a $\mathrm{GF}(p)N_{p'}H$-module then N_p is projective as a $\mathrm{GF}(p)H$-module. Therefore N_p is projective as a $\mathrm{GF}(p)H$-module if and only if N_p is projective as a $\mathrm{GF}(p)N_{p'}H$-module. Hence G and N_p satisfy the hypothesis of the theorem.

Now suppose that, for each prime p that divides the order of N, the Sylow p-subgroup N_p is projective as a $\mathrm{GF}(p)N_p'H$-module (that is, N_p is projective as a $\mathrm{GF}(p)H$-module). By [6, Corollary 5] and the p-group case, every partial complement of N in G is contained in a complement of N in G.

Now suppose that there exists a prime p that divides the order of N such that the Sylow p-subgroup N_p is not projective as a $\mathrm{GF}(p)N_p'H$-module and so N_p is not projective as a $\mathrm{GF}(p)H$-module. By [6, Corollary 5] and the p-group case, there exists a partial complement of N in G that is not contained in a complement of N in G. We have thus proved Theorem 1.

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