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# WHEN IS A MATRIX SIGN STABLE?

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**Introduction.** An $n \times n$ real matrix $A = (a_{ij})$ is called *stable* (resp. *semistable*) if each of its eigenvalues has negative (resp. nonpositive) real part. These notions are important because of their close connection with the stability of motion, which can be described especially simply for a system

(1) $\dot{x} = Ax$

of linear differential equations with constant coefficients. In fact (see 2.3.i of [**2**] and Theorem 4.1a of [**5**]), the equilibrium $x = 0$ of the system (1) is *stable* (meaning that for each $\epsilon > 0$ there exists $\delta > 0$ such that every positive half-trajectory starting within $\delta$ of the origin lies eventually within $\epsilon$ of the origin) if and only if the matrix $A$ is semistable and each of its purely imaginary eigenvalues is a simple root of $A$'s minimum polynomial. And the equilibrium $x = 0$ is *asymptotically stable* (meaning it is stable and every positive half-trajectory starting sufficiently close to the origin actually converges to the origin) if and only if the matrix $A$ is stable.

The matrix $A$ is called *sign stable* (resp. *sign semistable*) if each matrix $B$ of the same sign-pattern as $A$ (sgn $b_{ij}$ = sgn $a_{ij}$ for all $i, j$) is stable (resp. semistable), regardless of the magnitudes of the $b_{ij}$. These notions are of interest because there are situations in economics [**14**; **15**], ecology [**6**; **8**; **9**; **10**; **11**] and chemistry [**3**; **16**] in which an interaction matrix $A$ is known only qualitatively in the sense that its entries can be determined with reasonable confidence so far as their signs are concerned but little can safely be said about their magnitudes.

A central problem has been to characterize sign stability in finitely computable terms. That was accomplished by Quirk and Ruppert [**14**] for the case in which $a_{ii} \neq 0$ for all $i$. A general characterization was discovered by Jeffries [**6**] and used by Klee and van den Driessche [**7**] as the basis of an efficient algorithm for testing sign stability. However, [**6**] did not include full proofs and the purpose of this paper is to supply them. Our arguments make more explicit use of the system (1) than do those of [**12**; **13**; **14**], which are more purely matrix-theoretic in character. However, like [**12**; **13**] we also rely heavily on certain graphs associated with the matrix $A$.

**Characterizing conditions.** The "if" and "only if" parts of the following result appear in [**14**] and [**13**] respectively. The theorem is proved again here because of its essential role in the characterization of sign stability.

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THEOREM 1. (Quirk-Ruppert-Maybee). *An $n \times n$ real matrix $A = (a_{ij})$ is sign semistable if and only if it satisfies the following three conditions:*

($\alpha$) $a_{ii} \leqq 0$ *for all $i$;*

($\beta$) $a_{ij}a_{ji} \leqq 0$ *for all $i \neq j$;*

($\gamma$) $a_{i(1)\,i(2)} \ldots a_{i(k-1)\,i(k)}a_{i(k)\,i(1)} = 0$ *for each sequence of $k \geqq 3$ distinct indices $i(1), \ldots, i(k)$.*

Let $D_A$ denote the directed graph whose vertex-set is $\{1, \ldots, n\}$ and edge-set is $\{(i, j) : i \neq j \text{ and } a_{ij} \neq 0\}$. Then condition ($\gamma$) asserts $D_A$ has no $k$-cycle for $k \geqq 3$. The graph $D_A$ is used later.

Quirk and Ruppert [**14**] showed if $A$ is sign stable then

($\delta_*$) $a_{ii} < 0$ for at least one $i$,

while sign stability is equivalent to sign semistability when

($\delta^*$) $a_{ii} < 0$ for all $i$.

Alternatives to some of their arguments were supplied by Maybee and Quirk [**13**] and Maybee [**12**].

Plainly stability of $A$ implies

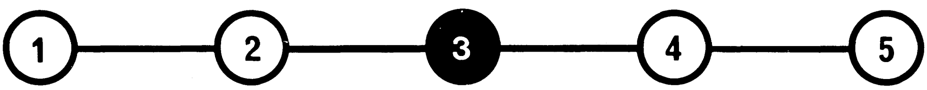
($\epsilon^*$) $A$ is nonsingular.

It was stated in [**13**; **14**] that if $A$ is indecomposable (for all $i \neq j$ there is in $D_A$ a path from $i$ to $j$) then conditions ($\alpha$) – ($\gamma$), ($\delta_*$) and ($\epsilon^*$) are sufficient for sign stability; the same claim was made in [**10**; **11**] for arbitrary $A$. However, the following example of Jeffries satisfies those conditions, is indecomposable, and has $\pm i$ among its eigenvalues.

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

An easy way to see directly that this matrix does not yield asymptotic stability for (1) is to consider the trajectory given by $x_1(t) = -x_5(t) = \cos t$, $x_2(t) = x_4(t) = \sin t$, $x_3(t) = 0$. It is a solution of (1) but obviously never approaches the origin.

For any $n \times n$ matrix $A$, let the undirected graph $G_A$ have $\{1, \ldots, n\}$ as its vertex-set and $\{\{i, j\} : i \neq j \text{ and } a_{ij} \neq 0 \neq a_{ji}\}$ as its edge-set. Thus the edges of $G_A$ correspond to 2-cycles in $D_A$. And let $R_A = \{i : a_{ii} \neq 0\}$. Below is the graph $G_A$ associated with the above matrix $A$, $R_A$ consisting of the black vertex 3.




An $R_A$-*coloring* of $G_A$ is a partition of its vertices into two sets, black and white (one of which may be empty), such that each vertex in $R_A$ is black, no black vertex has precisely one white neighbor, and each white vertex has at least one white neighbor. (The above figure displays such a coloring.) *A* $(V \sim R_A)$-*complete matching* in $G_A$ is a set $M$ of pairwise disjoint edges such that $V \sim R_A \subset \cup M$; in other words, an exact cover of the vertex-set $V = \{1, \ldots, n\}$ can be obtained by using the pairs in $M$ and certain singletons from $R_A$. (In the above example, the edges $\{1, 2\}$ and $\{4, 5\}$ form a $(V \sim R_A)$-complete matching.)

The main result of this paper is the following characterization of sign stability [**8**].

THEOREM 2 (Jeffries). *An* $n \times n$ *real matrix* $A = (a_{ij})$ *is sign stable if and only if it satisfies the following five conditions:*

($\alpha$) $a_{ii} \leqq 0$ *for all* $i$;

($\beta$) $a_{ij}a_{ji} \leqq 0$ *for all* $i \neq j$;

($\gamma$) *the directed graph* $D_A$ *has no* $k$*-cycle for* $k \geqq 3$;

($\delta$) *in every* $R_A$*-coloring of the undirected graph* $G_A$, *all vertices are black;*

($\epsilon$) *the undirected graph* $G_A$ *admits a* $(V \sim R_A)$*-complete matching.*

Note that ($\delta^*$) implies ($\delta$). And if ($\delta$) holds then $R_A$ intersects every nondegenerate component of $G_A$, whence ($\delta_*$) holds or $G_A$ has no edges. In the presence of ($\gamma$), ($\epsilon$) is equivalent to the condition, used in [**13**; **14**], that some term in the expansion of $A$'s determinant is different from 0. However, ($\epsilon$) has the advantage of suggesting the efficient computational test used in [**7**]. See [**1**] for another relationship between complete matchings and nonsingularity.

When $A$ is an $n \times n$ real matrix $(a_{ij})$, let $Q_A$ denote (as in [**13**]) the set of all $n \times n$ real matrices $B = (b_{ij})$ such that $\operatorname{sgn} b_{ij} = \operatorname{sgn} a_{ij}$ for all $i, j$. For $0 \leqq k \leqq n^2$, let $Q_A(k)$ denote the set of all $B \in Q_A$ such that $b_{ij} \neq a_{ij}$ for at most $k$ pairs $(i, j)$. Then

$$\{A\} = Q_A(0) \subset Q_A(1) \subset \ldots \subset Q_A(n^2) = Q_A.$$

Let $S(A)$ and $S'(A)$ denote respectively the number

inf $\{k$: some member of $Q_A(k)$ has an eigenvalue with nonnegative real part$\}$ and the number

inf $\{k$: some member of $Q_A(k)$ has an eigenvalue with positive real part$\}$,

so that $S(A)$ (resp. $S'(A)$) measures the robustness of $A$'s property of being stable (resp. semistable). In particular, $S(A) = 0$ when $A$ is not stable, $S(A) = 1$ when $A$ is stable but can be converted into a not stable matrix by changing the magnitude (without changing the sign) of a single entry $\ldots$, $S(A) = \infty$ when $A$ is sign stable. The following is a consequence of results established below.

THEOREM 3. *For any* $n \times n$ *real matrix* $A$,

$$S'(A) < \infty \Rightarrow S'(A) \leqq n, \text{ and}$$

$$S(A) < \infty \Rightarrow S(A) \leqq \tfrac{1}{2}(n^2 - n).$$

**Proofs of necessity.** The proofs of Theorems 1–3 are based on several lemmas.

LEMMA 1. *If $A$ fails to satisfy condition* ($\alpha$) (*resp.* ($\beta$), ($\gamma$)) *then some member of* $Q_A(1)$ (*resp.* $Q_A(2)$, $Q_A(n)$) *has an eigenvalue with positive real part.*

*Proof.* The argument is essentially that of [**13**, Lemma 5.1]. Failure of ($\alpha$) (resp. ($\beta$), ($\gamma$)) implies the existence for $k = 1$ (resp. $k = 2$, some $k \geqq 3$) of distinct indices $i(1), \ldots, i(k)$ and a number $p$ such that $a_{i(1)\,i(1)} = p > 0$ (resp. $a_{i(1)\,i(2)}a_{i(2)\,i(1)} = p > 0$, $a_{i(1)\,i(2)} \ldots a_{i(k-1)\,i(k)}a_{i(k)\,i(1)} = p \neq 0$). Let the matrix $C = (c_{ij})$ be such that $c_{ij} = a_{ij}$ for all pairs $(i, j)$ involved in the representation of $p$ and $c_{ij} = 0$ otherwise. For each positive integer $r$, the matrix $M_r = A + rC$ belongs to $Q_A(k)$. If the eigenvalues of $M_r$ all have nonpositive real part then the same is true of $(1/r)M_r$ and hence, by continuity, of $C$. But this contradicts the fact, verifiable by direct computation, that the eigenvalues of $C$ include $p$ (resp. $p^{1/2}$, all complex numbers $\lambda$ such that $\lambda^k = p$).

LEMMA 2. *If $A$ satisfies* ($\gamma$) *but not* ($\epsilon$) *then* 0 *is an eigenvalue of* $A$.

*Proof.* Each term in the expansion of $A$'s determinant is of the form $\pm a_{1\pi(1)} \ldots a_{n\pi(n)}$, where the permutation $\pi$ of $\{1, \ldots, n\}$ can be decomposed into cycles in the usual way. If 0 is not an eigenvalue of $A$ then $\det A \neq 0$ and some such term is nonzero. But if ($\gamma$) holds the corresponding permutation $\pi$ has no cycles of length $\geqq 3$, and with

$$M = \{\{i, j\} : i \neq j, \pi(i) = j \text{ and } \pi(j) = i\}$$

it is clear that

$$\{1, \ldots, n\} \sim \bigcup M = \{i : \pi(i) = i\} \subset R_A,$$

whence $M$ is a $(V \sim R_A)$-complete matching in $G_A$.

LEMMA 3. *If $A$ satisfies* ($\beta$) *and* ($\gamma$) *but not* ($\delta$) *then some member of* $Q_A(\frac{1}{2}(n^2 - n))$ *has an eigenvalue with nonnegative real part.*

*Proof.* Rather than dealing explicitly with eigenvalues, we use the fact that $A$ is stable if and only if the equilibrium $x = 0$ of the system (1) is asymptotically stable. Suppose that $A$ satisfies ($\beta$) and ($\gamma$) but not ($\delta$), and let $W$ denote the nonempty set of all white vertices in an $R_A$-coloring of $G_A$ corresponding to the failure of ($\delta$). Let $X$ denote the set of all twice-differentiable real-valued functions on the real line, $Y$ the set of all $x \in X$ such that $\ddot{x} = -x$, and $w$ an arbitrary point of $W$. With $V = \{1, \ldots, n\}$, a set $E_w \subset V \times V$ will be constructed such that

(2) for each $(i, j) \in V \times V$, $(i, j) \in E_w$ or $(j, i) \in E_w$ or both, and

(3) for each function $y \in Y$ and each matrix $B \in Q_A$ there exist functions $x_1, \ldots, x_n \in X$ and a matrix $C = (c_{ij}) \in Q_A$ such that

(3a) $x_w = y$,

(3b) $c_{ij} = b_{ij}$ for all $(i, j) \in E_w$, and

(3c) $\dot{x}_i = \sum_{j \in V} c_{ij}x_j$ for all $i \in V$.

It follows from (2) that $|E_w| \geqq n + \frac{1}{2}(n^2 - n)$, whence

$$C \in Q_B(\tfrac{1}{2}(n^2 - n))$$

by (3b). Let $y = \text{sine} \in Y$ and $B = A \in Q_A$. Then $C \in Q_A(\frac{1}{2}(n^2 - n))$ though it is clear from (3a) and (3c) that the equilibrium $x = 0$ of the system $\dot{x} = Cx$ is not asymptotically stable and hence the matrix $C$ is not stable. (The purpose of working with an arbitrary $y \in Y$ and $B \in Q_A$, rather than exclusively with the sine function and the matrix $A$, is to indicate the flexibility of the construction. Note that the same $E_w$ works for all $y$ and $B$.)

Now let $D_A$ denote the directed graph, defined earlier, whose vertex-set and edge-set are respectively $V$ and $\{(i, j) : i \neq j \text{ and } a_{ij} \neq 0\}$, and let $V$ be partitioned into three sets $V_i$ as follows, where it is understood $w \in V_1$:

$V_0 = \{v \in V : D_A$ admits no directed path from $v$ to $w\}$;

$V_1 = \{v \in V : D_A$ admits a directed path from $v$ to $w$ and also one from $w$ to $v\}$;

$V_2 = \{v \in V : D_A$ admits a directed path from $v$ to $w$ but none from $w$ to $v\}$.

(See the first part of the Appendix for an example.) It follows from condition ($\gamma$) that the induced subgraph of $G_A$ having $V_1$ as its vertex-set is a tree $T$ and any two vertices of $T$ that are joined by an edge of $D_A$ are in fact joined by two such (oppositely directed) edges and hence by an edge of $G_A$. Let the tree $T$ be rooted at $w$, and for each $v \in V_1 \sim \{w\}$ let $v^*$ denote $v$'s neighbor in the unique path that joins $v$ to $w$ in $T$; in other words, $v^*$ is $v$'s immediate predecessor in the partial ordering of $V_1$ induced by rooting $T$ at $w$. For each choice of $u, v \in V_1$ with $u \neq v$ it follows from ($\beta$) and ($\gamma$) that

$$\begin{aligned} &a_{uv} = 0 = a_{vu} \quad \text{or} \\ &a_{uv}a_{vu} < 0 \quad \text{and} \quad v = u^* \quad \text{or} \\ (4) \qquad &a_{uv}a_{vu} < 0 \quad \text{and} \quad u = v^*. \end{aligned}$$

Let the set $E_w$ consist of all ordered pairs in $V \times V$ except for the $(u, v) \in V_1 \times V_1$ that satisfy condition (4). Plainly (2) holds, and it remains to establish (3) by showing that for each $y \in Y$ and $B \in Q_A$ there exist $x_1, \ldots, x_n$ and $C$ as described. Note that $c_{uv}$ is required to be equal to $b_{uv}$ for all $(u, v) \in V \times V$, except that when $(u, v) \in V_1 \times V_1$ and (4) holds, $c_{uv}$ need merely be of the same sign as $b_{uv}$.

To begin the construction, let $x_w = y$, $x_i = 0$ for all $i \in V_0$, and $c_{ij} = b_{ij}$ for all $(i, j) \in E_w$. Then conditions (3a) and (3b) are satisfied. Also, (3c) holds for each $i \in V_0$, regardless of how construction of the $x_j$'s and $c_{ij}$'s is continued, because $c_{ij} = 0$ when $i \in V_0$ and $j \notin V_0$.

As an aid in continuing the construction, let $\tau_w = 0$ and let $N(w)$ denote the set of all $T$-neighbors of $w$ that belong to $W$. Since an $R_A$-coloring is involved, $N(w) \neq \emptyset$. For each $v \in V_1 \sim \{w\}$, let $N(v)$ denote the set of all

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$T$-neighbors of $v$ that belong to $W$ and are distinct from $v^*$, and let

$$\tau_v = 0 \quad \text{when} \quad v^* \notin W,$$
$$\tau_v \in ]0, 1[ \quad \text{when} \quad v^* \in W \quad \text{and} \quad N(v) \neq \emptyset,$$
$$\tau_v = 1 \quad \text{when} \quad v^* \in W \quad \text{and} \quad N(v) = \emptyset.$$

For each $v \in V_1 \sim W$, let $x_v = 0$ and $c_{vv^*} = b_{vv^*}$. It remains, in this stage of the construction, to define the function $x_v$ and the number $c_{vv^*}$ for each $v \in W \cap V_1 \sim \{w\}$. The definition is made inductively in such a way that for all such $v$,

(5) $\quad x_v \in Y \quad \text{and} \quad \tau_v \dot{x}_v = c_{vv^*} x_{v^*}.$

Let $V_1$ be linearly ordered so that $w$ comes first and each $v \in V_1 \sim \{w\}$ is preceded in the ordering by $v^*$. The definitions of $x_i$ and $c_{ij}$ are extended inductively with respect to the ordering, defining $x_v$ and $c_{vv^*}$ for $v \in W \cap V_1 \sim \{w\}$ as soon as $x_p$ has been defined for all predecessors $p$ of $v$ and $c_{pp}{}^*$ has been defined for all predecessors $p \neq w$. Note that if $v \in W \cap V_1 \sim \{w\}$ and $u = v^*$, then $b_{uv} \neq 0$ by (4), and $v \in N(u)$. Thus it is permissible to define $x_v$ and $c_{vu}$ as follows:

(6) if $u \notin W$ then $u \neq w$ and hence $u^*$ is well defined; let

$$x_v = -\frac{1}{b_{uv}|N(u)|} c_{uu^*} x_{u^*} \quad \text{and} \quad c_{vu} = b_{vu};$$

(7) if $u \in W$ then let

$$x_v = \frac{1 - \tau_u}{b_{uv}|N(u)|} \dot{x}_u \quad \text{and} \quad c_{vu} = -\tau_v \frac{1 - \tau_u}{b_{uv}|N(u)|}.$$

Starting from the fact that $x_w = y \in Y$, it is now possible to show inductively that (5) holds for all $v \in W \cap V_1 \sim \{w\}$. When (6) applies, the inclusion $x_v \in Y$ follows from the fact that if $u^* \in W$ then $x_{u^*} \in Y$, while if $u^* \notin W$ then $x_{u^*} = 0 \in Y$; for the second part of (5), note that $\tau_v = 0$ and $x_u = 0$. When (7) applies, $x_u \in Y$, whence $\dot{x}_u \in Y$ and $x_v \in Y$; for the second part of (5), note that

$$\tau_v \dot{x}_u = \tau_v \frac{1 - \tau_u}{b_{uv}|N(u)|} \ddot{x}_u = (-c_{vu})(-x_u).$$

To see that $c_{vu}$ has the desired sign when (7) applies, note that $\tau_v > 0$ because $v^* \in W$ and $\tau_u < 1$ because $N(u) \neq \emptyset$. Thus the sign of $c_{vu}$ is opposite the sign of $b_{uv}$ and by (4) is the same as the sign of $b_{vu}$.

Using (5)–(7) it can now be shown that

(8) $\quad \dot{x}_i = \sum_{j \in V_1} c_{ij} x_j \quad \text{for all } i \in V_1.$

Note first that, since $x_v = 0$ for all $v \in V_1 \sim W$, $c_{uv} = b_{uv}$ for all $(u, v) \in E_w$, and $b_{vv} = 0$ for all $v \in W$, (8) is equivalent to

(9) $\quad \dot{x}_i = c_{ii^*} x_{i^*} + \sum_{j \in N(i)} b_{ij} x_j$

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except that the first term on the right is missing when $i = w$. When $i \in W$ with $N(i) \neq \emptyset$, the final sum in (9) is transformed with the aid of (7) into

$$\sum_{j\in N(i)} b_{ij} \frac{1-\tau_i}{b_{ij}|N(i)|}\dot{x}_i = (1-\tau_i)\dot{x}_i;$$

if $i = w$, $\tau_i = 0$ and (9) follows, while if $i \neq w$ then $c_{ii^*}x_{i^*} = \tau_i\dot{x}_i$ by (5) and again (9) follows. When $i \in W$ with $N(i) = \emptyset$, then $i^* \in W$ because an $R_A$-coloring is involved, whence $\tau_i = 1$ and both (9) and (5) assert $\dot{x}_i = c_{ii^*}x_{i^*}$. When $i \notin W$, $\dot{x}_i = 0$, and (6) implies

$$x_j = -\frac{1}{b_{ij}|N(i)|}c_{ii^*}x_{i^*} \quad \text{for all } j \in N(i);$$

then (9) follows if $N(i) \neq 0$; if, on the other hand, $N(i) = \emptyset$, then $i^* \in V_1 \sim W$ because an $R_A$-coloring is involved, whence $x_{i^*} = 0$ and again (9) holds. Thus (8) has been established for all $i \in V_1$. But $x_j = 0$ for all $j \in V_0$ and (when $i \in V_1$) $c_{ij} = 0$ for all $j \in V_2$, so it follows from (9) that (3c) holds for all $i \in V_1$, regardless of how the construction is continued.

There remains only the definition of $x_i$ for $i \in V_2$. For each such $i$, let $z_i = \sum_{j\in V_0 \cup V_1} c_{ij}x_j$. Then by a standard theorem for linear systems (Theorem 4.1 on pp. 75–76 of [**4**]) there exist functions $x_i \in X$ for all $i \in V_2$ such that

$$\dot{x}_i = z_i + \sum_{j\in V_2} c_{ij}x_j \quad \text{for all } i \in V_2.$$

Condition (3$c$) is now satisfied for all $i \in V$ and the proof of Lemma 3 is complete.

That $S'(A) < \infty \Rightarrow S'(A) \leqq n$ follows from Lemma 1 in conjunction with the fact that conditions $(\alpha) - (\gamma)$ imply sign semistability. That $S(A) < \infty \Rightarrow S(A) \leqq \frac{1}{2}(n^2 - n)$ follows from Lemmas 1–3 in conjunction with the fact that conditions $(\alpha) - (\epsilon)$ imply sign stability. Thus the sufficiency arguments in the next section will complete the proofs of Theorems 1–3.

**Proofs of sufficiency.** Consider an $n \times n$ real matrix $A$ satisfying conditions $(\alpha) - (\gamma)$, and let the matrix $C$ be such that $c_{ij} = a_{ij}$ when $a_{ij} \neq 0 \neq a_{ij}$ and $c_{ij} = 0$ otherwise. The undirected graphs $G_A$ and $G_C$ are the same, and $C$ satisfies $(\alpha) - (\gamma)$. The directed graphs $D_A$ and $D_C$ may be different, but by $(\gamma)$ they have the same cycles. Since, moreover, $c_{ii} = a_{ii}$ for all $i$, it is easily seen that $A$ and $C$ have the same characteristic polynomial. We now show that $C$ is semistable.

By $(\gamma)$, each component of the graph $G_C$ is a tree. Let the set $U \subset V$ include precisely one vertex of each such tree, and for each $v \in V \sim U$ let $v^*$ denote $v$'s neighbor in the unique path that joins $v$ to an element of $U$ in $G_C$. Let $\lambda_u = 1$ for each $u \in U$, and when $v \in V \sim U$ is such that $\lambda_{v^*}$ has been defined, let $\lambda_v = -\lambda_{v^*}c_{v^*v}/c_{vv^*}$. Then

(10) $\lambda_i c_{ij} = -\lambda_j c_{ji}$ for all $i \neq j$.

For each $x = (x_1, \ldots, x_n) \in R^n$, let $\varphi(x) = \sum_1^n \lambda_i x_i^2$. Then $\varphi(x) > 0$ for all $x \neq 0$, because by ($\beta$) and (10) all $\lambda_i$'s are positive. For $x = x(t)$ such that $\dot{x} = Cx$, it follows with the aid of (10) that

$$(11) \quad \dot{\varphi}(x) = \sum_1^n 2\lambda_i x_i \dot{x}_i = 2\sum_{i=1}^n \sum_{j=1}^n \lambda_i x_i c_{ij} x_j = 2\sum_1^n \lambda_i c_{ii} x_i^2$$

and then by ($\alpha$) that $\dot{\varphi}(x) \leqq 0$ for all $x$. Thus the positive definite Lyapunov function $\varphi$ has a negative semidefinite derivative, and it follows from a well-known stability theorem ((7.2.i) of [**2**] and Theorem 25.1 of [**5**]) that the equilibrium $\dot{x} = Cx$ is stable and hence each positive half-trajectory is bounded. Thus the matrices $C$ and $A$ are both semistable. That completes the proof of Theorem 1, for conditions ($\alpha$) – ($\gamma$) are satisfied by all matrices $B \in Q_A$.

Now suppose that $A$ (and hence $C$) satisfies conditions ($\delta$) and ($\epsilon$) in addition to ($\alpha$) – ($\gamma$). We want to show $C$ is stable and thus complete the proof of Theorem 2.

By ($\gamma$) and the reasoning used to prove Lemma 2, there is a natural correspondence between the $(V \sim R_C)$-complete matchings in $G_C$ and the nonzero terms in the expansion of $C$'s determinant. Each such term is the product, for a permutation $\pi$ of $\{1, \ldots, n\}$ composed entirely of 1-cycles and 2-cycles, of factors $a_{ii}(< 0$ by ($\alpha$)) such that $\pi$ fixes $i$, factors $a_{ij}a_{ji}(< 0$ by ($\beta$)) such that $\pi$ interchanges $i$ and $j$, and the factor

$$(-1)^{\text{parity of } \pi}.$$

Thus all the nonzero terms in question have the same sign as $(-1)^n$, and since ($\epsilon$) implies there is a nonzero term the matrix $C$ is nonsingular.

We now require another basic result (Theorem 26.2 of [**5**]) on the direct method of Lyapunov, asserting that the equilibrium $x = 0$ of the system $\dot{x} = Cx$ is asymptotically stable (and the matrix $C$ is stable) if no positive half-trajectory other than $x = 0$ lies entirely in the set $\{x : \dot{\phi}(x) = 0\}$. Note that by (11), ($\alpha$), and the positivity of the $\lambda_i$'s,

$$(12) \quad \dot{\varphi}(x) = 0 \quad \Leftrightarrow \quad x_i = 0 \quad \text{for all } i \in R_C.$$

Now consider a positive half-trajectory $\{x(t) = (x_1(t), \ldots, x_n(t)) : t \geqq t_0\}$ for the system $\dot{x} = Cx$, and suppose it is different from $\{0\}$ and contained in $\{x : \dot{\varphi}(x) = 0\}$. Color a vertex $i \in V$ black if the function $x_i$ is constant, and otherwise color $i$ white. By (12), all members of $R_C$ are black, and plainly no black vertex has precisely one white neighbor. If all vertices are black (that is, all $x_i$'s are constant) the algebraic system $Cx = 0$ has a nontrivial solution and hence $C$ is singular, contradicting ($\epsilon$). Thus there exists a white vertex, whence by ($\delta$) the coloring is not an $R_C$-coloring and thus there is a white vertex $w$ that has no white neighbor. Since $c_{ww} = 0$, $\dot{x}_w$ is a nonzero constant, contradicting the fact that the positive half-trajectory in question is bounded. That completes the proof.

Since the Lyapunov function $\varphi$ was needed in the proof of Theorem 2, the above proof of the "if" part of Theorem 1 is well suited to our exposition.

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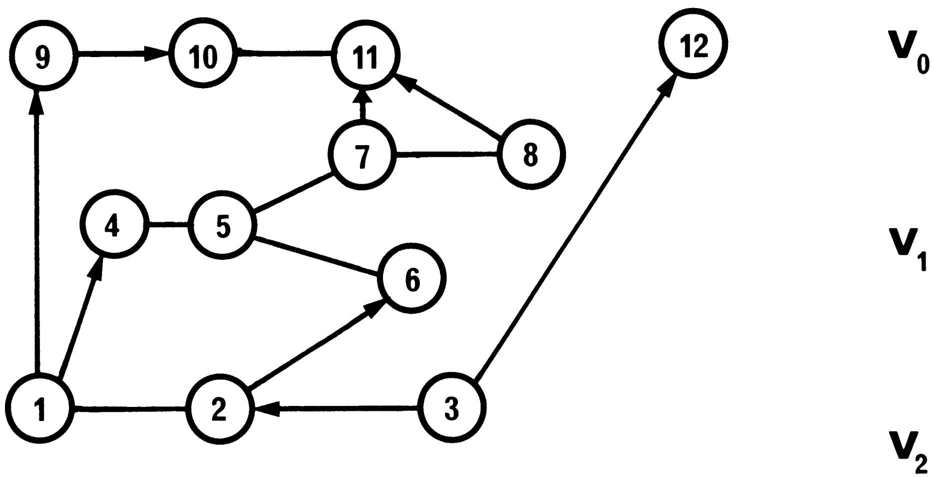
However, it should also be observed that in deriving $A$'s semistability from ($\alpha$) — ($\gamma$) it may be assumed without loss of generality that $A$ satisfies ($\delta$) and ($\epsilon$) as well. Simply note that arbitrarily small adjustment of $A$'s diagonal entries can be made to produce a nonsingular matrix $B$ with $b_{ii} < 0$ for all $i$ and $b_{ij} = a_{ij}$ for all $i \neq j$. That was the approach of Quirk and Rupert [**14**, Lemma 5].

## Appendix: Examples and comments.

*An example illustrating the sets $V_i$ used in proving Lemma* 3. In the illustration below, directed segments represent edges of $D_A$ and undirected segments represent edges of $G_A$ corresponding to pairs of oppositely directed edges of $D_A$. Thus, for example, $a_{45} \neq 0 \neq a_{54}$, $a_{19} \neq 0 = a_{91}$, and $a_{49} = 0 = a_{94}$. The sets $V_i$ are shown on successive levels corresponding to $w = 5$. Since $V_1$ is rooted at $w$, $4^* = 6^* = 7^* = 5$ and $8^* = 7$.




In the ecological terminology of [**6**], the edges of $G_A$ represent prey-predator links, $V_1$ is the predation community containing $w$, and the members of $R_A$ (not specified in the above example) correspond under ($\alpha$) to self-regulating species. Of course $w \notin R_A$, for $w$ is white and in an $R_A$-coloring all members of $R_A$ are black.

*A sufficient condition for sign stability.* For a matrix $A$ that satisfies conditions ($\alpha$) — ($\gamma$), the presence or lack of sign stability depends on the set $R_A = \{i : a_{ii} < 0\}$ of self-regulating species. For each $S \subset V = \{1, \ldots, n\}$, let the matrix $B(A, S) = (b_{ij})$ be such that $b_{ij} = a_{ij}$ for all $i \neq j$, $b_{ii} = -1$ when $i \in S$, and $b_{ii} = 0$ when $i \notin S$. Let $\mathscr{S}_A$ denote the set of all $S \subset V$ such that $B(A, S)$ is sign stable. It follows from Theorem 2 that if $S_1 \subset S_2 \subset V$ and $S_1 \in \mathscr{S}_A$ then $S_2 \in \mathscr{S}_A$; in other words, the sign stability of a matrix cannot be destroyed by merely changing non-self-regulating species to self-regulating.

It would be of interest to characterize the minimal members of $\mathscr{S}_A$ in terms of their location in the graph $G_A$. The following result picks out a member of $\mathscr{S}_A$ that is usually not minimal but is much smaller than $V$. When applied to ecological systems, it may help to explain the territorial behavior of species at the tops of food chains.

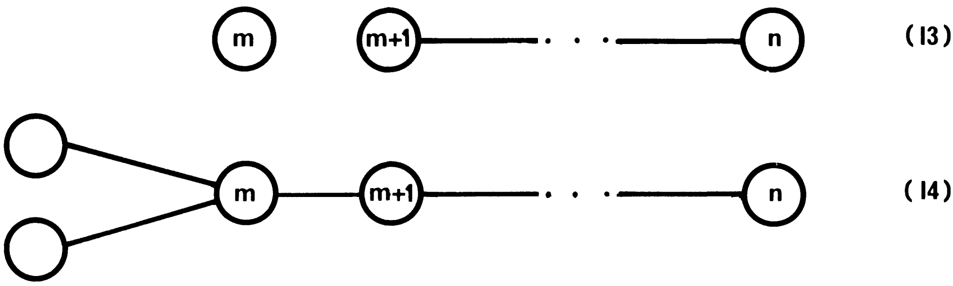
PROPOSITION 1. *If the $n \times n$ real matrix $A$ satisfies conditions $(\alpha) - (\gamma)$ and $a_{ii} < 0$ for each vertex $i$ of valence $\leqq 1$ in $G_A$ then $A$ is sign stable.*

*Proof.* The proposition follows readily from Theorem 2 when $G_A$ is a simple path, which is understood here to include the case of a single vertex. In the remaining case, $G_A$ is a forest but not a simple path, and the proof proceeds by induction on $n$. In the inductive step it may be assumed without loss of generality that there exists $m$ with $1 \leqq m < n$ such that the sequence $(m + 1, \ldots, n)$ determines a simple path that is an induced subgraph of $G_A$, all neighbors of $j$ are in the path for $m + 1 < j \leqq n$, and either

(13) all neighbors of $m + 1$ are in the path, or

(14) $m$ is the only neighbor of $m + 1$ not in the path and there are at least two neighbors of $m$ not in the path.

The cases (13) and (14) are depicted below, where all neighbors of $m + 1, \ldots, n$ are indicated but $m$ may have neighbors in addition to those indicated.




Let $B$ denote the matrix formed by the first $m$ rows and columns of $A$. Then $B$ satisfies conditions $(\alpha) - (\gamma)$ and $b_{ii} < 0$ for each vertex $i$ of valence $\leqq 1$ in $G_B$, so it follows from the inductive hypothesis that $B$ is sign stable. Since $n \in R_A$, the existence of a $(V \sim R_A)$-complete matching in $G_A$ follows readily from the existence of a $(V \sim R_B)$-complete matching in $G_B$; that is, $A$ satisfies condition $(\epsilon)$. To see that $A$ satisfies the coloring condition $(\delta)$, suppose that $G_A$ admits an $R_A$-coloring with at least one white vertex. Since $n \in R_A$, $n$ is black and it follows that each of $n - 1, \ldots, m + 1$ is black. But then the restriction of the coloring to $G_B$ is an $R_B$-coloring with at least one white vertex, contradicting the fact that $B$ satisfies $(\delta)$.

*The case in which $G_A$ is a simple path.* The next result provides an additional illustration of how Theorem 2 picks out which vertices are important as self-regulating ones.

PROPOSITION 2. *Suppose the graph $G_A$ is a simple path whose successive vertices are* $1, \ldots, n$, *and let* $R_A = \{i : a_{ii} < 0\}$. *Then the matching condition* ($\epsilon$) *is satisfied if and only if n is even or some member of* $R_A$ *is odd. And the coloring condition* ($\delta$) *is satisfied if and only if* $R_A$ *intersects* $\{1, 2, n - 1, n\}$ *or there are two members i and j of* $R_A$ *for which* $|i - j| \leqq 2$.

*Proof.* The proof concerning ($\epsilon$) is left to the reader, as is the "only if" part of the proof concerning ($\delta$). For the "if" part of the latter proof, suppose there exist $i, j \in R_A$ with $i < j \leqq i + 2$, and consider an arbitrary $R_A$-coloring of $G_A$. Of course $i$ and $j$ are black, and $i + 1$ is black when $j = i + 2$ because each white vertex has at least one white neighbor. Since no black vertex has precisely one white neighbor, the blackness of $i - 1, \ldots, 1$ and $j + 1, \ldots, n$ is then apparent. The reasoning is similar when $R_A$ intersects $\{1, 2, n - 1, n\}$.

It follows from Proposition 2 that if $A$ satisfies conditions ($\alpha$) – ($\gamma$), $G_A$ is a simple path as described, and $R_A$ consists of a single vertex $r \in \{1, \ldots, n\}$, then $A$ is sign stable if and only if $n$ is even and $r \in \{1, 2, n - 1, n\}$ or $n$ is odd and $r \in \{1, n\}$.

*When is a matrix sign quasistable*? Let us say that a square real matrix $A$ is *quasistable* if the equilibrium $x = 0$ of the system $\dot{x} = Ax$ is stable. Plainly

$$\text{stable} \Rightarrow \text{quasistable} \Rightarrow \text{semistable}.$$

And $A$ is *sign quasistable* if each matrix $B$ having the same sign-pattern as $A$ is quasistable. It would be of interest to characterize sign quasistability in finitely computable terms and to develop an efficient algorithm for testing sign quasistability. If the matrix $A$ satisfies conditions ($\alpha$) and ($\gamma$), and if $A$ is combinatorially skew-symmetric in the sense that

$$(\beta^*) \ \operatorname{sgn} a_{ji} = -\operatorname{sgn} a_{ij} \quad \text{for all } i \neq j,$$

then $A$ is equal to the matrix $C$ of the preceding section and it follows from the reasoning there that $A$ is sign quasistable. Other sufficient conditions for sign quasistability have been communicated to us by Bruce Clarke. Any general characterization of sign quasistability must consider edges of $D_A$ that join different components of $G_A$, edges that have been unimportant in the study of sign stability and sign semistability. Note, for example, that the matrices

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

have the same characteristic polynomial but not the same minimal polynomial. The first is sign quasistable but the second is not, and their graphs are

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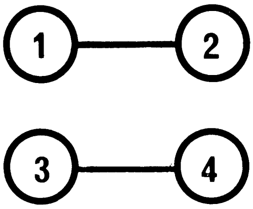
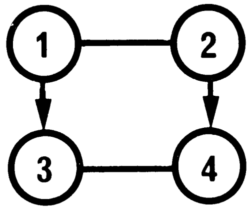



and




respectively.

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