*J. Inst. Math. Jussieu* (2024), **23**(1), 495–519 495
doi:10.1017/S1474748022000524 

# SPECIAL VALUES OF ZETA-FUNCTIONS OF REGULAR SCHEMES

STEPHEN LICHTENBAUM

*Department of Mathematics, Brown University, Providence, RI 02912*
(stephen_lichtenbaum@brown.edu)

(Received 22 February 2021; revised 21 September 2022; accepted 10 October 2022; first published online 6 December 2022)

*Abstract* We formulate a conjecture on the special values of zeta functions of regular arithmetic schemes in terms of Weil-étale cohomology…

*2020 Mathematics Subject Classification*: 11G40, 14G10, 19F27

## Introduction

In this paper, we will give a conjectural formula (Conjecture 3.1) for the special values $\zeta^*(X,r)$ of the scheme zeta-function of a regular scheme $X$ projective and flat of dimension d (so relative dimension $d-1$) over Spec $\mathbb{Z}$ at a rational integer $r$ in terms of singular, de Rham and Weil-étale motivic cohomology, valid up to sign and powers of 2. (Let $G$ be any meromorphic function on $\mathbb{C}$, let $r$ be a rational integer and let $a_r$ be the order of the zero of $G(s)$ at $s=r$. Let $G^*(r)$ be the limit as $s$ approaches $r$ of $G(s)(s-r)^{-a_r}$. If $G$ is a zeta-function, $G^*(r)$ is referred to as a special value of $G$.)

We can factor the map from $X$ to Spec $\mathbb{Z}$ uniquely through Spec $O_K$, where $K$ is a number field and the generic fiber of $X$ over Spec $O_K$ is a smooth connected algebraic variety over $K$. We will construct complexes made up of variants of these cohomology groups, and the conjectured formula will give the special value as a product of Euler characteristics of these complexes, equipped with suitable integral structures. We will prove that, if $d \le 2$, this conjecture is compatible with Serre's conjectured functional equation [19] for the zeta-function, and if $d > 2$, this compatibility is true modulo two previously existing conjectures which are true in dimension $\le 2$.

We will discuss how this relates to previous work on this subject. In order to make everything precise, we need to recall the definitions of the scheme zeta-function and what we will call here the Hasse–Weil zeta-function. If $X$ is a scheme of finite type over Spec $\mathbb{Z}$, the scheme zeta-function is defined to be $\zeta(X,s) = \prod(1-N(x))^{-s})^{-1}$, where the product is taken over all closed points $x$ of $X$ and $N(x)$ is the number of elements in the residue field $\kappa(x)$. Recall that $x$ is closed if and only if $\kappa(x)$ is finite. This product converges for $Re(s) > d$, where $d$ is the Krull dimension of $X$. It is a well-known conjecture that $\zeta(X,s)$

extends to a meromorphic function on the entire plane. If $Y$ is an integral scheme of dimension $d-1$ which is projective and smooth over Spec $K$, where $K$ is a number field, and $m$ is an integer between 0 and $2d-2$, Serre in [19] defined an L-function $L_m(Y,s)$. Serre also conjectured the exact form of a functional equation involving this L-function. We define the Hasse–Weil zeta-function of X to be $\prod_{m=0}^{2d-2} L_m(X_0,s)^{(-1)^m}$, where $X_0$ is the generic fiber of $X$. It is also conjectured that the L-functions can be continued to meromorphic functions. If $X$ is smooth over Spec $O_K$, then the Hasse–Weil zeta-function of $X$ is equal to the scheme zeta-function of $X$.

Beilinson, building on a previous conjecture of Deligne, gave a special values conjecture [24] for $r \leq 0$ up to a rational number. More specifically, Beilinson gave a conjectured formula for the special value $L^*_m(X,r)$ up to a rational number. Bloch and Kato [23] gave a formula up to sign for this special value when the weight $m-2r \leq -3$. Fontaine and Perrin-Riou [8] gave a conjectured formula for all integers $r$. All of these conjectures actually involve the L-function $L_m(X,s)$ and so by taking products give rise to a conjecture involving the Hasse–Weil zeta-function of $X$.

Fontaine and Perrin-Riou first introduced various (conjecturally) finite-dimensional vector spaces and made a conjecture giving the special value of the L-functions up to a rational number in terms of determinants of maps between these vector spaces tensored with $\mathbb{R}$. They then used spaces taken from p-adic Hodge theory to refine these conjectures to be valid up to sign. Taking the alternating product, we get a conjecture about the special values of the Hasse–Weil zeta-function of $X$, which we will call Conjecture FPR.

A crude description of our Conjecture 3.1 would be to say that it refines Conjecture FPR by using canonical integral models for the vector spaces used in that conjecture, avoiding the necessity for p-adic Hodge theory. In fact, the difference between the scheme zeta-function and the Hasse–Weil zeta-function forces minor changes in the vector spaces to be considered.

The original version of Conjecture 3.1 was announced in 2017 [17]. This was preceded in 2016 by a special-values conjecture by Flach and Morin [5] for the scheme zeta-function, which we will refer to as conjecture FM1. Conjecture FM1 does make use of p-adic Hodge theory and is more closely related to Conjecture FPR than is to our Conjecture 3.1. In fact, Conjecture FM1 was shown by Flach and Morin to be equivalent to Conjecture FPR if $X$ is smooth over Spec $\mathbb{Z}$.

In 2019, Flach and Morin made a new conjecture (referred to here as Conjecture FM2) [6], which avoids p-adic Hodge theory and so is more closely modeled on Conjecture 3.1 and less related to Conjecture FPR. In Appendix B, we discuss the relation between Conjecture 3.1 and Conjecture FM2. In this paper, we show that Conjecture 3.1 is compatible with a form of the functional equation for the scheme zeta-function. Flach and Morin [6] also showed this for Conjecture FM2. This has not been shown for either Conjecture FM1 or Conjecture FPR.

Conjecture 3.1 is a bit more ad hoc than Conjecture FM2 but has the advantage that it involves much less elaborate machinery. Of course, as previously mentioned, Conjecture FM2 is more precise since it includes powers of 2, but we hope that it is possible to remedy this by modifying Conjecture 3.1 using the Artin–Verdier topology.

In this paper, as in the papers [5, 6] of Flach and Morin, we only consider the zeta-function and not the associated L-functions. We believe that the L-function conjectures are probably not completely correct, basically because of torsion phenomena in cohomology, which necessitate correction terms in analogous formulas for special values of zeta-functions of varieties over finite fields. We note that the wonderful formula [12] of Bloch, Kato and T. Saito, which plays an extremely important role in the proof of compatibility, is only valid for Euler characteristics and not for individual cohomology groups, which forces the restriction to zeta-functions. We also remark again that throughout this paper we work with the scheme zeta-function of $X$.

Recently, Niranjan Ramachandran and I [14] have shown that, if $X$ is an arithmetic surface and $r = 1$, Conjecture 3.1 is equivalent to the conjecture of Birch and Swinnerton-Dyer for the Jacobian of $X$. A similar result was proved by Flach and Siebert [7] for Conjecture FM2.

There are two basic approaches to zeta-function conjectures: One (the Tamagawa approach) involves writing the formula as a product of local formulas (one for each prime p) and then using the product formula to show that, although the individual factors may depend on choices (possibly of a differential), the product does not. This approach is used by Tate in his Bourbaki talk [22] on the conjecture of Birch and Swinnerton-Dyer for abelian varieties, by Fontaine and Perrin-Riou [8] and by Flach and Morin. [5].

The other involves just working with the infinite primes and, for example, choosing a particularly good differential. This approach was used by the author in his conjectures on the Dedekind zeta-function [16], by Silverman in stating the conjecture of Birch and Swinnerton-Dyer for elliptic curves in his book [21, 20] on elliptic curves and is used in this paper and in [6]. The basic idea of this paper is that, by making the infinite prime part of the formula of Fontaine and Perrin -Riou more precise, we can dispense with the detailed local p-adic analysis.

We should make it clear that, even to state our conjecture, we have to assume the validity of other previous conjectures.

First, we need the conjecture, which is very far from being proved, that the zeta-function of $X$, which converges for $Re(s) > d$, can be meromorphically continued to the entire plane, so we can talk about $\zeta^*(X,r)$ for $r < d$.

Second, we assume that the ètale motivic cohomology groups that we will define are finitely generated.

Third, we assume that the Beilinson regulator maps and various Arakelov intersection pairings induce isomorphisms on the complex vector spaces.

For the proof of compatibility, we need the theorem that the groups $H^{2r+1}_{et}(X,\mathbb{Z}(r))$ and $H^{2(d-r)+1}_{et}(X,\mathbb{Z}(d-r))$ are finite and Pontriagin dual to each other. This was proved by Flach and Morin [5] if $d \leq 2$ and under some restrictions in the general case.

We also need, for the full Bloch–Kato–Saito theorem, resolution of singularities for arithmetic schemes.

The product defining $\zeta(X,s)$ is well known to converge for $Re(s) > d$ and is conjectured to have a meromorphic continuation to the entire plane. We will tacitly assume this conjecture in what follows. It is further conjectured [19, 2] that there exists a Γ-factor

$\Gamma(X,s)$ and a positive rational number $A$, the conductor, such that if we let $\phi(X,s) = A^{s/2}\zeta(X,s)\Gamma(X,s)$, then $\phi(X,s)$ satisfies the functional equation $\phi(X,s) = \pm\phi(X,d-s)$.

Now, let $X$ be regular, and projective and flat over Spec $\mathbb{Z}$. The basic idea behind our conjectured formula is to start with Fontaine's 'Deligne–Beilinson' conjectures [8], which give the special values up to a rational number in terms of determinants of maps of complex vector spaces with given rational structures. These complex vector spaces come from singular and de Rham cohomology, and from Weil-étale motivic cohomology, and have to be slightly modified to reflect the difference between the scheme zeta-function and the Hasse–Weil zeta-function. We replace the rational structures by integral structures and take determinants with respect to these. The singular cohomology of course has a natural integral structure, and the Weil-étale groups conjecturally do also. We define an integral structure on the de Rham groups by using derived exterior powers. We should note that these derived exterior powers have an important role to play even in the number ring case ($d = 1$).

We also introduce the orders of naturally occurring finite cohomology groups into the picture. Finally, we replace the period maps in Fontaine's picture by 'modified' period maps, where we divide by special values of the gamma function.

Our conjectural formula expresses the special values of the zeta-function in terms of the product of Euler characteristics of exact sequences of complex vector spaces with integral structures. The complex vector spaces will be derived from singular cohomology, de Rham cohomology and Weil-ètale motivic cohomology. For the exact formula, see Section §3. The maps between them arise from Beilinson's conjectures, Arakelov height pairings and periods.

As we move along, we will explain how our definitions of groups and maps relate to those of Fontaine and Perrin-Riou [9].

## 1. Integral structures and Euler characteristics

Let $V_0, V_1, \cdots, V_n$ be finite-dimensional complex vector spaces, and

$$V_* = 0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$$

be an exact sequence.

Let $B_i$ be a lattice spanned by a basis for the vector space $V_i$. Let $\Lambda V$ denote the highest exterior power of $V$ and $\Lambda B$ denote the highest exterior power of $B$. The alternating tensor product of the $\Lambda V_i' s$ is canonically isomorphic to $\mathbb{C}$, and the alternating tensor product of the $\Lambda B_i' s$ is isomorphic to $\mathbb{Z}$. The natural inclusions of $B_i$ in $V_i$ induce a map from $\mathbb{Z}$ to $\mathbb{C}$ and the determinant $det(V_*, B_*)$ in $\mathbb{C}^*/\pm 1$ of the pair $(V_*, B_*)$ is defined to be the image of a generator of $\mathbb{Z}$ in $\mathbb{C}$.

**Definition 1.1.** Let $V_*$ be a sequence of finite-dimensional complex vector spaces. An integral structure on $V_*$ is a sequence of pairs $(A_*, a_*)$, where $A_*$ is a lattice in $V_*$ and $a_*$ is a positive rational number.

An example of an integral structure on a finite-dimensional complex vector space $V$ comes from a finitely generated abelian group $M$ with a homomorphism from $M$ to $V$

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

whose image is a lattice $M_0$ in $V$ and whose kernel is the torsion subgroup $M_{tor}$ of $M$. The integral structure is then $(M_0,|M_{tor}|)$.

**Definition 1.2.** An integral structure $(A_*,a_*)$ is torsion-free if each $a_j$ is equal to 1.

**Definition 1.3.** Let $(A_*,a_*)$ be an integral structure on the finite exact complex $V_*$. We define the Euler characteristic of $(A_*,a_*)$ to be det $(V_*,A_*)\prod a_j^{(-1)^{j+1}}$.

**Definition 1.4.** Let $(A_*,a_*)$ be an integral structure on the exact complex $V_*$, $(B_*,b_*)$ be an integral structure on the exact complex $W_*$ and $\phi_*$ be a map of complexes from $V_*$ to $W_*$ such that $\phi_j$ is an isomorphism for all $j$. Let $det_j$ be the determinant of $\phi_j$ with respect to the lattices $A_j$ and $B_j$, and let

$$\chi(\phi_j)=\frac{b_j\ det_j}{a_j}.$$

Define the Euler characteristic

$$\chi(\phi):=\prod_{j=0}^{n}(\chi(\phi_j))^{(-1)^{j+1}}.$$

**Proposition 1.5.** *The Euler characteristic of the dual of a torsion-free integral structure is equal to either to the Euler characteristic of the original integral structure or to its inverse, depending on whether n is odd or even.*

**Proof.** Straightforward. □

## 2. The groups and maps involved in the conjecture

### 2.1. Weil-étale motivic cohomology

As before, let $X$ be a regular scheme, projective and flat over Spec $\mathbb{Z}$. Let $X_0$ be the fiber of $X$ over Spec $\mathbb{Q}$, and let $K$ be the algebraic closure of $\mathbb{Q}$ in the function field of $X$. Let $O_K$ be the ring of integers in $K$. We may regard $X$ as a scheme projective and flat over Spec $O_K$.

We will first define Weil-étale motivic cohomology groups and then discuss their relation to the groups defined by Fontaine and Perrin-Riou [8, 4].

Let $r$ be an integer and $j$ a nonnegative integer. We would like to define a Weil-étale site and complexes of sheaves $\mathbb{Z}(r)$ on this site whose cohomology groups $H^j_W(X,\mathbb{Z}(r))$ would be Weil-étale motivic cohomology, but unfortunately we do not know how to do this. Instead, for $j\le 2r$ we define $H^j_W(X,\mathbb{Z}(r))$ to be the hypercohomology groups $H^j_{et}(X,\mathbb{Z}(r))$, where $\mathbb{Z}(r)$ denotes Bloch's higher Chow group complex sheafified for the étale topology [1, 13]. Sometimes, these groups are referred to as étale motivic cohomology. For $j\ge 2r+1$, we define $H^j_W(X,\mathbb{Z}(r))$ to be $h^j(RHom(R\Gamma_{et}(X,\mathbb{Z}(d-r)),\mathbb{Z}[-2d-1])$, so we have the exact sequence

$$0\to \mathrm{Ext}^1(H^{2d+2-j}_{et}(X,\mathbb{Z}(d-r)),\mathbb{Z})\to$$
$$H^j_W(X,\mathbb{Z}(r))\to \mathrm{Hom}(H^{2d+1-j}_{et}(X.\mathbb{Z}(d-r)),\mathbb{Z})\to 0.$$

If we had our hypothetical Weil-étale site, with a global sections functor denoted by $\Gamma_W$, this would follow, up to 2-torsion, from a duality theorem which asserted that $R\Gamma_W(X,\mathbb{Z}(d-r))$ was isomorphic to $RHom(R\Gamma_W(X,\mathbb{Z}(r)),\mathbb{Z}[-2d-1])$. The analogue of this theorem, assuming the usual conjectures, is true for Weil-étale cohomology in the geometric case, as shown in [10]. We note here that in [5], Flach and Morin have constructed such a complex of abelian groups, which satisfies this duality theorem assuming that standard finiteness conjectures hold.

The group $H^{2r}_W(X,\mathbb{Z}(r))$ is by definition $H^{2r}_{et}(X,\mathbb{Z}(r))$, and by standard arguments this agrees with the group $H^{2r}_{Zar}(X,\mathbb{Z}(r))$ of codimension $r$ cycles on $X$ modulo rational equivalence after tensoring with $\mathbb{Q}$. Hence, there is a cycle map $\phi$ from $H^{2r}_W(X,\mathbb{Z}(r))$ to singular cohomology with rational coefficients. Let $H^{2r}_W(X,\mathbb{Z}(r))^1$ denote Ker $\phi$ (cycles homologous to zero) and $H^{2r}_W(X,\mathbb{Z}(r))^2$ denote Image $\phi$ (cycles modulo homological equivalence.).

We have the exact sequence

$$0 \to H_W^{2(d-r)}(X,\mathbb{Z}(d-r))^1 \to H_W^{2(d-r)}(X,\mathbb{Z}(d-r)) \to H_W^{2(d-r)}(X,\mathbb{Z}(d-r))^2 \to 0.$$

**Conjecture 2.1.** *The groups $H^j_{et}(X,\mathbb{Z}(r))$ are finitely generated for $j \leq 2r+1$, and finite for $j = 2r+1$.*

This implies

**Conjecture 2.2.** *The cohomology groups $H^j_W(X,\mathbb{Z}(r))$ are finitely generated for all $j$.*

Assuming the validity of Conjecture 2.2, we give the complex vector space $H^j_W(X,\mathbb{Z}(r))_{\mathbb{C}}$ the standard integral structure $H^j_W(X,\mathbb{Z}(r))$. We also need the following.

**Conjecture 2.3.** *The finite groups $H^{2r+1}_{et}(X,\mathbb{Z}(r))$ and $H^{2(d-r)+1}_{et}(X,\mathbb{Z}(d-r))$ are Pontriagin duals.*

Flach and Morin showed in [5, Proposition 3.4] that this follows from Conjecture 2.2 for $d \leq 2$ and, under some restrictions, in the general case.

### 2.2. Singular and de Rham cohomology

We also will have need of singular cohomology groups. Let $X_{\mathbb{C}} = X \times_{\mathbb{Z}} \mathbb{C}$. Complex conjugation $c$ acts on $H^j_B(X_{\mathbb{C}},\mathbb{Z})$ via the natural action of conjugation on $\mathbb{C}$. If $r$ is even (resp. odd), let $\tilde{H}^j_B(X,\mathbb{C}(r))^+$ and $\tilde{H}^j_B(X,\mathbb{Z}(r))^+$ be the set of elements $y$ in $H^j_B(X_{\mathbb{C}},\mathbb{C})$ and $H^j_B(X_{\mathbb{C}},\mathbb{Z})$ such that $c(y) = y$ (resp. $c(y) = -y$). We define $H^j_B(X,\mathbb{C}(r))$ (resp. $H^j_B(X,\mathbb{C}(r))^+$) to be $H^j_B(X_{\mathbb{C}},\mathbb{C})$ (resp. $\tilde{H}^j_B(X_{\mathbb{C}},\mathbb{C})^+$), and its standard integral structure is given by mapping $H^j_B(X_{\mathbb{C}},\mathbb{Z})$ (resp. $\tilde{H}^j_B(X_{\mathbb{C}},\mathbb{Z})^+$) to $H^j_B(X_{\mathbb{C}},\mathbb{C})$ (resp $\tilde{H}^j_B(X_{\mathbb{C}},\mathbb{C})^+$) via the natural map followed by multiplication by $(2\pi i)^r$.

If $r$ is even (resp. odd), let $\tilde{H}^j_B(X,\mathbb{C}(r))^-$ and $\tilde{H}^j_B(X,\mathbb{Z}(r))^-$ be the set of elements $y$ in $H^j_B(X_{\mathbb{C}},\mathbb{C})$ and $H^j_B(X_{\mathbb{C}},\mathbb{Z})$ such that $c(y) = -y$ (resp. $c(y) = y$). We define $H^j_B(X,\mathbb{C}(r))^-$ to be $\tilde{H}^j_B(X_{\mathbb{C}},\mathbb{C})^-$, and its standard integral structure is given by mapping $\tilde{H}^j_B(X_{\mathbb{C}},\mathbb{Z})^-$ to $\tilde{H}^j_B(X_{\mathbb{C}},\mathbb{C})^-$ via the natural map followed by multiplication by $(2\pi i)^r$.

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

We will also need the following Euler characteristics:

$$\chi(X_{\mathbb{C}}) = \sum_j (-1)^j \dim\, H_B^j(X_{\mathbb{C}}), \quad \chi^+(X_{\mathbb{C}}) = \sum_j (-1)^j \dim\, H_B^{j,+}(X_{\mathbb{C}},\mathbb{C}(0)),$$

$$\chi^-(X_{\mathbb{C}}) = \sum_j (-1)^j \dim\, H_B^{j,-}(X_{\mathbb{C}},\mathbb{C}(0)).$$

Let $\chi^+(X_{\mathbb{C}},Z(r)) = \chi^+(X_{\mathbb{C}})$ if $r$ is even and $\chi^-(X_{\mathbb{C}})$ if $r$ is odd.
Let $\chi^-(X_{\mathbb{C}},Z(r)) = \chi^-(X_{\mathbb{C}})$ if $r$ is even and $\chi^+(X_{\mathbb{C}})$ if $r$ is odd.

Let $\Omega = \Omega_{X_{\mathbb{C}}/\mathbb{C}}$. The de Rham cohomology group $H_{DR}^j(X_{\mathbb{C}},\mathbb{C})$ has the Hodge decomposition

$$\prod_{i+k=j} H^i(X_{\mathbb{C}},\Lambda^k\Omega)$$

which gives rise to the Hodge filtration $G_m = \prod_{k\geq m} H^{j-k}(X,\Lambda^k\Omega)$.

Then, $H_{DR}^j(X_{\mathbb{C}},\mathbb{C}(r))$ is defined to be $H_{DR}^j(X_{\mathbb{C}},\mathbb{C})$ but with the Hodge filtration $F$ given by $F_m = G_{m+r}$. If $M$ is $H^j(X_{\mathbb{C}},\mathbb{Z}(r))$, we define $t_M = t_{j,r}$ to be

$$H_{DR}^j(X_{\mathbb{C}},\mathbb{C}(r))/F_0 := \prod_{k<r} H^{j-k}(X_{\mathbb{C}},\Lambda^k\Omega) = \prod_\sigma \prod_{k<r} H^{j-k}(X_\sigma,\Lambda^k\Omega).$$

Here, $\sigma$ runs through all embeddings of the number field $K$ into $\mathbb{C}$. and $X_\sigma = X \times_{O_K} \mathbb{C}$ where the map from $O_K$ to $\mathbb{C}$ is induced by $\sigma$. The standard integral structure on $t_M$ is given by

$$\prod_\sigma \prod_{k<r} H^{j-k}(X,\lambda^k\Omega_{X/O_K}),$$

where $\lambda^k$ denotes the $k$th derived exterior power. (See Appendix A for a discussion of derived exterior powers).

## 2.3. Maps between cohomology groups

Let $M = M_{j,r}$ be the motive $h^j(X_{\mathbb{C}},\mathbb{Q}(r))$. Let $M^* = H_B^{2d-2-j}(X_{\mathbb{C}},\mathbb{Z}(d-1-r))$, and $N = M^*(1) = H_B^{2d-2-j}(X_{\mathbb{C}},\mathbb{Z}(d-r))$. The classical period map $\beta_{j,0}$ maps $H_B(M_{j,0})_{\mathbb{C}} = H_B^j(X_{\mathbb{C}},\mathbb{C})$ to $H_{DR}(M_{j,o})_{\mathbb{C}} = H_{DR}^j(X_{\mathbb{C}},\mathbb{C})$. Let $\pi_{j,r}$ be the projection of $H_{DR}(M)_{\mathbb{C}}$ onto $(t_M)_{\mathbb{C}}$, and let $\alpha_{j,r} = \pi_{j,r} \circ (2\pi i)^r \beta_{j,0})$ from $H_B(M)_{\mathbb{C}}^+ = H_B^j(X,\mathbb{Z}(r))_{\mathbb{C}}^+$ to $(t_M)_{\mathbb{C}}$. We now define a new map $\gamma_M$ (which we call the enhanced period map) as follows: $H_{DR}(M)$ has a decreasing Hodge filtration $F_{q'}(M)$. Let $H^{q'} = F_{q'}/F_{q'+1}$. Let $h_{q'}$ be the dimension of $H_{q'}$. Then $H_{DR}(M)$ has the direct sum Hodge decomposition $\coprod H^{q'}$. We decompose $\alpha_M = \alpha_{j,r}$ into the direct sum of the maps $\alpha^{q'}(M)$, where $\alpha^{q'}$ is the map $\alpha_M$ followed by the projection onto $H^{q'}$. Let $\Gamma$ be the usual gamma-function. Recall that the weight $w(M)$ of $M$ is equal to $j-2r$, Now, let $\gamma^{q'}(M)$ be $\Gamma^*(-w(M)+q'))\alpha^{q'}(M)$ and let $\gamma_{j,r}$ be the isomorphism $\tilde{\gamma}_M = \coprod_{q'} \gamma^{q'}(M)$. Since $h_{q'} = h(p,q)$, where $p+q=j$ and $q' = q-r$, the determinant of $\gamma_{j,r}$ is equal to the determinant of $\alpha_{j,r}$ multiplied by $\prod_{q'} \Gamma^*(-w(M)+q')^{h_{q'}}$ which is equal to $\prod_p \Gamma^*(r-p)^{h(p,q)}$, where $p+q=j$ and the product is over all $p$ between 0 and $j$.

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

Consider the following diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (H_B(M)^+)_{\mathbb{C}} & \xrightarrow{i} & (H_B(M))_{\mathbb{C}} & \xrightarrow{p} & (H_B(M)^-)_{\mathbb{C}} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \tilde{\gamma}_M} & & & & \\ 0 & \longrightarrow & (F_0(M))_{\mathbb{C}} & \xrightarrow{j} & (H_{DR}(M))_{\mathbb{C}} & \xrightarrow{q} & (t_M)_{\mathbb{C}} & \longrightarrow & 0 \end{array}$$

Let $\gamma_M = q \circ \tilde{\gamma}_M \circ i$. Let $\beta_M = p \circ \tilde{\gamma}_M^{-1} \circ j$. Diagram-chasing immediately shows that $\tilde{\gamma}_M$ induces isomorphisms from Ker $\gamma_M$ to Ker $\beta_M$ and from Coker $\gamma_M$ to Coker $\beta_M$.

**Proposition 2.4.** *The exact sequence of complex vector spaces*

$$0 \to Ker\ \gamma_M \to (H_B(M)^+)_{\mathbb{C}} \to (t_M)_{\mathbb{C}} \to Coker\ \gamma_M \to 0 \qquad (2.4.1(M))$$

*is dual to the exact sequence*

$$0 \to Ker\ \beta_N \to (F_0(N))_{\mathbb{C}} \to (H_B^-(N))_{\mathbb{C}} \to Coker\ \beta_N \to 0. \qquad (2.4.2(N))$$

**Proof.** $F_0(N)_{\mathbb{C}}$ is the Serre dual of $(t_M)_{\mathbb{C}}$. $H_B(N)^-$ may be identified with $H_B(M^*)^+$, which is the Poincarè dual of $H_B(M)^+$. $H_B^{2d-2}(X,\mathbb{C}(d-1))$ may be canonically identified with $H_{DR}^{2d-2}(X,\mathbb{C})$. Poincarè duality is compatible with Serre duality, which implies the proposition. □

We from now on choose an arbitrary basis for Ker $\gamma_M$, the basis for Ker $\beta_M$ induced by the isomorphism between Ker $\gamma_M$ and Ker $\beta_M$, the basis for Coker $\gamma_N$ induced by the above duality and the basis for Coker $\beta_N$ induced by the isomorphism between Coker $\gamma_N$ and Coker $\beta_N$ We will use these integral structures on the various kernels and cokernels,

If $A$ is a finitely generated abelian group, let $A_{tor}$ denote the torsion subgroup of $A$ and $A_{tf}$ denote $A/A_{tor}$.

If $\phi : A \to B$ is a homomorphism of finitely generated abelian groups, let $\phi_{tf}$ be the induced homomorphism from $A_{tf}$ to $B_{tf}$ and let $\phi_{tor}$ be the induced homomorphism from $A_{tor}$ to $B_{tor}$.

**Lemma 2.5.** *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence of finitely generated abelian groups. There is a natural isomorphism from* $Ker\ g_{tf}/Im f_{tf}$ *to* $Coker\ g_{tor}$, *and the determinant of* $0 \to A_{tf} \to B_{tf} \to C_{tf} \to 0$ *is equal to the Euler characteristic* $|B_{tor}|/|A_{tor}||C_{tor}|$ *of* $0 \to A_{tor} \to B_{tor} \to C_{tor} \to 0$.

**Proof.** Exercise □

**Proposition 2.6.** $\chi(2.4.2(M)) = \chi(\tilde{\gamma_M})\chi(2.4.1(M))$

**Proof.** Let $\Lambda$ denote highest exterior power. Let $A_1^* = H_B(M)^+$, let $A_2^* = H_B(M)$, and let $A_3^* = H_B(M)^-$. Let $A_j$ be a generator of $\Lambda((A_j^*)_{tf})$. Let $B_1^* = F_0(M)$, $B_2^* = H_{DR}(M)$, and $B_3^* = t_M$. Let $B_j$ be a generator of $\Lambda((B_j^*)_{tf})$.

Let $A = A_2/A_1A_3$ and $B = B_2/B_1B_3$. Let $a_j = |(A_j^*)_{tor}|$ and $b_j = |(B_j^*)_{tor}|$. Let $\chi(A_{tor}) = a_1a_3/a_2$ and $\chi(B_{tor}) = b_1b_3/b_2$. $\chi_{tor}(2.4.1(M)) = b_3/a_1$, and $\chi_{tor}(2.4.2(M)) = a_3/b_1$.

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

We have $\chi_{tor}(2.4.1(M))/\chi_{tor}(2.4.2(M)) = b_1b_3/a_1a_3$, and

$$\begin{aligned}\frac{\chi(2.4.1(M))}{\chi(2.4.2(M))} &= \frac{det(2.4.1(M))}{det(2.4.2(M))}\frac{\chi_{tor}(2.4.2(M))}{\chi_{tor}(2.4.1(M))}\\ &= \frac{B_1B_3a_1a_3}{A_1A_3b_1b_3}\\ &= \frac{B_2\ det(B)}{A_2\ det(A)}\frac{\chi(A_{tor})a_2}{\chi(B_{tor})b_2}.\end{aligned}$$

Lemma 2.5 implies that this equals $B_2a_2/b_2A_2$, which is $\chi(\bar{\gamma}_M)^{-1}$. □

Beilinson defines Chern class maps from the algebraic K-theory groups to Deligne cohomology. Let $\gamma_{j,r} = \gamma_M$, where $M = h^j(X_{\mathbb{C}}, \mathbb{Q}(r))$. In our language, Beilinson's map becomes a map $c_{j,r}$ (for $j \le 2r-2$) from $H_W^{j+1}(X,\mathbb{Z}(r))_{\mathbb{C}}$ to Coker $\gamma_{j,r}$. Let $N$ be the motive $M^*(1) = h^{2d-2-j}(X_{\mathbb{C}}, \mathbb{Q}(d-r))$. Since Ker $\gamma_N$ may be identified with the dual of Coker $\gamma_{2d-2-j,d-r}$, we also have the dual $b_{j,r}$ (for $j \ge 2d-2r$) of Beilinson's Chern class map which maps Ker $\gamma_{2d-2-j,d-r}$ to $H_W^{2d-j}(X,\mathbb{Z}(d-r))_{\mathbb{C}}$. We have:

**Conjecture 2.7** (Beilinson). *If $j \le 2r-3$ the maps $c_{j,r}$ and $b_{j,r}$ are isomorphisms.*

Recall that by definition $H_W^{2r+1}(X,\mathbb{Z}(r))_{\mathbb{C}}$ is dual to $H_{et}^{2(d-r)}(X,\mathbb{Z}(d-r))_{\mathbb{C}}$ which is the same as $H_{Zar}^{2(d-r)}(X,\mathbb{Z}(d-r))_{\mathbb{C}}$, that is, codimension (d-r) cycles on $X$ modulo rational equivalence. Recall that $H_{et}^{2(d-r)}(X,\mathbb{Z}(d-r))^1$ denote cycles homologically equivalent to zero, and $H_{et}^{2(d-r)}(X,\mathbb{Z}(d-r))^2$ denote cycles modulo homological equivalence. Let $H_W^{2r+1}(X,\mathbb{Z}(r))_{\mathbb{C}}^1$ be the dual of $H_{et}^{2d-r}(X,\mathbb{Z}(d-r))_{\mathbb{C}}^1$ and $H_W^{2r+1}(X,\mathbb{Z}(r))_{\mathbb{C}}^2$ be the dual of $H_{et}^{2d-r}(X,\mathbb{Z}(d-r))_{\mathbb{C}}^2$.

**Conjecture 2.8** (Beilinson). *There is an exact sequence*

$$0 \to H_W^{2r-1}(X,\mathbb{Z}(r))_{\mathbb{C}} \stackrel{c_{2r-2,r}}{\to} Coker(\gamma_M) \to H_W^{2r+1}(X,\mathbb{Z}(r))_{\mathbb{C}}^2 \to 0$$

*with $M = H^{2r-2}(X,\mathbb{Z}(r))$.*

This is a slightly different but more natural variant of Beilinson's original conjecture, and it is implicitly used by Fontaine [8].

**Conjecture 2.9.** *There is an exact sequence*

$$0 \to H_W^{2r}(X,\mathbb{Z}(r))_{\mathbb{C}}^2 \to Ker(\gamma_N) \to H_W^{2r+2}(X,\mathbb{Z}(r))_{\mathbb{C}} \to 0.$$

This is the dual of Conjecture 2.8, with $M$ replaced by

$$N = M^*(1) = H^{2d-2r}(X,\mathbb{Z}(d-r)).$$

**Conjecture 2.10** (Beilinson). *The Arakelov intersection pairing induces an isomorphism from $(H_W^{2r+1}(X,\mathbb{Z}(r))^1)_{\mathbb{C}}$ to $(H_W^{2r}(X,\mathbb{Z}(r))^1)_{\mathbb{C}}$.*

This is the nondegeneracy of the Arakelov intersection pairing restricted to finite cycles homologous to zero, where it is independent of metrics.

## 3. The statement of the conjecture

We would like to first explain the relationship between Weil-étale motivic cohomology groups and the groups which occur in Fontaine's Deligne–Beilinson conjecture [8, 9]. We look at the motive $M = h^j(X_{\mathbb{C}},\mathbb{Q}(r))$, Recall that $N = M^*(1)$ is $h^{2d-2-j}(X_{\mathbb{C}},\mathbb{Q}(d-r))$.

Fontaine starts with a projective nonsingular algebraic variety $X_0$ over Spec $\mathbb{Q}$. He chooses a regular model $X$ for $X_0$ projective and flat over Spec $\mathbb{Z}$. He conjectures that the following six-term sequence is always exact:

$$0 \to H^0_f(M)_{\mathbb{C}} \to Ker(\gamma_M) \to H^1_c(M)_{\mathbb{C}} \to H^1_f(M)_{\mathbb{C}} \to Coker(\gamma_M) \to H^2_c(M)_{\mathbb{C}} \to 0.$$

If $j \leq 2r-2$, Fontaine's $H^1_f(M)$ is our $H^{j+1}_W(X,\mathbb{Z}(r))_{\mathbb{Q}}$; we are using motivic cohomology instead of algebraic K-theory, but these two groups agree after tensoring with $\mathbb{Q}$. (Actually, Fontaine's group is the image of $K(X)$ in $K(X_0)$, but we conjecture that the natural map is always injective.)

If $j = 2r-1$, Fontaine's $H^1_f(M)$ is the group of codimension $r$ cycles on $X_0$ homologically equivalent to zero, tensored with $\mathbb{Q}$,

If $j \geq 2r$, $H^1_f(M) = 0$.

Fontaine's $H^0_f(M)$ is zero unless $j = 2r$, in which case it equals the group of codimension $r$ cycles on $X_0$ modulo homological equivalence, tensored with $\mathbb{Q}$.

Fontaine's $H^i_c(M)$ is the $\mathbb{Q}$-dual of $H^{2-i}_f(N)$ for $i = 1,2$.

For each $j$ and $r$ with $j \leq min(2d-1,2r-3)$, we will define a sequence of integral structures $A(j,r)$. For each $j$ and $r$ with $j \geq max(0,2r+1)$, we will define a sequence of integral structures $A'(j,r)$. $A(j,r)$ is given by:

$$(j \leq 2r-3) \qquad c_{j,r} : H^{j+1}_W(X,\mathbb{Z}(r))_{\mathbb{C}} \to Coker(\gamma_{j,r})$$

while $A'(j,r)$ is given by:

$$(j \geq 2r+1) \qquad b_{j,r} : Ker(\gamma_{j,r}) \to H^{j+2}_W(X,\mathbb{Z}(r))_{\mathbb{C}}.$$

If $1 < r < d$, we define an integral structure $C(r)$ given by: $C(r) =$

$$0 \to H^{2r-1}_W(X,\mathbb{Z}(r))_{\mathbb{C}} \overset{c_{2r-2,r}}{\to} Coker(\gamma_{2r-2,r}) \to H^{2r+1}_W(X,\mathbb{Z}(r))_{\mathbb{C}} \overset{e_r}{\to}$$
$$\to H^{2r}_W(X,\mathbb{Z}(r))_{\mathbb{C}} \to Ker(\gamma_{2r,r}) \overset{b_{2r,r}}{\to} H^{2r+2}_W(X.\mathbb{Z}(r))_{\mathbb{C}} \to 0.$$

Here, $e_r$ is induced by the Arakelov intersection pairing. We give these vector spaces the standard integral structures previously defined in §2.2.

Conjectures 2.7, 2.8, 2.9 and 2.10 imply that these sequences are exact.

We give degrees to the terms of these complexes by requiring that $Ker(\gamma_M)$ has even degree and $Coker(\gamma_M)$ has odd degree. (These sequences are all truncations of modified versions of Fontaine's six-term sequence in [8], and this convention makes the degrees agree)

Finally, we define exact sequences $B(j,r)_{\mathbb{C}}$ given for all $j$ and $r$ by

$$0 \to Ker(\gamma_M) \to (H^j_B(X_{\mathbb{C}},\mathbb{Z}(r))^+)_{\mathbb{C}} \overset{\gamma_M}{\to} (t_{j,r})_{\mathbb{C}} \to Coker(\gamma_M) \to 0.$$

We put $Ker(\gamma_M)$ in degree zero.

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

The integral structures on the cohomology groups here are induced by the standard integral structures defined in §2.2. Let

$$\chi_{A,C}(X,r) = \chi(C(r)) \prod_{j=0}^{min(2d-1,2r-3)} (\chi(A(j,r))^{(-1)^j} \prod_{j=max(0,2r+1)}^{2d-1} \chi(A'(j,r))^{(-1)^j},$$

$$\chi_B(X,r) = \prod_{j=0}^{\infty} \chi(B(j,r))^{(-1)^j},$$

and let

$$\chi(X,r) = \frac{\chi_{A,C}(X,r)}{\chi_B(X,r)}.$$

**Conjecture 3.1.** *Give all groups in the above exact sequences their standard integral structures. Then*

$$\zeta^*(X,r) = \chi(X,r)$$

*up to sign and powers of 2.*

*(Note that $B(j,r)$ is torsion for $j \geq 2d-1$ and zero for $j$ large).*

If $j \neq 2r-1$, each of the terms Ker $(\gamma_M)$ and Coker $(\gamma_M)$ occurs exactly twice in the conjecture with degrees of opposite parity, so the conjecture is independent of the choice of integral structure. If $j = 2r-1$, Ker $(\gamma_M)$ and Coker $(\gamma_M)$ are both zero.

**Proposition 3.2.** a) *If $0 \leq j \leq 2r-3$, then $2(d-r)+1 \leq 2d-2-j \leq 2d-2$ and the integral structures $A'(2d-2-j,d-r)_{tf}$ and $A(j,r)_{tf}$ are dual. Hence, $det(A(j,r)) = det(A'(2d-2-j,d-r))$.*

b) *The integral structures $C(r)_{tf}$ and $C(d-r)_{tf}$ are dual. Hence, $det(C(r)) = det(C(d-r))$.*

**Proposition 3.3.** $\chi_{A,C}(X,r) = \chi_{A,C}(X,d-r)$.

**Proof.** $\chi(C(r)) = det(C(r))/|(C(r)_{tor}|$. $C(r)_{tf}$ is dual to $C(d-r)_{tf}$, so $det(C(r)) = det(C(d-r))$. $\chi(A(j,r) = det(A(j,r))/|A(j,r)_{tor}|$. $\chi(A'(j,r)) = det(A'(j,r))/|(A'(j,r)_{tor}|$. $A(j,r)_{tf}$ is dual to $A'(2d-2-j,d-r)_{tf}$ so $det(A(j,r)) = det(A'(2d-2-j,d-r))$ and $det(A'(j,r) = det(A(2d-2-j,d-r))$.

On the other hand,

$$|C(r)_{tor}| \prod_{j=0}^{2r-3} |A(j,r)_{tor}|^{(-1)^j} \prod_{j=2r+1}^{2d-1} |A'(j,r)_{tor}|^{(-1)^j} = \prod_{j=1}^{2d+1} |H^j(X,\mathbb{Z}(r))_{tor}|^{(-1)^j}$$

which is equal by duality to

$$\prod_{j=1}^{2d+1} |H^j(X,\mathbb{Z}(d-r))_{tor}|^{(-1)^j}$$

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

which is equal to

$$|C(d-r)_{tor}| \prod_{j=0}^{2d-2r-3} |A(j,d-r)_{tor}|^{(-1)^j} \prod_{j=2d-2r+1}^{2d-1} |A'(j,d-r)_{tor}|^{(-1)^j}. \qquad \square$$

## 4. The Euler characteristic of the period map

Let $K$ be the integral closure of $\mathbb{Q}$ in the function field of $X$. Then we can view $X$ as a scheme over $S = \text{Spec } O_K$, and $X_{\mathbb{C}}$ is canonically isomorphic to $\prod_\sigma X \times_S \text{Spec } \mathbb{C}$, where the product is taken over all embeddings $\sigma$ of $K$ in $\mathbb{C}$. Let $\kappa(v)$ be the residue field of the closed point $v$, and let $X_v = X \times_S \text{Spec } \kappa(v)$

Recall that $A$ is the positive rational number which appears in the conjectured functional equation $A^{s/2}\Gamma(X,s)\zeta(X,s) = \pm A^{(d-s)/2}\Gamma(X,d-s)\zeta(X,d-s)$ for $\zeta(X,s)$. Let $\omega_{X/S}$ be the relative canonical class of $X$ over $S$.

**Definition 4.1.** Let $A'_v = (\Delta_X.\Delta_X)_v = (-1)^d c_{d_{X_v}^X}(\Omega_{X_v/O_v}) \in CH_0(X_{\kappa(v)})$. Let $A' = (\Delta_X.\Delta_X)_S = \prod_v A'_v$.

(This definition is taken from [12]. We will not use it directly in what follows. What we need is stated in Theorems 4.2 and 4.3.)

Here, the product is taken over all closed points $v$ of $S$ such that $X$ is not smooth over $S$ at $v$. Note that the Chern class $c_{d_{X_v}^X}(\Omega)$ is equal to 1 if X is smooth over $S$ at $v$.

**Theorem 4.2.** *(Bloch–Kato–Saito). If $d \le 3, A' = A$. If strong resolution of singularities holds for schemes of finite type over Spec $\mathbb{Z}$, then $A' = A$ in general.*

**Proof.** This is the main result [12, Theorem 6.2.3] of [12]. $\square$

Let $O_v = O_K$ localized at $v$. Let $X_v = X \times_{O_K} O_v$.

Let $\lambda^k$ denote the $k$-th derived exterior power.

**Theorem 4.3.** *Consider the cone $C_{m,v}$ of the map from $R\Gamma(X_v, \lambda^m \Omega_{X_v/O_v})$ to $R\Gamma(X_v, RHom(\lambda^{d-1-m}\Omega_{X_v/O_v}, \omega_{X_v/O_v}))$ induced by Serre duality; its Euler characteristic is equal to $(A')_v^{(-1)^{m+1}}$.*

**Proof.** This is [18, Corollary 4.9]. $\square$

**Theorem 4.4.** *The Euler characteristic of the classical period map $\alpha$ from de Rham to singular cohomology of $X_{\mathbb{C}}$ with respect to the canonical integral structures is $(A')^{d/2}(2\pi i)^{\chi(X_{\mathbb{C}})(d-1)/2}$.*

This theorem will follow from the following propositions.

**Proposition 4.5.** *Let $H^j$ be $H^j_{DR}(X_{\mathbb{C}}, \mathbb{C}) = \prod_\sigma H^j_{DR}(X_\sigma.\mathbb{C})$ with its canonical integral structure. Let $G^j$ be $H^j_{DR}(X_{\mathbb{C}}, \mathbb{C})$ with the integral structure given by $\prod_\sigma \prod_{k=0}^{j} Ext^{2d-2-j+k}_{X_\sigma}(\lambda^{d-1-k}\Omega_{X/O_K}, \omega_{X/O_K})$. Then the Euler characteristic $\chi_I$ of the identity map from $H^*$ to $G^*$ with respect to the given integral structures is $(A')^d$.*

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

**Proof.** Let $\Omega = \Omega_{X_{\mathbb{C}}/\mathbb{C}}$ and $\Omega_K = \Omega_{X/O_K}$, By a spectral sequence argument, $H^{j-k}(X_{\mathbb{C}},\Lambda^k\Omega)$ is canonically isomorphic to $Ext^{j-k}_{X_{\mathbb{C}}}(\Lambda^{d-1-k}\Omega,\omega_{\mathbb{C}})$. So $A_{j,k} = H^{j-k}(X, \lambda^k\Omega_K)$ and $B_{j,k} = Ext^{j-k}_X(\lambda^{d-1-k}\Omega_K,\omega)$ give two different integral structures on $H^{j-k}(X_{\mathbb{C}},\Lambda^k\Omega)$. By Serre duality, $A_{j,k}$ is dual to $B_{2d-2-j,d-1-k}$. Let $(x_{i.j})$ be a basis for $\coprod_k A_{j.k}$, and let $(y_{i,j})$ be a basis for $\coprod_k B_{j,k}$.

Let $d_{j,k}$ be the determinant of the identity map of $H^{j-k}(X_{\mathbb{C}},\Lambda^k\Omega)$ with respect to these two integral structures. (This is independent of the choice of basis for the integral structures, up to sign.) Then Theorem 4.3 asserts that

$$\prod_j (d_{j,k})^{(-1)^{j-k}}\chi((A_{j,k})_{tor})^{-1}\chi((B_{j,k})_{tor}) = (A')$$

if $k$ is even and is equal to $(A')^{-1}$ if $k$ is odd. By Serre duality, $\chi((A_{j,k})_{tor}) = (\chi((B_{j,k})_{tor})^{-1}$.

We conclude easily from this that $\prod_k\prod_j(d_{j,k})^{(-1)^j}(\chi(\lambda^k(\Omega))_{tor}))^{-2} = (A')^d$, and the product on the left is the Euler characteristic of the identity map. □

Let $Q_{DR}$ denote the de Rham cup product and $Q_B$ denote the cup product in singular cohomology. Let $\alpha^j : H^j_{DR}(X_{\mathbb{C}},\mathbb{C}) \to H^j_B(X_{\mathbb{C}},\mathbb{C})$ be the classical period map, We know that $Q_{DR}$ and $Q_B$ are compatible, that is, $Q_B(\alpha^j(a),\alpha^k(b)) = \alpha^{j+k}(Q_{DR}(a,b))$.

**Proposition 4.6.** *Let $[v_{i,j}]$ be a basis of $H^j_{DR}(X_{\mathbb{C}},\mathbb{C})$. Let $[u_{i,j}]$ be a basis of $H^j_B(X_{\mathbb{C}},\mathbb{C})$ coming from a basis of $H^j_B(X_{\mathbb{C}},\mathbb{Z})$ modulo torsion. Let $E^j$ be the matrix of $\alpha^j$ with respect to the bases $[v_{i,j}]$ and $[u_{i,j}]$. Then*

$$(\prod_j (detE^j)^{(-1)^j})^2 = \prod_j det(Q_{DR}(v_{i,j},v_{k,2d-2-j}))^{(-1)^j}(2\pi i)^{\chi(X_{\mathbb{C}})(d-1)}.$$

**Proof.** Fix $j$. On the top-dimensional cohomology group $H^{2d-2}_{DR}(X_{\mathbb{C}},\mathbb{C})$, $\alpha(1_{DR}) = (2\pi i)^{d-1}1_B$. Therefore, $\alpha(Q_{DR}(v_{i,j},v_{k,2d-2-j})) = Q_B(\alpha(v_{i,j}),\alpha(v_{k,2d-2-j}))$ implies

$$\begin{aligned}(2\pi i)^{(d-1)B_j}det(Q_{DR}(v_{i,j},v_{k,2d-2-j})) &= detQ_B(\alpha(v_{i,j}),\alpha(v_{k,2d-2-j})) = \\ &= det(E^j)det(E^{2d-2-j})det(Q_B(u_{i,j},u_{k,2d-2-j})) = \pm det(E^j)det(E^{2d-2-j}).\end{aligned}$$

Raising to the $(-1)^j$ power and taking products over $j$, we obtain the proposition. □

**Proposition 4.7.** *Let $d_j = \prod_k d_{j,k}$ be the determinant of the identity map with respect to the two given integral structures on $H^j_{DR}(X_{\mathbb{C}},\mathbb{C})$. Then*

a) $d_j = det(Q_{DR}(x_{i,j},x_{k,2d-2-j}))$.

b) $\prod d_j^{(-1)^j} = \prod det((Q_{DR}(x_{i,j},x_{k,2d-2-j})^{(-1)^j})$.

**Proof.** By the compatibility of cup product and Serre duality, $det(Q_{DR}(x_{i,j},y_{k,2d-2-j}))$ is equal to 1. Then 4.7 a follows from the definition of $d_j$, and of course implies 4.7 b. □

**Proof of Theorem 4.4.** Recall that by definition and Serre duality $\chi(\alpha) = \prod_j(detE^j)^{(-1)^j}\chi(H^*_{DR}(X)_{tor})^{-1}$

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

and $\chi_I = \prod d_j^{(-1)^j} \chi(H^*_{DR}(X)_{tor})^{-2}$. Substituting $x$ for $v$ in Proposition 4.6 and using Proposition 4.5, we obtain

$$(A')^d(\chi(H^*_{DR}(X)_{tor}))^2 = \prod_j det(Q_{DR}([x_{i,j}],[x_{k,2d-2-j}])^{(-1)^j}.$$

Proposition 4.6 implies

$$(A')^d(\chi(H^*_{DR}(X)_{tor})^2 = (2\pi i)^{-\chi(X_{\mathbb{C}})(d-1)}(\prod det(\alpha^j)^{(-1)^j})^{-2}$$

which implies

$$(2\pi i)^{\chi(X_{\mathbb{C}})(d-1)/2}(A')^{d/2} = \det(\alpha^*)\chi((H^*_{DR}(X))_{tor})^{(-1)}.$$

By Poincarè duality, $\chi(H^*_B(X_{\mathbb{C}},\mathbb{Z})_{tor}) = 1$, so

$$\det(\alpha^*)\chi((H^*_{DR}(X))_{tor})^{(-1)}\chi((H^*_B(X_{\mathbb{C}},\mathbb{Z})_{tor}) = (2\pi i)^{\chi(X_{\mathbb{C}})(d-1)/2}(A')^{d/2}$$

$$\chi(\alpha) = (2\pi i)^{\chi(X_{\mathbb{C}})(d-1)/2}(A')d/2,$$

which is Theorem 4.4. □

**Corollary 4.8.** *The Euler characteristic $\chi_r$ of the classical period map $\alpha_r$ from $H^*_{DR}(X_{\mathbb{C}},\mathbb{C}(r))$ to $H^*_B(X_{\mathbb{C}},\mathbb{C}(r))$ is $(A')^{d/2}(2\pi i)^{\chi(X_{\mathbb{C}})((d-1)/2-r)}$.*

This follows from the definition of twisting by $r$.

Recall that $\Gamma_{r,j} = \prod_{p+q=j}\Gamma^*(r-p)^{h(p,q)}$. Let $\Gamma_r = \prod \Gamma_{r,j}^{(-1)^{j+1}}$.

**Corollary 4.9.** *The Euler characteristic $\chi(\gamma_r)$ of the enhanced period map $\gamma_r$ is*

$$\Gamma_r(A')^{d/2}(2\pi i)^{\chi(X_{\mathbb{C}})((d-1)/2-r)}.$$

**Proof.** By the remarks at the beginning of §2.3, $\chi(\gamma_r)$ is equal to $\chi(\alpha_r)$ multiplied by $\prod_{j=0}^{2d-2}\Gamma_{r,j}^{(-1)^{j+!}} = \Gamma_r$. □

## 5. Serre's functional equation and Γ-function identities

Let $X_0$ be a smooth projective algebraic variety of dimension $d-1$ over the number field $K$. Let $j$ be a nonnegative integer, and let $L^j(X_0,s)$ be the L-function attached by Serre in [19] to the $j$-th cohomology group of $X_0$,

Let $\sigma$ be an embedding of $K$ into $\mathbb{C}$, and let $v$ be the place of $K$ induced by $\sigma$. Let $K_v$ be the completion of $K$ at $v$. Let $X_v = X_0 \times_K \mathbb{C}$, where $\sigma$ maps $K$ into $\mathbb{C}$, and let $\Omega = \Omega_{X_v/\mathbb{C}}$. Recall that Hodge theory gives us a decomposition $H^j_{DR}(X_v) = \coprod H_v^{p.q}$, where the sum is taken over pairs $(p.q)$ such that $p+q=j$ and $H_v^{p.q} = H^q(X_v,\Omega^p)$. Let $c$ be the automorphism of $X_v$ induced by complex conjugation acting on $\mathbb{C}/K_v$. Then if $j$ is even and equal to $2n$, $c$ acts as an involution on $H_v^{n,n}$. Let $h_v(p,q)$ be the dimension of $H_v^{p,q}$.

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

Then $H_v^{n,n} = H_v^{(n,+)} \oplus H_v^{(n,-)}$, where

$$H_v^{(n,+)} = \{x \in H_v^{n,n}, c(x) = (-1)^n x\}$$
$$H_v^{(n,-)} = \{x \in H_v^{n,n}, c(x) = (-1)^{(n+1)} x\}$$

Let $h_v(n,+) = \dim H_v^{(n,+)}$ and $h_v(n,-) = \dim H_v^{(n,-)}$.

Let $B_v^j$ be the rank of $H^j(X_v,\mathbb{Z})$, and let $(B_v^j)^+$ be the rank of the subgroup of $H^j(X_v,\mathbb{Z})$ left fixed by $c$. Let $(B_v^j)^- = B_v^j - (B_v^j)^+$. Note that if $j = 2n$, $(B_v^j)^+$ is equal to $\Sigma h_v(p,q) + h_v(n,+)$ if $n$ is even and is equal to $\Sigma h_v(p,q) + h_v(n,-)$ if $n$ is odd, where the sum is taken over all pairs $(p,q)$ where $p < q$ and $p+q = j$. Let $(B_v^{j,r})^-$ be $(B_v^j)^-$ if $r$ is even and $(B_v^j)^+$ if $r$ is odd.

Let $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$.

Serre [19] gives the functional equation $\phi^j(s) = \pm\phi(j+1-s)$, where $\phi(s) = L^j(s)A_j^{s/2}\Gamma^j(s)$, $A_j$ is a certain positive integer, and $\Gamma^j(s)$ is described as follows:

$\Gamma^j(s) = \prod_v \Gamma_v^j(s)$, where $\Gamma_v^j(s) = \prod_{p+q=j}(\Gamma_{\mathbb{C}}(s - \inf(p,q))^{h_v(p,q)}$ if $v$ is a complex place of $K$,

$\Gamma_v^j(s) = \Gamma_{\mathbb{R}}(s-n)^{h_v(n,+)}\Gamma_{\mathbb{R}}(s-n+1)^{h_v(n,-)}\prod_{p<j-p}\Gamma_{\mathbb{C}}(s-p)^{h_v(p,j-p)}$, if $v$ is a real place of $K$.

We observe that it is an easy computation that $\Gamma_v^j(s) = \Gamma_v^{2d-2-j}(s+d-j-1)$ so that with our earlier observation that at least in the smooth case $L^j(s) = L^{2d-2-j}(s+d-j-1)$ we obtain that Serre's functional equation is equivalent to the functional equation $\phi^j(s) = \pm\phi^{2d-2-j}(d-s)$.

**Theorem 5.1.** *Let $v$ be a real place of $K$.*

*If $j$ is even, $(\Gamma_v^j)^*(r)/(\Gamma_v^{2d-2-j})^*(d-r)$ is equal up to sign and powers of 2 to $\prod_p(\Gamma^*(r-p))^{h_v(p,q)}\pi^{-B_v^j(r-j/2)+(B_v^{j,r})^-}$. (The products run over $0 \le p \le B_v^j$).*

*If $j$ is odd, $(\Gamma_v^j)^*(r)/(\Gamma_v^{2d-2-j})^*(d-r)$ is equal up to sign and powers of 2 to $\prod_p \Gamma^*(r-p)^{h_v(p,q)}\pi^{-B_v^j(r-(j+1)/2)}$*

**Proof.** We consider the case when $j$ is even. (The case when $j$ is odd is similar but simpler.) Let $j = 2n$. Fix $v$, and let $q = j-p$. First look at terms where $p \neq q$. Let $p' = d-1-p$ and $q' = d-1-q$, so $p'+q' = 2d-2-j$. We have

$$\frac{\prod_{p<q}\Gamma_{\mathbb{C}}^*(r-p)^{h_v(p,q)}}{\prod_{p'<q'}\Gamma_{\mathbb{C}}^*(d-r-p')^{h_v(p',q')}} = \frac{\prod_{p<q}\Gamma_{\mathbb{C}}^*(r-p)^{h_v(p,q)}}{\prod_{p>q}\Gamma_{\mathbb{C}}^*(1-r+p)^{h_v(p,q)}}$$

since $h_v(p,q) = h_v(p',q')$ by Serre duality. By definition of $\Gamma_{\mathbb{C}}$, this is equal to

$$\frac{\prod_{p<q}\Gamma^*(r-p)^{h_v(p,q)}}{\prod_{p>q}\Gamma^*(1-r+p)^{h_v(p,q)}}$$

multiplied by

$$(2\pi)^{-(\Sigma_{p<q}(h_v(p,q)(r-p))-\Sigma_{p>q}(h_v(p,q)(1-r+p))))}.$$

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

This product is then equal to

$$\pm \prod_{p\neq n} \Gamma^*(r-p)^{h_v(p,q)}(2\pi)^{-(\Sigma_{p<q}(h_v(p,q)((r-p)-(1-r+j-p))))}$$

because of the relation $\Gamma^*(r) = \pm\Gamma^*(1-r)^{-1}$ for integral $r$ which follows immediately from the functional equation for the Gamma function. We then obtain:

$$\pm \prod_{p\neq n} \Gamma^*(r-p)^{h_v(p,q)}(2\pi)^{-(B_v^j-h_v(n,n))(r-(j+1)/2)} \tag{5.1}$$

We now look at the terms involving $n$ with $v$ still fixed. We first observe that the functional equation for the gamma function implies that
$\Gamma^*(a/2)\Gamma^*((2-a)/2)$ equals $\pm\pi$ if $a$ is an odd integer and equals $\pm 1$ if $a$ is an even integer.

We compute:

$$\Gamma^*_{\mathbb{R}}((r-n))^{h_v(n,+)}\Gamma^*_{\mathbb{R}}((r-n+1))^{h_v(n,-)}$$

multiplied by

$$\Gamma^*_{\mathbb{R}}((d-r-(d-1-n))^{-h_v(n,+)}\Gamma^*_{\mathbb{R}}(d+1-r-(d-1-n))^{-h_v(n,-)}$$

which we rewrite as

$$(\Gamma^*((r-n)/2)(\Gamma^*(1-r+n)/2)^{-1})^{h_v(n,+)} \tag{5.2}$$

multiplied by

$$\Gamma^*((r-n+1)/2)(\Gamma^*((n+2-r)/2)^{-1})^{h_v(n,-)} \tag{5.3}$$

multiplied by

$$\pi^{-(h_v(n,+)(2r-2n-1)/2+h_v(n,-)(2r-2n-1)/2)}. \tag{5.4}$$

First, assume that $r-n$ is odd. By the functional equation,

$$\Gamma^*((1-r+n)/2)^{-1} = \pm\Gamma^*(r-n+1)/2 \tag{5.5}$$

$$\Gamma^*((n+2-r)/2)^{-1} = \pm\pi^{-1}\Gamma^*((r-n)/2). \tag{5.6}$$

Now, recall the duplication formula for the gamma function:

$$\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2)/\sqrt{\pi}. \tag{5.7}$$

Then equation (5.2) becomes (using equations (5.5) and (5.7))

$$(\pm 2^{n-r+1}\Gamma^*(r-n)\sqrt{\pi})^{h_v(n,+)}$$

and equation (5.3) becomes

$$(\pm 2^{n-r}\Gamma^*(r-n)\sqrt{\pi})^{-h_v(n,-)}$$

while equation (5.4) is

$$\pi^{-h_v(n,n)(r-(j+1)/2)}. \tag{5.8}$$

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

So, up to sign and powers of 2, our product has become

$$\Gamma^*(r-n)^{h_v(n,n)}\pi^{(h_v(n,n)(r-(j+1)/2)+h_v(n,+)/2-h_v(n,-)/2} \tag{5.9}$$

Multiplying equation (5.1) by equation (5.9), we get

$$\prod_p \Gamma^*(r-p)^{h_v(p,q)}\pi^{-(B_v^j(r-(j+1/2)))+h_v(n,+)/2-h_v(n,-)/2)} \tag{5.10}$$

which equals

$$\prod_p \Gamma^*(r-p)^{h_v(p,q)}\pi^{-(B_v^j(r-j/2)+(B_v^j)^+} \tag{5.11}$$

if $n$ is even (so $r$ odd) and equals

$$\prod_p \Gamma^*(r-p)^{h_v(p,q)}\pi^{-B_v^j(r-j/2)+(B_v^j)^-} \tag{5.12}$$

if $n$ is odd (so $r$ even) since $(B_j - h_v(n,n)/2) + h_v(n,+)$ is equal to $(B_v^j)^+$ if $n$ is even and $(B_v^j)^-$ if $n$ is odd.

The proof for $r-n$ even is identical, except for switching $h_v(n,+)$ and $h_v(n,-)$. □

**Theorem 5.2.** *Let $v$ be a complex place of $K$. Then $(\Gamma_v^j)^*(r)/(\Gamma_v^{2d-2-j})^*(d-r)$ is equal up to sign and powers of 2 to $(\prod_p \Gamma^*(r-p))^{2h(p,q)}\pi^{-B_v^j(2r-(j+1))}$.*

**Proof.** Let $q=j-p$, $p'=d-1-p$ and $q'=d-1-q$. Note that $0\le p,q,p',q'\le d-1$

We write $(\Gamma^j)_v^*(r)/(\Gamma^{2d-2-j})_v^*(d-r) =$

$$\frac{\prod_p \Gamma_{\mathbb{C}}^*(r-\inf(p,q))^{h_v(p,q)}}{\prod_{p'} \Gamma_{\mathbb{C}}(d-r-\inf(p',q'))^{h_v(p',q')}}$$

which is equal to

$$\frac{\prod_p \Gamma^*(r-\inf(p,q))^{(h_v(p,q)}}{\prod_{p'} \Gamma^*(d-r-\inf(p',q'))^{h_v(p',q')}} \tag{5.13}$$

multiplied by

$$(2\pi)^{-\Sigma_p h_v(p.q)(r-\inf(p,q))-\Sigma_{p'}h_v(p',q')(d-r-\inf(p',q'))}. \tag{5.14}$$

Since $h_v(p',q') = h_v(p,q)$, the functional equation for the gamma function transforms equation (5.13) into (up to sign)

$$\prod_p \Gamma^*(r-\inf(p,q))^{h_v(p,q)}\prod_p \Gamma^*(r-sup(p,q))^{h_v(p,q)}, \tag{5.15}$$

which in turn equals

$$(\prod_p \Gamma^*(r-p)^{h_v(p,q)})^2.$$

Now, equation (5.14) easily transforms into $(\pi^{2r-j-1})^{-\Sigma_p h_v(p,q)}$, which becomes $\pi^{-(B_v^j(2r-(j+1))}$. □

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

## 6. Compatibility of the conjecture with the functional equation

Starting with Serre's conjectured functional equation for cohomological L-functions described in the previous section, Bloch, Kato and T. Saito conclude that the following functional equation holds for the zeta-function of $X$;

**Conjecture 6.1.** *Let $\phi(X,s) = \zeta(X,s)A^{s/2}\Gamma(X,s)$. Then $\phi(X,s) = \pm\phi(X,d-s)$*

Here, the constant $A = A(X)$ is obtained by taking the alternating product of the constants $A_j$ which occur in the conjectured functional equation for Serre's L-function $L_j$ and modifying it by terms coming from degenerate fibers. It is still a positive rational number.

The generic fiber $X_0$ of $X$ is a projective algebraic variety smooth over a number field $K = H^0(X_0, O_{X_0})$. Let $\Gamma(X,s) = \prod_j \prod_\sigma \Gamma(X^j_{v(\sigma)}, s)^{(-1)^j}$, where $\Gamma^j_v$ was defined in the previous section. If we rewrite our conjecture as $\zeta^*(X,r) = \chi(X,r)$, then what we want to show is that $\zeta^*(X,r)/\zeta^*(X,d-r) = \chi(X,r)/\chi(X,d-r)$, up to sign and powers of 2, where the left-hand side is computed by the functional equation.

**Proposition 6.2.** *Conjecture 6.1 implies*

$$\frac{\zeta^*(X,r)}{\zeta^*(X,d-r)} = A^{d/2-r}\prod_j \Big(\prod_{p+q=j} \Gamma^*(r-p)^{h(p,q)}\Big)^{(-1)^j} \pi^{-\chi(X_{\mathbb{C}})(r-(d-1)/2)+\chi^-(X_{\mathbb{C}},\mathbb{Z}(r))}.$$

**Proof.** This is an immediate consequence of Theorems 5.1 and 5.2, remembering that $B_j = B_{2d-2-j}$. □

We now wish to compute $\chi(X,r)/\chi(X,d-r)$ and show that it agrees with the expression in Proposition 6.2. We first recall that $\chi(X,r) = \chi_{A,C}(X,r)\chi_B(X,r)$ and that $\chi_{A,C}(X,r)/\chi_{A,C}(X,d-r) = 1$, by Proposition 3.3.

We now have to look at $\chi_B(X,r)/\chi_B(X,d-r)$.

**Lemma 6.3.** *Let $\chi_{i,k}$ be the Euler characteristic of the identity map from the complex vector space $H^i(X_{\mathbb{C}}, \Lambda^k\Omega_{X_{\mathbb{C}}})$ with the integral structure $H^i(X, \lambda^k\Omega_X)$ to the same vector space with the integral structure $\underline{RHom}(H^{d-1-i}(X, \lambda^{d-1-k}\Omega_X), \omega)$. Then $\prod_i \chi_{i,k}^{(-1)^i} = (A')^{(-1)^k}$.*

**Proof.** This follows immediately from Theorem 4.3. □

**Lemma 6.4.** *Let $\theta_{j,r}$ be the Euler characteristic of the identity map from the complex vector space $t_M = \coprod_{0\le k<r} H^{j-k}(X_{\mathbb{C}}, \Lambda^k\Omega_{X_{\mathbb{C}}})$ with the integral structure $\coprod_{0\le k<r} H^{j-k}(X, \lambda^k\Omega_X)$ to the same vector space with the integral structure given by*

$$\underline{RHom}(H^{d-1-j+k}(X, \lambda^{d-1-k}\Omega_X), \omega).$$

*Then $\prod_j \theta_{j,r}^{(-1)^j} = (A')^r$.*

**Proof.** This follows immediately from Lemma 6.3. □

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

**Lemma 6.5.** *Let $\eta_j$ be the Euler characteristic of the identity map from $H^j_B(X_{\mathbb{C}},\mathbb{C})^+$ (if r is odd) or $H^j_B(X_{\mathbb{C}},\mathbb{C})^-$ (if r is even) with the integral structure $H^j_B(X,\mathbb{Z}(r-1))^-$ to the same vector space with the integral structure $H^j_B(X,\mathbb{Z}(r))^+$ or $H^j_B(X,\mathbb{Z}(r))^-$. Then $\prod \eta_j^{(-1)^j} = (2\pi i)^{\chi^-(X_{\mathbb{C}},\mathbb{Z}(r))}$*

**Proof.** This follows immediately from the definitions. □

Let $\gamma(j,r)$ denote the integral structure

$$0 \to Ker\ \gamma_* \to H^j_B(X,\mathbb{C}(r))^+ \to t(j,r)_{\mathbb{C}} \to Coker\ \gamma_* \to 0.$$

Let $\beta(j,r)$ denote the integral structure

$$0 \to Ker\ \beta_* \to F_0(j,r)_{\mathbb{C}} \to H^j_B(X,\mathbb{C}(r))^- \to Coker\ \beta_* \to 0.$$

**Proposition 6.6.** a) *The integral structure $\beta(j,r)$ is dual to the integral structure $\delta(2d-2-j,d-r)$ given as follows:*

$$0 \to Ker\ \gamma_* \to (H^{2d-2-j}_B(X,\mathbb{Z}(d-1-r))_{\mathbb{C}})^- \to t(2d-2-j,d-r)_{\mathbb{C}} \to Coker\ \gamma_* \to 0,$$

*where the singular cohomology group has its standard structure and t(2d-2-j, d-r) has the integral structure dual to the standard one on $F_0(j,r)$.*

b) *det* $(\beta(j,r)) = det(\delta(2d-2-j,d-r))$.

**Proof.** Part a) follows immediately from the compatibility of Serre and Poincarè duality, and then b) follows immediately. □

Let

$$\chi(\gamma(r)) = \prod_{j=0}^{2d-2} \chi(\gamma(j,r))^{(-1)^j}, \quad \chi(\beta(r)) = \prod_{j=0}^{2d-2} \chi(\beta(j,r))^{(-1)^j},$$

and

$$\chi(\epsilon(d-r)) = \prod_{j=0}^{2d-2} \chi(\epsilon(j,d-r))^{(-1)^j}.$$

**Corollary 6.7.** $\chi(\beta(r)) = \chi(\delta(d-r))$

**Proof.** This follows from the preceding proposition and a consideration of torsion, exactly as in Proposition 3.3. □

Note that $\gamma(2d-2-j,d-r)$ is the integral structure $\delta(2d-2-j,d-r)$ except that the singular cohomology group now has the integral structure given by $H^{2d-2-j}_B(X,\mathbb{Z}(d-r)_{\mathbb{C}})^+$ and the tangent space has the standard integral structure rather than the one coming from duality.

**Proposition 6.8.** $\chi(\gamma(d-r)) = (A')^{d-r}(2\pi i)^{-\chi^-(X_{\mathbb{C}},\mathbb{Z}(r))}\chi(\delta(d-r))$

**Proof.** This follows immediately from Lemmas 6.4 and 6.5 and the previous remark. Note also that $\chi^-(X_{\mathbb{C}},\mathbb{Z}(d-1-r)) = \chi^-(X_{\mathbb{C}},\mathbb{Z}(r))$ because Poincarè duality is compatible with complex conjugation. □

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

**Proposition 6.9.** *One has*

$$\frac{\chi(\gamma(r))}{\chi(\beta(r))} = \Gamma_r(A')^{d/2}(2\pi i)^{\chi(X_{\mathbb{C}})((d-1)/2-r)}.$$

**Proof.** This follows from Proposition 2.6 and Corollary 4.9. □

**Proposition 6.10.** *One has*

$$\frac{\chi(\gamma(d-r))}{\chi(\gamma(r))} = \Gamma_r(A')^{d/2-r}(2\pi i)^{\chi(X_{\mathbb{C}})((d-1)/2-r)+\chi^-(X_{\mathbb{C}},\mathbb{Z}(r))}.$$

**Proof.** This follows immediately from Propositions 6.8 and 6.9 and Corollary 6.7. □

**Theorem 6.11.** *One has*

$$\frac{\chi_B(X,d-r)}{\chi_B(X,r)} = \Gamma_r(A')^{d/2-r}(2\pi i)^{\chi(X_{\mathbb{C}})(r-(d-1)/2)+\chi^-(X_{\mathbb{C}},\mathbb{Z}(r))}.$$

**Proof.** Observe that $\chi_B(X,d-r)/\chi_B(X,r) = \chi(\gamma(d-r))/\chi(\gamma(r))$. □

Proposition 6.2 and Theorem 6.11 immediately imply the compatibility of our conjecture with the functional equation if we replace $A$ by $A'$.

**Theorem 6.12.** *If $d \leq 2$, Conjecture 3.1 is compatible with the functional equation.*

**Proof.** This follows from Theorem 6.11 and the main theorem of [12]. □

## 7. The case of number rings

Let F be a number field with ring of integers $O_F$, class number $h$, number of roots of unity $w$ and discriminant $d_F$. Let $X = \text{Spec } O_F$. We will explain how our conjecture for $X$ and $r$ reduces to standard theorems if $r = 0$ or $r = 1$ and well-known conjectures if $r < 0$ or $r > 1$.

We begin with $r = 0$.

We know that $H^j_{et}(X,\mathbb{Z}(1)) = 0$ if $j < 1$, $H^1_{et}(X,\mathbb{Z}(1)) = O^*_F$, $H^2_{et}(X,\mathbb{Z}(1)) = Pic(X)$, $H^3(X,\mathbb{Z}(1)) = 0$ (up to 2-torsion) and $H^0_{et}(X,\mathbb{Z}(0)) = \mathbb{Z}$.

It immediately follows from the definitions that $H^0_W(X,\mathbb{Z}(0)) = \mathbb{Z}$, $H^1_W(X,\mathbb{Z}(0)) = 0$, and $H^3(X,\mathbb{Z}(0)) = \mu_F^\vee$, the dual of the roots of unity in $F$.

We also have the exact sequence $0 \to Pic(X)^\vee \to H^2_W(X,\mathbb{Z}(0)) \to Hom(O^*_F,\mathbb{Z}) \to 0$.

We also have $t_{j,0} = 0$ for all $j$. It follows that $A(j,0)$ is always equal to 0.

We also see that $A'(j,0) = 0$ unless $j = 1$, when $A(1,0)$ reduces to $(\mu_F)^\vee$ in degree 2, so $\chi(A'(1,0)) = w$.

$C(0)$ becomes $0 \to \mathbb{C} \to \mathbb{C}^{r_1+r_2} \to Hom(O^*_F,\mathbb{C}) \to 0$, with the integral structure on the last term being $H^2_W(X,\mathbb{Z}(2))$ and the second map being the dual of the classical regulator. So $\chi(C(0)) = hR$, and $\chi(X,0) = hR/w$. As is well known, $\zeta^*(X,0) = -hR/w$.

We now consider the case when $r = 1$.

$A(j,1)$ is easily seen to be zero for $j < -1$.

The complex part of $\mathbb{C}(1)$ is given by $0 \to O^*_F \otimes \mathbb{C} \to \mathbb{C}^{r_1+r_2} \to \mathbb{C} \to 0$, with the last two terms getting the standard bases and the first term a basis coming from $O^*_F$, so $det(\mathbb{C}(1)$

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

is the classical regulator $R$. The Euler characteristic of $\mathbb{C}(1)_{tor}$ is

$$|H^1_W(X,\mathbb{Z}(1))_{tor}|/|H^2(X,\mathbb{Z}(1))_{tor}| = w/h.$$

It follows that the Euler characteristic of $\mathbb{C}(1)$ is $hR/w$.

Since $t_{j,1} = 0$ for $j \geq 0$ and $H^j_W(X,\mathbb{Z}(1)) = 0$ for $j \geq 5$, $A'(j,1) = 0$ for $j \geq 3$.

Finally, $B(j.1) = 0$ if $j \neq 0$, and $B(0,1)$ is given by

$$0 \to \mathbb{C}^{r_2} \to O_F \otimes \mathbb{C} \to \mathbb{C}^{r_1+r_2} \to 0. \tag{7.1}$$

Note that we have the usual map $\theta$ mapping $O_F \otimes \mathbb{C}$ to $\mathbb{C}^{r_1+2r_2}$ by sending $x \otimes 1$ to the collection of $\sigma(x)$ as $\sigma$ runs through the embeddings of $F$ in $\mathbb{C}$. The map from $\mathbb{C}^{r_2}$ to $O_F \otimes \mathbb{C}$ is given by the natural inclusion of $\mathbb{C}^{r_2}$ in $\mathbb{C}^{r_1+2r_2}$ multiplied by $2\pi i$, followed by the inverse of $\theta$. Since the determinant of $\theta$ with respect to the usual bases is $\sqrt{d_F}$, we see that the determinant of equation (7.1) is equal to $(2\pi i)^{r_2}/\sqrt{d_F}$. Hence, the Euler characteristic $\chi(X,1)$ is equal to $hR(2\pi i)^{r_2}/w\sqrt{d_F}$ which is equal to the usual formula $hR(2\pi)^{r_2}2^{r_1}/w\sqrt{|d_F|}$ for $\zeta^*(X,1)$ up to a power of 2.

Now, let $r < 0$.

The only nonzero groups $H^k_W(X,\mathbb{Z}(r))$ occur when $k = 2$ or $k = 3$, so $A(j,r)$ is equal to zero for all $j$ in the appropriate range, $C(r) = 0$, and $A'(j,r) = 0$ unless $j = 0$ or $j = 1$.

$B(j,r) = 0$ unless $j = 0$. $B(0,r)$ reduces to the isomorphism $Ker(b_{0,r}) \to H^2_W(X,\mathbb{Z}(r))^+$ so we give $Ker(b_{0,r})$ the integral structure induced from $H^2_W(X,\mathbb{Z}(r))^+$, and $\chi(B(0,r)) = 1$.

$A'(0,r)_{\mathbb{C}}$ is given by $Ker(b_{0,r}) \to H^2_W(X,\mathbb{Z}(r))_{\mathbb{C}}$. $H^2_W(X,\mathbb{Z}(r))_{\mathbb{C}}$ is dual to $H^1_W(X,\mathbb{Z}(1-r))_{\mathbb{C}}$ which is canonically isomorphic to $K_{1-2r}(O_F)_{\mathbb{C}}$, and $Ker(b_{0,r})$ is canonically dual to $Coker(c_{0,1-r})$, where $c_{0,1-r}$ is the Beilinson regulator map. We conclude that $det(A'(0,r))$ is dual to the determinant $D_{1-r}$ of the Beilinson regulator map with respect to the natural bases coming from singular cohomology and K-theory.

Then $\chi(X,r)$ is equal to $D_{1-r}|H^2_W(X,\mathbb{Z}(r))_{tor}|/|H^3_W(X,\mathbb{Z}(r))|$, which is equal to $D_{1-r}|H^2_W(X,\mathbb{Z}(1-r))|/|H^1_W(X,\mathbb{Z}(1-r))_{tor}|$ which in turn is equal to

$$\frac{D_{1-r}|K_{-2r}(O_F)|}{|K_{1-2r}(O_F)_{tor}|},$$

up to 2-torsion. This is essentially what was conjectured in [16] to be $\zeta^*(X,r)$.

Finally, let $r > 1$. By definition, since $j$ has to be between $2r+1$ and $2d-1$, there is no contribution from $A'(j,r)$

$A(j,r)_{\mathbb{C}}$ is the complex $H^{j+1}_W(X,\mathbb{Z}(r))_{\mathbb{C}} \to Coker(\gamma_{j,r})$, which is only nonzero when $j = 0$, in which case the map is the Beilinson regulator $c_{0,r}$, Since $j$ has to be either 0 or 1, the torsion Euler characteristic is $|H^2(X,\mathbb{Z}(r))|/|H^1(X,\mathbb{Z}(r))_{tor}$ which up to 2-torsion is $K_{2r-2}(O_F)|/|K_{2r-1}(O_F)_{tor}|$. So letting $R_r$ be the determinant of the Beilinson regulator map, the Euler characteristic

$$\chi(A,r) = \frac{|K_{2r-2}(O_F)}{|K_{2r-1}(O_F)_{tor}|}R_r,$$

up to 2-torsion.

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

Now, $B(j,r)_{\mathbb{C}}$ obviously is zero if $j \neq 0$. Let $s(r) = r_2$ If $r$ is even and $s(r) = r_1 + r_2$ if $r$ is odd. Then by the same arguments as in the case when $r = 1$ we have $det(B(0,r)_{\mathbb{C}}) = (2\pi i)^{rs(r)}\sqrt{d_F}$.

The integral structure on $t_{j,r}$ is given by $\coprod_{0\leq k<r} H^{j-k}(X,\lambda^k(\Omega))$ In the appendix, we compute that $\lambda^k(\Omega)$ is equal to $\Omega[1-k]$. So $t_{j,r} = \coprod_{k<r} H^{j-k}(X,\Omega[1-k])$. Since $\Omega$ only has cohomology in dimension 0, we see that $t_{j,r} = 0$ unless $j = 1$, and the order of $H^0(X,\Omega)$ is $d_F$ so $\chi_{tor}(t_{j,r}) = d_F^{-r}$ and $\chi(B(j,r)) = (2\pi i)^{rs(r)} d_F^{(r-1/2)}$.

Putting everything together, we get that our conjecture says that up to 2-torsion $\zeta(X,r) = \chi(X,r)$, where

$$\chi(X,r) = \frac{|K_{2r-2}(O_F)|}{|K_{2r-1}(O_F)_{tor}|}(2\pi i)^{rs(r)} d_F^{r-1/2} R_r$$

since) $H^{k-1}(X,\lambda^k\Omega) = H^{k-1}(X,\Omega[1-k]) = d_F$, and $H^j(X,\lambda^k\Omega) = 0$ if $j \neq k-1$. This is compatible with our conjecture for $s = 1-r$ via the functional equation, which of course here is a well-known theorem.

## Appendix A. Derived Exterior Powers

Let $\mathcal{A}$ be an abelian category. Let $S\mathcal{A}$ denote the category of simplicial objects of $\mathcal{A}$ and $C\mathcal{A}$ denote the category of homological chain complexes of objects of $\mathcal{A}$ ending in degree zero. There are well-known functors $N : S\mathcal{A} \to C\mathcal{A}$ and $K : C\mathcal{A} \to S\mathcal{A}$ such that $NK$ is the identity and $KN$ is naturally equivalent to the identity functor. $N$ and $K$ also preserve homotopies. Let $\Lambda^k$ denote $k$-th exterior power. Let $X$ be a scheme and $\mathcal{A}$ be the category of coherent locally free sheaves on $X$. If $Q.$ is in $S\mathcal{A}$ with $Q_n$ a locally free sheaf on $X$ for all $n$, we define $\Lambda^k Q.$ to be $\Lambda^k(Q_n)$ in $S\mathcal{A}$.

**Proposition A.1.** *Let $X$ be a regular scheme projective over Spec $\mathbb{Z}$. Write $X$ as a closed subscheme of a projective space $P = (P^n)_{\mathbb{Z}}$ such that $I$ is the sheaf of ideals defining $X$. Then the complex of locally free sheaves $C_{X,P} = I/I^2 \to \Omega_{P/\mathbb{Z}}$ defines an element in the derived category of locally free sheaves on $X$ which is independent of the choice of embedding of $X$ into $P$.*

**Proof.** If we have two different embeddings of $X$ in $P_1$ and $P_2$, take the Segré embedding of $P_1 \times P_2$ in $P_3$ and compare successively the complexes defined by the embeddings into $P_1$ and $P_2$ with the product embedding into $P_3$. (For details, see [15].) □

**Definition A.2.** $\lambda^k\Omega_{X/\mathbb{Z}} = N\Lambda^k K C_{X,P}$.

We see easily that this is independent of the choice of embedding.

We begin by recalling the following fact [11, Exercise 5.16 (d))]:

**Lemma A.3.** *Let $(X,O_X)$ be a ringed space, and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of locally free sheaves of $O_X$-modules. Then there exists a finite filtration of $\Lambda^r F$:*

$$\Lambda^r F = G^0 \supseteq G^1 \supseteq \cdots \supseteq G^r \supseteq G^{r+1} = 0$$

*with quotients $G^p/G^{p+1} = \Lambda^p F' \otimes \Lambda^{r-p} F''$.*

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

**Theorem A.4.** *Let $(X,O_X)$ be a scheme such that every coherent sheaf on $X$ has a finite resolution by locally free sheaves. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of coherent sheaves on $X$. Let $r$ be a positive integer, Then there exist objects in the derived category $G^0, G^1, \ldots G^{r+1}$ and maps from $G^p$ to $G^{p+1}$ such that we have $G^0 = \lambda^r(F)$, $G^{r+1} = 0$, and exact triangles $G^p \to G^{p+1} \to \lambda^p(F') \otimes^L \lambda^{r-p}(F'') \to G^p[1]$.*

**Proof.** This is an easy corollary of the definition of derived exterior power and the previous lemma. □

**Theorem A.5.** *Let $A$ be a ring and $M$ an $A$-module.* a) *$\lambda^0 M$ is canonically homotopic to the complex consisting only of $A$ in degree* 0.

b) *$\lambda^1 M$ is canonically homotopic to the complex consisting only of $M$ in degree 0.*

c) *If $M$ has projective dimension $r$, then $\lambda^k M$ is represented by a complex of length $kr$.*

**Proof.** a) and b) are obvious, and c) follows immediately from a theorem of Dold and Puppe [3]. □

**Theorem A.6.** *Let $X$ =Spec $O_F$, and $\Omega = \Omega_{O_F/\mathbb{Z}}$. Then $\lambda^k(\Omega)$ is isomorphic to $\Omega[1-k]$*

**Proof.** Let $D$ be the inverse different of $O_F$ over $\mathbb{Z}$. We use induction on $k$, applying Theorem A4 to the exact sequence $0 \to O_F \to D \to \Omega \to 0$, Since $\lambda^k(O_F) = \lambda^k(D) = 0$ for $k > 1$, $\lambda^0\Omega = O_F$, and $\lambda^1\Omega = \Omega$, Theorem A4 reduces to the triangle $\lambda^k\Omega \to 0 \to \lambda^{k-1}\Omega \to \lambda^k\Omega[1]$, so to the equality $\lambda^k\Omega = \lambda^{k-1}\Omega[-1]$. □

## Appendix B. The relation between our conjecture and the second Flach–Morin conjecture

Both these conjectures involve Euler characteristics constructed from Weil-étale cohomology groups, singular cohomology groups, derived de Rham cohomology groups and maps (after tensoring with the real or complex numbers) between them given by Beilinson regulators, period isomorphisms and height pairings. In both papers, a naive Euler characteristic is constructed and then modified by using values of the gamma-function to give a formula which is compatible with a modified version of Serre's functional equation. We observe that it is not immediately apparent that the definition of Weil-étale cohomology groups in both papers is the same, but this is easily seen to follow from Lemma 3.7 and Proposition 3.4 of [FM1], taking into account Conjecture 3.2 of the same paper. Flach and Morin need to assume this finite generation hypothesis in order even to define Weil-étale cohomology, but this is harmless since the finite generation hypothesis is necessary in both papers in order to state the conjectures.

There seems to be little doubt that the unmodified Euler characteristics in both papers are the same, but this is less clear with regard to the modifications.

In our paper, the factorials modify the standard period map, so we get a modification of Fontaine's six-term sequence of cohomology groups and use this Euler characteristic instead. This depends on the map from $H_B^+(M)$ to $t_M$ and so does not lend itself to an explicit numerical formula. We only get a numerical result by computing the ratio between the formula for a cohomology group and the one for its dual. Flach and Morin

https://doi.org/10.1017/S1474748022000524 Published online by Cambridge University Press

ignore the map from $H_B^+(M)$ to $t_M$ and give a correction factor which only depends on Hodge numbers so does lead to a numerical formula. Both these modifications, amazingly, are compatible with the functional equation. In the cases where one knows more, that is, if $X$ is the spectrum of a number ring or if we are looking at a curve with $r=1$ (the case which is related to the conjecture of Birch and Swinnerton-Dyer), the two approaches agree. There are good, although wildly different, reasons for each of the two approaches, and the reader should decide what he or she thinks is more plausible.

**Acknowledgements.** We would like to thank Spencer Bloch, Matthias Flach, Thomas Geisser, Baptiste Morin, Niranjan Ramachandran and Takeshi Saito for many helpful conversations. We thank the referee for a very detailed report.

**Competing Interests.** None.

## References

[1] S. Bloch, 'Algebraic cycles and higher $K$-theory', *Adv. in Math.* **61**(3) (1986), 267–304.
[2] S. Bloch, 'de Rham cohomology and conductors of curves', *Duke Math. J.* **54**(2) (1987), 295–308.
[3] A. Dold and D. Puppe, 'Homologie nicht-additiver Funktoren. Anwendungen', *Ann. Inst. Fourier (Grenoble)* **11** (1961), 201–312.
[4] M. Flach, 'The equivariant Tamagawa number conjecture: a survey', in *Stark's Conjectures: Recent Work and New Directions,* Contemp. Math., vol. **358** (Amer. Math. Soc., Providence, RI, 2004), 79–125. With an appendix by C. Greither.
[5] M. Flach and B. Morin, 'Weil-étale cohomology and zeta-values of proper regular arithmetic schemes', *Doc. Math.* **23** (2018), 1425–1560.
[6] M. Flach and B. Morin, 'Compatibility of special value conjectures with the functional equation of zeta functions', *Doc. Math.* **26** (2021), 1633–1677.
[7] M. Flach and D. Siebel, 'Special values of the zeta function of an arithmetic surface', *Journal of the Institute of Mathematics of Jussieu* (2021), 1–49.
[8] J.-M. Fontaine, 'Valeurs spéciales des fonctions $L$des motifs', in *Séminaire Bourbaki,* **1991/92**, Astérisque, 206 (SMF, Paris, 1992), 205–249.
[9] J.-M. Fontaine and B. Perrin-Riou, 'Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions $L'$, in *Motives (Seattle, WA, 1991),* Proc. Sympos. Pure Math., vol. **55** (Amer. Math. Soc., Providence, RI, 1994), 599–706.
[10] T. Geisser, 'Arithmetic cohomology over finite fields and special values of $\zeta$-functions', *Duke Math. J.* **133**(1) (2006), 27–57.
[11] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, no. 52 (Springer-Verlag, New York-Heidelberg, 1977).
[12] K. Kato and T. Saito, 'On the conductor formula of Bloch', *Publ. Math. Inst. Hautes Études Sci.* **100** (2004), 5–151.
[13] M. Levine, 'Techniques of localization in the theory of algebraic cycles', *J. Algebraic Geom.* **10**(2) (2001), 299–363.
[14] S. Lichtenbaum and N. Ramachandran, 'Values of zeta-functions of arithmetic surfaces at $s=1'$, *Journal of the Institute of Mathematics of Jussieu* (2022), 1–42.
[15] S. Lichtenbaum and M. Schlessinger, 'The cotangent complex of a morphism', *Trans. Amer. Math. Soc.* **128** (1967), 41–70.

[16] S. Lichtenbaum, 'Values of zeta-functions, étale cohomology, and algebraic $K$-theory', in *Algebraic $K$-Theory, II: "Classical" Algebraic $K$-Theory and Connections with Arithmetic (Proc. Conf., (Battelle Memorial Inst., Seattle, WA, 1972)*, Lecture Notes in Math., vol. **342** (Springer-Verlag, New York, 1973), 489–501.

[17] S. Lichtenbaum, 'Special values of zeta functions of scheme', Preprint, 2017, arXiv:1704.00062.

[18] T. Saito, 'Parity in Bloch's conductor formula in even dimension', *J. Théor. Nombres Bordeaux* **16**(2) (2004),403–421.

[19] J.-P. Serre, 'Facteurs locaux des fonctions zêta des varietés algébriques (définitions et conjectures)', in *Séminaire Delange-Pisot-Poitou. 11e année: 1969/70. Théorie des nombres. no. 1* (Secrétariat Math, Paris, 1970).

[20] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, vol. **151** (Springer-Verlag, New York, 1994).

[21] J. H. Silverman, *The Arithmetic of Elliptic Curves*, second edn, Graduate Texts in Mathematics, vol. **106** (Springer, Dordrecht, 2009).

[22] J. Tate, 'On the conjectures of Birch and Swinnerton-Dyer and a geometric analog', in *Séminaire Bourbaki: Années 1965/66, exposé 9* (Société mathématique de France, Paris, 1966).

[23] S. Bloch and K. Kato, 'L-functions and Tamagawa numbers of motives', in *The Grothendieck Festschrift.*, vol. **I**, Vol. 86 of Progr. Math. (Birkhäuser, Boston, 1990), 333–400.

[24] M. Rapoport, N. Schappacher and P. Schneider, 'Beilinson's conjectures on special values of L-functions', in *Perspectives in Mathematics*, vol. **4** (Academic Press, Boston, 1988), xxiv+373. ISBN: 0-12-581120-9.