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HILBERT-KUNZ MULTIPLICITY OF THREE-DIMENSIONAL LOCAL RINGS

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Abstract. In this paper, we investigate the lower bound $s_{\mathrm{HK}}(p,d)$ of Hilbert-Kunz multiplicities for non-regular unmixed local rings of Krull dimension d containing a field of characteristic $p > 0$. Especially, we focus on three-dimensional local rings. In fact, as a main result, we will prove that $s_{\mathrm{HK}}(p,3) = 4/3$ and that a three-dimensional complete local ring of Hilbert-Kunz multiplicity $4/3$ is isomorphic to the non-degenerate quadric hypersurface $k[[X,Y,Z,W]]/(X^2+Y^2+Z^2+W^2)$ under mild conditions.

Furthermore, we pose a generalization of the main theorem to the case of $\dim A \geq 4$ as a conjecture, and show that it is also true in case $\dim A = 4$ using the similar method as in the proof of the main theorem.

Introduction

Let A be a commutative Noetherian ring containing an infinite field of characteristic $p > 0$ with unity. In [15], Kunz proved the following theorem, which gives a characterization of regular local rings of positive characteristic.

Kunz' Theorem. ([15]) *Let $(A,\mathfrak{m},k)$ be a local ring of characteristic $p > 0$. Then the following conditions are equivalent*:

(1) *A is a regular local ring.*

(2) *A is reduced and is flat over the subring $A^p = \{a^p : a \in A\}$. In other words, the Frobenius map $F : A \to A$ ($a \mapsto a^p$) is flat.*

(3) *$l_A(A/\mathfrak{m}^{[q]}) = q^d$ for any $q = p^e$, $e \geq 1$, where $\mathfrak{m}^{[q]} = (a^q : a \in \mathfrak{m})$ and $l_A(M)$ denotes the length of an A-module M.*

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Furthermore, in [16], Kunz observed that $l_A(A/\mathfrak{m}^{[q]})/q^d$ $(q = p^e)$ is a reasonable measure for the singularity of a local ring. Based on the idea of Kunz, Monsky [18] proved that there exists a constant $c = c(A)$ such that

$$l_A(A/\mathfrak{m}^{[q]}) = cq^d + O(q^{d-1})$$

and defined the notion of *Hilbert-Kunz multiplicity* by $e_{\mathrm{HK}}(A) = c$. In 1990's, Han and Monsky [10] have given an algorism to compute the Hilbert-Kunz multiplicity for any hypersurface of Briskorn-Fermat type

$$A = k[X_0, \dots, X_n]/(X_0^{d_0} + \cdots + X_n^{d_n}).$$

See e.g. [1], [2], [4], [24] about the other examples. Hochster and Huneke [11] have given a "Length Criterion for Tight Closure" in terms of Hilbert-Kunz multiplicity (see Theorem 1.8) and indicated the close relation between tight closure and Hilbert-Kunz multiplicity. In [22], the authors proved a theorem which gives a characterization of regular local rings in terms of Hilbert-Kunz multiplicity:

Theorem A. ([22, Theorem 1.5]) *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of positive characteristic. Then A is regular if and only if $e_{\mathrm{HK}}(A) = 1$.*

Many researchers have tried to improve this theorem. For example, Blickle and Enescu [3] recently proved the following theorem:

Theorem B. (Blickle-Enescu [3]) *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$. Then the following statements hold:*

(1) *If $e_{\mathrm{HK}}(A) < 1 + \frac{1}{d!}$, then A is Cohen-Macaulay and F-rational.*
(2) *If $e_{\mathrm{HK}}(A) < 1 + \frac{1}{p^d d!}$, then A is regular.*

So it is natural to consider the following problem:

Problem C. *Let $d \geq 2$ be any integer. Determine the lower bound $(s_{\mathrm{HK}}(p, d))$ of Hilbert-Kunz multiplicities for d-dimensional non-regular unmixed local rings of characteristic p. Also, characterize the local rings A for which $e_{\mathrm{HK}}(A) = s_{\mathrm{HK}}(p, d)$ holds.*

In case of one-dimensional local rings, it is easy to answer to this problem. In fact, $s_{\mathrm{HK}}(p, 1) = 2$; $e_{\mathrm{HK}}(A) = 2$ if and only if $e(A) = 2$. In case of two-dimensional Cohen-Macaulay local rings, the authors [23] have given a complete answer to this problem. Namely, we have $s_{\mathrm{HK}}(p, 2) = \frac{3}{2}$ by the theorem below.

THEOREM D. (see also Corollary 2.6) *Let* $(A, \mathfrak{m}, k)$ *be a two-dimensional Cohen-Macaulay local ring of positive characteristic. Put* $e = e(A)$, *the multiplicity of* A. *Then the following statements hold*:

(1) $e_{\mathrm{HK}}(A) \geq \frac{e+1}{2}$.

(2) *Suppose that* $k = \overline{k}$. *Then* $e_{\mathrm{HK}}(A) = \frac{e+1}{2}$ *holds if and only if the associated graded ring* $\mathrm{gr}_{\mathfrak{m}}(A)$ *is isomorphic to the Veronese subring* $k[X, Y]^{(e)}$.

In the following, let us explain the organization of this paper. In Section 1, we recall the notions of Hilbert-Kunz multiplicity and tight closure etc. and gather several fundamental properties of them. In particular, Goto-Nakamura's theorem (Theorem 1.9) is important because it plays a central role in the proof of the main result (Theorem 3.1).

In Section 2, we give a key result to estimate Hilbert-Kunz multiplicities for local rings of lower dimension. Indeed, Theorem 2.2 is a refinement of the argument in [23, Section 2]. Also, the point of our proof is to estimate $l_A(\mathfrak{m}^{[q]}/J^{[q]})$ (where J is a minimal reduction of $\mathfrak{m}$) using volumes in $\mathbb{R}^d$.

In Section 3, we prove the following theorem as the main result in this paper.

THEOREM 3.1. *Let* $(A, \mathfrak{m}, k)$ *be a three-dimensional unmixed local ring of characteristic* $p > 0$. *Then the following statements hold.*

(1) *If* A *is not regular, then* $e_{\mathrm{HK}}(A) \geq \frac{4}{3}$.

(2) *Suppose that* $k = \overline{k}$ *and* $\mathrm{char}\, k \neq 2$. *Then the following conditions are equivalent*:

  (a) $e_{\mathrm{HK}}(A) = \frac{4}{3}$.

  (b) $\widehat{A} \cong k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)$.

Also, we study lower bounds on $e_{\mathrm{HK}}(A)$ for local rings A having a given (small) multiplicity e. In particular, we will prove that any three-dimensional unmixed local ring A with $e_{\mathrm{HK}}(A) < 2$ is F-rational.

In Section 4, we consider a generalization of Theorem 3.1 and pose the following conjecture:

CONJECTURE 4.2. *Let* $d \geq 1$ *be an integer and* $p > 2$ *a prime number. Put*

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2).$$

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Let $(A, \mathfrak{m}, k)$ be a d-dimensional unmixed local ring with $k = \overline{\mathbb{F}_p}$. Then the following statements hold.

(1) *If A is not regular, then $e_{\mathrm{HK}}(A) \geq e_{\mathrm{HK}}(A_{p,d}) \geq 1 + \frac{c_d}{d!}$ (see 4.2 for the definition of c_d). In particular, $s_{\mathrm{HK}}(p, d) = e_{\mathrm{HK}}(A_{p,d})$.*

(2) *If $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(A_{p,d})$, then the $\mathfrak{m}$-adic completion $\widehat{A}$ of A is isomorphic to $A_{p,d}$ as local rings.*

Also, we prove that this is true in case of $\dim A = 4$. Namely we will prove the following theorem.

Theorem 4.3. *Let $(A, \mathfrak{m}, k)$ be a four-dimensional unmixed local ring of characteristic $p > 0$. Also, suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$. Then $e_{\mathrm{HK}}(A) \geq \frac{5}{4}$ if $e(A) \geq 3$. Also, the following statements hold.*

(1) *If A is not regular, then $e_{\mathrm{HK}}(A) \geq e_{\mathrm{HK}}(A_{p,4}) = \frac{29p^2+15}{24p^2+12}$.*

(2) *The following conditions are equivalent*:

  (a) *Equality holds in* (1).

  (b) $e_{\mathrm{HK}}(A) < \frac{5}{4}$.

  (c) *$\widehat{A}$ is isomorphic to $A_{p,4}$.*

§1. Preliminaries

Throughout this paper, let A be a commutative Noetherian ring with unity. Furthermore, we assume that A has a positive characteristic p, that is, it contains a prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, unless otherwise specified. For every positive integer e, let $q = p^e$. If I is an ideal of A, then $I^{[q]} = (a^q : a \in I)A$. Also, we fix the following notation: $l_A(M)$ (resp. $\mu_A(M)$) denotes the length (resp. the minimal number of generators) of M for any finitely generated A-module M.

First, we recall the notion of Hilbert-Kunz multiplicity (see [15], [16], [18]). Also, see [17] or [20] for usual multiplicity.

Definition 1.1. (multiplicity, Hilbert-Kunz multiplicity) Let $(A, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ with $\dim A = d$. Let I be an $\mathfrak{m}$-primary ideal of A, and let M be a finitely generated A-module. The *(Hilbert-Samuel) multiplicity* $e(I, M)$ of I with respect to M is defined by

$$e(I, M) = \lim_{n\to\infty} \frac{d!}{n^d}\, l_A(M/I^n M).$$

The *Hilbert-Kunz multiplicity* $e_{\mathrm{HK}}(I, M)$ of I with respect to M is defined by

$$e_{\mathrm{HK}}(I, M) = \lim_{q \to \infty} \frac{l_A(M/I^{[q]}M)}{q^d}.$$

By definition, we put $e(I) = e(I, A)$ (resp. $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(I, A)$) and $e(A) = e(\mathfrak{m})$ (resp. $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(\mathfrak{m})$).

We recall several basic results on Hilbert-Kunz multiplicity.

PROPOSITION 1.2. (Fundamental properties (cf. [13], [15], [16], [18], [22])) *Let $(A, \mathfrak{m}, k)$ be a local ring of positive characteristic. Let I, I' be $\mathfrak{m}$-primary ideals of A, and let M be a finitely generated A-module. Then the following statements hold.*

(1) *If $I \subseteq I'$, then $e_{\mathrm{HK}}(I) \geq e_{\mathrm{HK}}(I')$.*

(2) $e_{\mathrm{HK}}(A) \geq 1$.

(3) $\dim M < d$ *if and only if* $e_{\mathrm{HK}}(I, M) = 0$.

(4) *If $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated A-modules, then*

$$e_{\mathrm{HK}}(I, M) = e_{\mathrm{HK}}(I, L) + e_{\mathrm{HK}}(I, N).$$

(5) (Associative formula)

$$e_{\mathrm{HK}}(I, M) = \sum_{\mathfrak{p} \in \mathrm{Assh}(A)} e_{\mathrm{HK}}(I, A/\mathfrak{p}) \cdot l_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}),$$

where $\mathrm{Assh}(A)$ *denotes the set of prime ideals* $\mathfrak{p}$ *of* A *with* $\dim A/\mathfrak{p} = \dim A$.

(6) *If J is a parameter ideal of A, then $e_{\mathrm{HK}}(J) = e(J)$. In particular, if J is a minimal reduction of I (i.e., J is a parameter ideal which is contained in I and $I^{r+1} = JI^r$ for some integer $r \geq 0$), then $e_{\mathrm{HK}}(J) = e(I)$.*

(7) *If A is regular, then $e_{\mathrm{HK}}(I) = l_A(A/I)$.*

(8) (Localization) $e_{\mathrm{HK}}(A_{\mathfrak{p}}) \leq e_{\mathrm{HK}}(A)$ *holds for any prime ideal* $\mathfrak{p}$ *such that* $\dim A/\mathfrak{p} + \operatorname{height} \mathfrak{p} = \dim A$.

(9) *If $x \in I$ is A-regular, then $e_{\mathrm{HK}}(I) \leq e_{\mathrm{HK}}(I/xA)$.*

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(10) *If* $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ *is a flat local ring homomorphism such that* $B/\mathfrak{m}B$ *is a field, then* $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(IB)$.

Remark 1. Also, the similar result as above (except (6), (7)) holds for usual multiplicities.

Let $(A, \mathfrak{m}, k)$ be any local ring of positive dimension. *The associated graded ring* $\mathrm{gr}_{\mathfrak{m}}(A)$ of A with respect to $\mathfrak{m}$ is defined as follows:

$$\mathrm{gr}_{\mathfrak{m}}(A) := \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

Then $G = \mathrm{gr}_{\mathfrak{m}}(A)$ is a homogeneous k-algebra such that $\mathfrak{M} := G_+$ is the unique homogeneous maximal ideal of G. If $\operatorname{char} A = p > 0$ and $\dim A = d$, then $G_{\mathfrak{M}}$ is also a local ring of characteristic p with $\dim G_{\mathfrak{M}} = d$.

Proposition 1.3. ([22, Theorem (2.15)]) *Let* $(A, \mathfrak{m}, k)$ *be a local ring of positive characteristic. Let* $G = \mathrm{gr}_{\mathfrak{m}}(A)$ *the associated graded ring of* A *with respect* $\mathfrak{m}$ *as above. Then* $e_{\mathrm{HK}}(A) \leq e_{\mathrm{HK}}(G_{\mathfrak{M}}) \leq e(A)$.

Remark 2. We use the same notation as in the above proposition. Although $e(A) = e(G_{\mathfrak{M}})$ always holds, $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(G_{\mathfrak{M}})$ seldom holds.

Proposition 1.4. (cf. [13]) *Let* $(A, \mathfrak{m}, k)$ *be a local ring of positive characteristic with* $d = \dim A$. *Let* I *be an* $\mathfrak{m}$*-primary ideal of* A. *Then*

$$\frac{e(I)}{d!} \leq e_{\mathrm{HK}}(I) \leq e(I).$$

Also, if $d \geq 2$, *then the inequality in the left-hand side is strict*; *see* [9].

We say that a local ring A is *unmixed* if $\dim \widehat{A}/\mathfrak{p} = \dim \widehat{A}$ holds for any associated prime ideal $\mathfrak{p}$ of $\widehat{A}$. The following theorem is an analogy of Nagata's theorem ([20, (40.6)]), which is a starting point in this article.

Theorem 1.5. ([22, Theorem (1.5)]) *Let* $(A, \mathfrak{m}, k)$ *be an unmixed local ring of positive characteristic. Then* A *is regular if and only if* $e_{\mathrm{HK}}(A) = 1$.

It is not so easy to compute Hilbert-Kunz multiplicities in general. However, one has simple formulas for them in case of quotient singularities and in case of binomial hypersurfaces; see below or [4, Theorem 3.1].

THEOREM 1.6. (cf. [22, Theorem (2.7)]) *Let* $(A, \mathfrak{m}) \hookrightarrow (B, \mathfrak{n})$ *be a module-finite extension of local domains of positive characteristic. Then for every* $\mathfrak{m}$*-primary ideal* I *of* A*, we have*

$$e_{\mathrm{HK}}(I) = \frac{e_{\mathrm{HK}}(IB)}{[Q(B) : Q(A)]} \cdot [B/\mathfrak{n} : A/\mathfrak{m}],$$

where $Q(A)$ *denotes the fraction field of* A*.*

Now let us see some examples of Hilbert-Kunz multiplicities which are given by the above formula. First, we consider the Veronese subring A defined by

$$A = k[[X_1^{i_1} \cdots X_d^{i_d} : i_1, \dots, i_d \geq 0, \textstyle\sum i_j = r]].$$

Applying Theorem 1.6 to $A \hookrightarrow B = k[[X_1, \dots, X_d]]$, we get

$$e_{\mathrm{HK}}(A) = \frac{1}{r}\binom{d+r-1}{r-1}. \tag{1.1}$$

In particular, if $d = 2$, $r = e(A)$, then $e_{\mathrm{HK}}(A) = \frac{e(A)+1}{2}$.

Next, we consider the homogeneous coordinate rings of quadric hypersurfaces in $\mathbb{P}_k^3$. Let k be a field of characteristic $p > 2$, and let R be the homogeneous coordinate ring of the hyperquadric Q defined by $q = q(X, Y, Z, W)$. Put $\mathfrak{M} = R_+$, the unique homogeneous maximal ideal of R, and $A = R_{\mathfrak{M}} \otimes_k \overline{k}$. By suitable coordinate transformation, we may assume that $\widehat{A}$ is isomorphic to one of the following rings:

$$\begin{cases} k[[X, Y, Z, W]]/(X^2), & \text{if } \mathrm{rank}(q) = 1, \\ k[[X, Y, Z, W]]/(X^2 - YZ), & \text{if } \mathrm{rank}(q) = 2, \\ k[[X, Y, Z, W]]/(XY - ZW), & \text{if } \mathrm{rank}(q) = 3. \end{cases} \tag{1.2}$$

Then $e_{\mathrm{HK}}(A) = 2$, $\frac{3}{2}$, or $\frac{4}{3}$, respectively.

In order to state other important properties of Hilbert-Kunz multiplicity, the notion of tight closure is very important. See [11], [12], [13] for definition and the fundamental properties of tight closure. In particular, the notion of F-rational ring is essential in our argument.

DEFINITION 1.7. ([6], [11], [12]) Let $(A, \mathfrak{m}, k)$ be a local ring of positive characteristic. We say that A is *weakly* F*-regular* (resp. F*-rational*) if every ideal (resp. every parameter ideal) is tightly closed. Also, A is F*-regular* (resp. F*-rational*) if any local ring of A is weakly F-regular (resp. F-rational).

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Note that an F-rational local ring is normal and Cohen-Macaulay.

Hochster and Huneke have given the following criterion of tight closure in terms of Hilbert-Kunz multiplicity.

THEOREM 1.8. (Length Criterion for Tight Closure (cf. [11, Theorem 8.17])) *Let $I \subseteq J$ be $\mathfrak{m}$-primary ideals of a local ring $(A, \mathfrak{m}, k)$ of positive characteristic.*

(1) *If $I^* = J^*$, then $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(J)$.*

(2) *Suppose that A is excellent, reduced and equidimensional. Then the converse of (1) is also true.*

The following theorem plays an important role in studying Hilbert-Kunz multiplicities for non-Cohen-Macaulay local rings.

THEOREM 1.9. (Goto-Nakamura [8]) *Let $(A, \mathfrak{m}, k)$ be an equidimensional local ring which is a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$. Then*

(1) *If J is a parameter ideal of A, then $e(J) \geq l_A(A/J^*)$.*

(2) *Suppose that A is unmixed. If $e(J) = l_A(A/J^*)$ for some parameter ideal J, then A is F-rational (hence is Cohen-Macaulay).*

The next corollary is well-known in case of Cohen-Macaulay local rings (e.g. see [13]).

COROLLARY 1.10. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$. Suppose that $e(A) = 2$. Then $\widehat{A}$ is F-rational if and only if $e_{\mathrm{HK}}(A) < 2$. When this is the case, A is an F-rational hypersurface.*

Proof. Since any Cohen-Macaulay local ring of multiplicity 2 is a hypersurface, it suffices to prove the first statement.

We may assume that A is complete and k is infinite. We can take a minimal reduction J of $\mathfrak{m}$. First, suppose that $e_{\mathrm{HK}}(A) < 2$. Then we show that A is Cohen-Macaulay, F-rational. By Goto-Nakamura's theorem, we have $2 = e(J) \geq l_A(A/J^*)$. If equality does not hold, then $l_A(A/J^*) = 1$, that is, $J^* = \mathfrak{m}$. Then $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(J^*) = e_{\mathrm{HK}}(J) = e(J) = 2$ by Proposition 1.2. This is a contradiction. Hence $e(J) = l_A(A/J^*)$. By Goto-Nakamura's theorem again, we obtain that A is Cohen-Macaulay, F-rational.

Conversely, suppose that A is a complete F-rational local ring. Then since A is Cohen-Macaulay and $J^* = J \neq \mathfrak{m}$, we have $e_{\mathrm{HK}}(A) < e_{\mathrm{HK}}(J) = e(J) = 2$ by the Length Criterion for Tight Closure. ∎

The next question is open in general. However, we will show that it is true for $\dim A \leq 3$; see Section 3.

QUESTION 1.11. *If A is an unmixed local ring with $e_{\mathrm{HK}}(A) < 2$, then is it F-rational?*

§2. Estimate of Hilbert-Kunz multiplicities

In this section, we will prove the key result to find a lower bound on Hilbert-Kunz multiplicities. Actually, it is a refinement of the argument which appeared in [22, Section 5] or in [23, Section 2]. The point is to use the tight closure J^* instead of "a parameter ideal J itself". This enables us to investigate Hilbert-Kunz multiplicities of non-Cohen-Macaulay local rings. In Sections 3, 4, we will apply our method to unmixed local rings with $\dim A = 3$, 4.

Before stating our theorem, we introduce the following notation: Fix $d > 0$. For any positive real number s, we put

$$v_s := \mathrm{vol}\left\{(x_1, \ldots, x_d) \in [0,1]^d : \sum_{i=1}^{d} x_i \leq s\right\}, \quad v'_s := 1 - v_s,$$

where $\mathrm{vol}(W)$ denotes the volume of $W \subseteq \mathbb{R}^d$. Then it is easy to see the following fact.

FACT 2.1. *Let s be a positive real number. Using the same notation as above, we have*

(1) $v_s + v'_s = 1$.

(2) $v'_{d-s} = v_s$.

(3) $v_{d/2} = v'_{d/2} = \frac{1}{2}$.

(4) *If* $0 \leq s \leq 1$, *then* $v_s = \frac{s^d}{d!}$.

Using the above notaion, the key result in this paper can be written as follows:

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Theorem 2.2. *Let* $(A, \mathfrak{m}, k)$ *be an unmixed local ring of characteristic* $p > 0$. *Put* $d = \dim A \geq 1$. *Let* J *be a minimal reduction of* $\mathfrak{m}$, *and let* r *be an integer with* $r \geq \mu_A(\mathfrak{m}/J^*)$, *where* J^* *denotes the tight closure of* J. *Also, let* $s \geq 1$ *be a rational number. Then we have*

$$e_{\mathrm{HK}}(A) \geq e(A)\left\{v_s - r \cdot \frac{(s-1)^d}{d!}\right\}. \tag{2.1}$$

Remark 3. When $1 \leq s \leq 2$, the right-hand side in Equation (2.1) is equal to $e(A)(v_s - r \cdot v_{s-1})$.

Before proving the theorem, we need the following lemma. In what follows, for any positive real number α, we define $I^\alpha := I^n$, where n is the minimum integer which does not exceed α.

Lemma 2.3. *Let* $(A, \mathfrak{m}, k)$ *be an unmixed local ring of characteristic* $p > 0$ *with* $\dim A = d \geq 1$. *Let* J *be a parameter ideal of* A. *Using the same notation as above, we have*

$$\lim_{q\to\infty} \frac{l_A(A/J^{sq})}{q^d} = \frac{e(J)s^d}{d!}, \quad \lim_{q\to\infty} l_A\left(\frac{J^{sq} + J^{[q]}}{J^{[q]}}\right) = e(J) \cdot v'_s.$$

Proof. First, note that our assertion holds if A is regular and $J = \mathfrak{m}$. We may assume that A is complete. Let $x_1, \ldots, x_d$ be a system of parameters which generates J, and put $R := k[[x_1, \ldots, x_d]]$, $\mathfrak{n} = (x_1, \ldots, x_d)R$. Then R is a complete regular local ring and A is a finitely generated R-module with $A/\mathfrak{m} = R/\mathfrak{n}$. Since the assertion is clear in case of regular local rings, it suffices to show the following claim.

Claim. *Let* $\mathcal{I} = \{I_q\}_{q=p^e}$ *be a set of ideals of* A *which satisfies the following conditions:*

(1) *For each* $q = p^e$, $I_q = J_q A$ *holds for some ideal* $J_q \subseteq R$.

(2) *There exists a positive integer* t *such that* $\mathfrak{n}^{tq} \subseteq J_q$ *for all* $q = p^e$.

(3) $\lim_{q\to\infty} l_R(R/J_q)/q^d$ *exists.*

Then

$$\lim_{q\to\infty} \frac{l_A(A/I_q)}{q^d} = e(J) \cdot \lim_{q\to\infty} \frac{l_R(R/J_q)}{q^d}.$$

In fact, since A is unmixed, it is a torsion-free R-module of rank $e := e(J)$. Take a free R-module F of rank e such that $A_W \cong F_W$, where $W = R \setminus \{0\}$. Since F and A are both torsion-free, there exist the following short exact sequences of finitely generated R-modules:

$$0 \to F \to A \to C_1 \to 0, \quad 0 \to A \to F \to C_2 \to 0,$$

where $(C_1)_W = (C_2)_W = 0$. In particular, $\dim C_1 < d$ and $\dim C_2 < d$.

Applying the tensor product $- \otimes_R R/J_q$ to the above two exact sequences, respectively, we get

$$\begin{aligned} l_A(A/I_q) &\leq l_R(F/J_qF) + l_R(C_1/J_qC_1), \\ l_R(F/J_qF) &\leq l_A(A/I_q) + l_R(C_2/J_qC_2). \end{aligned}$$

In general, if $\dim_R C < d$, then

$$\frac{l_R(C/J_qC)}{q^d} \leq \frac{l_R(C/\mathfrak{n}^{tq}C)}{q^d} \to 0 \quad (q \to \infty).$$

Thus the required assertion easily follows from the above observation. ∎

Proof of Theorem 2.2. For simplicity, we put $L = J^*$ and $e = e(A)$. We will give an upper bound of $l_A(\mathfrak{m}^{[q]}/J^{[q]})$. First, we have the following inequality:

$$\begin{aligned} l_A(\mathfrak{m}^{[q]}/J^{[q]}) &\leq l_A\left(\frac{\mathfrak{m}^{[q]} + \mathfrak{m}^{sq}}{J^{[q]}}\right) \\ &= l_A\left(\frac{\mathfrak{m}^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + \mathfrak{m}^{sq}}\right) + l_A\left(\frac{L^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + J^{sq}}\right) \\ &\qquad + l_A\left(\frac{L^{[q]} + J^{sq}}{J^{[q]} + J^{sq}}\right) + l_A\left(\frac{J^{[q]} + J^{sq}}{J^{[q]}}\right) \\ &=: \ell_1 + \ell_2 + \ell_3 + \ell_4. \end{aligned}$$

Next, we see that $\ell_1 \leq r \cdot l_A(A/J^{(s-1)q}) + O(q^{d-1})$. By our assumption, we can write $\mathfrak{m} = L + Aa_1 + \cdots + Aa_r$. Since $\mathfrak{m}^{(s-1)q}a_i^q \subseteq \mathfrak{m}^{sq} \subseteq \mathfrak{m}^{sq} + L^{[q]}$, we have

$$\begin{aligned} \ell_1 = l_A\left(\frac{\mathfrak{m}^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + \mathfrak{m}^{sq}}\right) &\leq \sum_{i=1}^{r} l_A\left(\frac{Aa_i^q + L^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + \mathfrak{m}^{sq}}\right) \\ &= \sum_{i=1}^{r} l_A\Big(A/(L^{[q]} + \mathfrak{m}^{sq}) : a_i^q\Big) \\ &\leq r \cdot l_A(A/\mathfrak{m}^{(s-1)q}). \end{aligned}$$

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Since J is a minimal reduction of $\mathfrak{m}$, we have $l_A(\mathfrak{m}^{(s-1)q}/J^{(s-1)q}) = O(q^{d-1})$. Thus we have the required inequality. Similarly, we get

$$\ell_2 = l_A\left(\frac{L^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + J^{sq}}\right) \le l_A(\mathfrak{m}^{sq}/J^{sq}) = O(q^{d-1}).$$

Also, we have $l_A(L^{[q]}/J^{[q]}) = O(q^{d-1})$ by Length Criterion for Tight Closure. Hence $\ell_3 = O(q^{d-1})$ and thus

$$l_A(\mathfrak{m}^{[q]}/J^{[q]}) \le r \cdot l_A(A/J^{(s-1)q}) + l_A\left(\frac{J^{[q]} + J^{sq}}{J^{[q]}}\right) + O(q^{d-1}).$$

It follows from the above argument that

$$\begin{aligned} e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(\mathfrak{m}) &\le r \cdot \lim_{q\to\infty} \frac{l_A(A/J^{(s-1)q})}{q^d} + \lim_{q\to\infty} \frac{1}{q^d}\, l_A\left(\frac{J^{[q]} + J^{sq}}{J^{[q]}}\right) \\ &= r \cdot e \cdot \frac{(s-1)^d}{d!} + e \cdot v'_s. \end{aligned}$$

Since $e_{\mathrm{HK}}(J) = e(J) = e$, $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(\mathfrak{m})$ and $v'_s = 1 - v_s$, we get the required inequality. □

The following fact is known, which gives a lower bound on Hilbert-Kunz multiplicities for hypersurface local rings.

Fact 2.4. (cf. [1], [2], [22]) *Let $(A, \mathfrak{m}, k)$ be a hypersurface local ring of characteristic $p > 0$ with $d = \dim A \ge 1$. Then*

$$e_{\mathrm{HK}}(A) \ge \beta_{d+1} \cdot e(A),$$

where β_{d+1} is given by the following equivalent formulas:

(a) $\displaystyle \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\theta}{\theta}\right)^{d+1} d\theta\,;$

(b) $\displaystyle \frac{1}{2^d d!}\sum_{\ell=0}^{[\frac{d}{2}]}(-1)^\ell (d+1-2\ell)^d \binom{d+1}{\ell}\,;$

(c) $\displaystyle \mathrm{vol}\left\{\underline{x} \in [0,1]^d : \frac{d-1}{2} \le \sum x_i \le \frac{d+1}{2}\right\} = 1 - v_{\frac{d-1}{2}} - v'_{\frac{d+1}{2}}.$

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Table 1.

d	0	1	2	3	4	5	6
β_{d+1}	1	1	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{115}{192}$	$\frac{11}{20}$	$\frac{5633}{11520}$

Remark 4. The above inequality is *not* best possible in general. In case of $d \geq 4$, one cannot prove the formula in the above fact as a corollary of our theorem. See also Proposition 3.9 and Theorem 4.3.

The following lemma is an analogy of Sally's theorem: If A is a Cohen-Macaulay local ring, then $\mu_A(\mathfrak{m}/J) = \mu_A(\mathfrak{m}) - \dim A \leq e(A) - 1$.

Lemma 2.5. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of positive characteristic, and let J be a minimal reduction of $\mathfrak{m}$.*

(1) $\mu_A(\mathfrak{m}/J^*) \leq e(A) - 1$.

(2) *If A is not F-rational, then $\mu_A(\mathfrak{m}/J^*) \leq e(A) - 2$.*

Proof. We may assume that A is complete and thus is a homomorphic image of a Cohen-Macaulay local ring.

(1) By Goto-Nakamura's Theorem, we have that $\mu_A(\mathfrak{m}/J^*) \leq l_A(\mathfrak{m}/J^*) \leq e(J) - 1 = e - 1$.

(2) If A is not F-rational, then $l_A(A/J^*) \leq e(J) - 1 = e - 1$. Thus $\mu_A(\mathfrak{m}/J^*) \leq e - 2$, as required. ∎

Using Theorem 2.2 and Lemma 2.5, one can prove the following corollary, which has been already proved in [23] in the case of Cohen-Macaulay local rings.

Corollary 2.6. (cf. [23]) *Let $(A, \mathfrak{m}, k)$ be a two-dimensional unmixed local ring of characteristic $p > 0$. Put $e = e(A)$. Then*

$$e_{\mathrm{HK}}(A) \geq \frac{e+1}{2}. \tag{2.2}$$

Also, suppose $k = \overline{k}$. Then the equality holds if and only if $\mathrm{gr}_{\mathfrak{m}}(A)$ is isomorphic to the Veronese subring $k[X,Y]^{(e)} = k[X^e, X^{e-1}Y, \ldots, XY^{e-1}, Y^e]$.

Moreover, if A is not F-rational, then we have

$$e_{\mathrm{HK}}(A) \geq \frac{e^2}{2(e-1)}.$$

Example 2.7. (Fakhruddin-Trivedi [7, Corollary 3.19]) Let E be an elliptic curve over a field $k = \overline{k}$ of characteristic $p > 0$, and let $\mathcal{L}$ be a very ample line bundle on E of degree $e \geq 2$. Let R be the homogeneous coordinate ring (the section ring of $\mathcal{L}$) defined by

$$R = \bigoplus_{n \geq 0} H^0(E, \mathcal{L}^{\otimes n}).$$

Also, put $A = R_{\mathfrak{M}}$, where $\mathfrak{M}$ be the unique homogeneous maximal ideal of R. Then we have $e_{\mathrm{HK}}(A) = \frac{e^2}{2(e-1)}$.

§3. Lower bounds in the case of three-dimensional local rings

In this section, we prove the main theorem in this paper, which gives the lower bound of Hilbert-Kunz multiplicities for non-regular unmixed local rings of dimension 3.

Theorem 3.1. *Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$. Then*

(1) *If A is not regular, then $e_{\mathrm{HK}}(A) \geq \frac{4}{3}$.*
(2) *Suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$. Then the following conditions are equivalent:*
   (a) $e_{\mathrm{HK}}(A) = \frac{4}{3}$.
   (b) $\widehat{A} \cong k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)$.
   (c) $\mathrm{gr}_{\mathfrak{m}}(A) \cong k[X, Y, Z, W]/(X^2+Y^2+Z^2+W^2)$. *That is,* $\mathrm{gr}_{\mathfrak{m}}(A) \cong k[X, Y, Z, W]/(XY - ZW)$.

Proposition 3.2. *Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$. If $e_{\mathrm{HK}}(A) < 2$, then A is F-rational.*

From now on, we divide the proof of Theorem 3.1 and Proposition 3.2 into several steps. In the following, we assume the following condition.

(#): Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$, and $e(A) = e$, the multiplicity of A. Also, suppose that $\mathfrak{m}$ has a minimal reduction J.

Suppose that A is not regular under the assumption (#). Then $e = e(A)$ is an integer with $e \geq 2$. Thus the first assertion of Theorem 3.1 follows from the following lemma. Also, this implies that if $e_{\mathrm{HK}}(A) = \frac{4}{3}$ then $e(A) = 2$ without extra assumptions.

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LEMMA 3.3. *Under the assumption (#), we have*

(1) *If* $e \geq 5$, *then* $e_{\mathrm{HK}}(A) > 2$.
(2) *If* $e = 4$, *then* $e_{\mathrm{HK}}(A) \geq \frac{7}{4} > \frac{4}{3}$.
(3) *If* $e = 3$, *then* $e_{\mathrm{HK}}(A) \geq \frac{13}{8} > \frac{4}{3}$.
(4) *If* $e = 2$, *then* $e_{\mathrm{HK}}(A) \geq \frac{4}{3}$.

Remark 5. The lower bounds of $e_{\mathrm{HK}}(A)$ in Lemma 3.3 are not best possible.

Proof. We may assume that A is complete. By Lemma 2.5(1), we can apply Theorem 2.2 with $r = e-1$. Namely, if $1 \leq s \leq 2$, then

$$e_{\mathrm{HK}}(A) \geq e(v_s - (e-1)v_{s-1}) = e\left(\frac{s^3}{6} - (e+2)\frac{(s-1)^3}{6}\right). \tag{3.1}$$

Define the real-valued function $f_e(s)$ by the right-hand side of Eq. (3.1). Then one can easily calculate $\max_{1 \leq s \leq 2} f_e(s)$. In fact, if $e \geq 2$, then

$$\max_{1 \leq s \leq 2} f_e(s) = f\left(\frac{e+2+\sqrt{e+2}}{e+1}\right) = \frac{e}{6}\left(\frac{e+2+\sqrt{e+2}}{e+1}\right)^2.$$

But, in order to prove the lemma, it is enough to use the following values only:

s	$\frac{3}{2}$	$\frac{7}{4}$	2
$f_e(s)$	$\frac{e(25-e)}{48}$	$\frac{e(289-27e)}{384}$	$\frac{e(6-e)}{6}$

(1) We show that $e_{\mathrm{HK}}(A) > 2$ if $e \geq 5$. If $e \geq 13$, then by Proposition 1.4,

$$e_{\mathrm{HK}}(A) \geq \frac{e}{3!} \geq \frac{13}{6} > 2.$$

So we may assume that $5 \leq e \leq 12$. Applying Eq. (3.1) for $s = \frac{3}{2}$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(25-e)}{48} \geq \frac{5(25-5)}{48} = \frac{25}{12} > 2.$$

(2) Suppose that $e = 4$. Applying Eq. (3.1) for $s = \frac{3}{2}$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(25-e)}{48} = \frac{7}{4}.$$

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(3) Suppose that $e = 3$. Applying Eq. (3.1) for $s = \frac{7}{4}$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(289 - 27e)}{384} = \frac{13}{8}.$$

(4) Suppose that $e = 2$. Applying Eq. (3.1) for $s = 2$,

$$e_{\mathrm{HK}}(A) \geq \frac{e(6 - e)}{6} = \frac{4}{3},$$

as required. ∎

Before proving the second assertion of Theorem 3.1, we prove Proposition 3.2. For that purpose, we now focus non-F-rational local rings.

Now suppose that A is not F-rational. If $e = 2$, then $e_{\mathrm{HK}}(A) = 2$ by Lemma 1.10. On the other hand, if $e \geq 5$, then $e_{\mathrm{HK}}(A) > 2$ by Lemma 3.3. Thus in order to prove Proposition 3.2, it is enough to investigate the cases of $e = 3, 4$. Namely, Proposition 3.2 follows from the following lemma.

LEMMA 3.4. *Suppose that A is not F-rational under the assumption* (#). *Then*

(1) *If $e = 3$, then $e_{\mathrm{HK}}(A) \geq 2$.*
(2) *If $e = 4$, then $e_{\mathrm{HK}}(A) > 2$.*

Proof. By Lemma 2.5(2), we can apply Theorem 2.2 for $r = e - 2$. Thus if $1 \leq s \leq 2$, then

$$e_{\mathrm{HK}}(A) \geq e\left(\frac{s^3}{6} - (e+1)\frac{(s-1)^3}{6}\right). \tag{3.2}$$

(1) Suppose that $e = 3$. Applying Eq. (3.2) for $s = 2$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(7 - e)}{6} = 2.$$

(2) Suppose that $e = 4$. Applying Eq. (3.2) for $s = \frac{7}{4}$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(316 - 27e)}{384} = \frac{13}{6} > 2,$$

as required. ∎

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EXAMPLE 3.5. Let $R = k[T, xT, xyT, yT, x^{-1}yT, x^{-2}yT, \dots, x^{-n}yT]$ be a rational normal scroll and put $\mathfrak{m} = (T, xT, xyT, yT, x^{-1}yT, \dots, x^{-n}yT)$. Then $A = R_{\mathfrak{m}}$ is a three-dimensional Cohen-Macaulay F-rational local domain with $e(A) = n + 2$, and

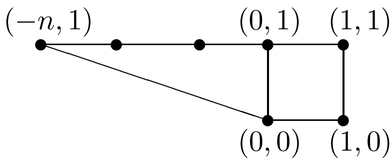
$$e_{\mathrm{HK}}(A) = \frac{e(A)}{2} + \frac{e(A)}{6(n+1)}.$$

Proof. Let $\mathcal{P} \subseteq \mathbb{R}$ be a convex polytope with vertex set

$$\Gamma = \{(0,0), (1,0), (1,1), (0,1), (-1,1), \dots, (-n,1)\},$$

and put $\widetilde{\mathcal{P}} := \{(\alpha, 1) \in \mathbb{R}^3 : \alpha \in \mathcal{P}\}$ and $d\mathcal{P} := \{d \cdot \alpha : \alpha \in \mathcal{P}\}$ for every integer $d \geq 0$. Also, if we define a cone $\mathcal{C} = \mathcal{C}(\widetilde{\mathcal{P}}) := \{r\beta : \beta \in \widetilde{\mathcal{P}}, 0 \leq r \in \mathbb{Q}\}$ and regard R as a homogeneous k-algebra with $\deg x = \deg y = 0$ and $\deg T = 1$, then the basis of R_d corresponds to the set $\{(\alpha, d) \in \mathbb{Z}^3 : \alpha \in \mathbb{Z}^2 \cap d\mathcal{P}\} = \{(\alpha, d) \in \mathbb{Z}^3 : \alpha \in \mathbb{Z}^2\} \cap \mathcal{C}$.

$\mathcal{P}$ =



If we put $\Gamma_q = \{(0,0), (q,0), (q,q), (0,q), (-q,q), \dots, (-nq,q)\}$, then $\mathfrak{m}^{[q]} = (x^a y^b T^q : (a,b) \in \Gamma_q)$. Since $[\mathfrak{m}^{[q]}]_d = \sum_{(a,b)\in\Gamma_q} R_{d-q}\, x^a y^b T^q$, we have

$$\begin{aligned} e_{\mathrm{HK}}(A) &= \lim_{q\to\infty} \frac{1}{q^3}\, l_A(A/\mathfrak{m}^{[q]}) \\ &= \lim_{q\to\infty} \frac{1}{q^3}\, \#\Big\{\mathbb{Z}^3 \cap \Big(\mathcal{C} \setminus \bigcup_{(a,b)\in\Gamma_q} (a,b,q) + \mathcal{C}\Big)\Big\}, \end{aligned}$$

that is,

$$e_{\mathrm{HK}}(A) = \lim_{q\to\infty} \frac{1}{q^3} \left[\sum_{d=0}^{\infty} \#\Big\{\mathbb{Z}^2 \cap \Big(d\mathcal{P} \setminus \bigcup_{(a,b)\in\Gamma_q} (a,b) + \max\{0, d-q\}\mathcal{P}\Big)\Big\}\right].$$

Also, if we define a real continuous function $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) = \text{the volume of } \left[t\mathcal{P} \setminus \bigcup_{(a,b)\in\Gamma} (a,b) + \max\{0, t-1\}\mathcal{P}\right] \text{ in } \mathbb{R}^2,$$

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then $e_{\mathrm{HK}}(A) = \displaystyle\int_0^\infty f(t)\,dt$. Let us denote the volume of $M \subseteq \mathbb{R}^2$ by $\mathrm{vol}(M)$. To calculate $e_{\mathrm{HK}}(A)$, we need to determine $f(t)$. Namely, we need to show the following claim.

Claim.

$$f(t) = \begin{cases} \mathrm{vol}(t\mathcal{P}), & 0 \le t < 1; \\ \mathrm{vol}(t\mathcal{P}) - (n+4)\,\mathrm{vol}((t-1)\mathcal{P}), & 1 \le t < \frac{n+2}{n+1}; \\ \frac{(n+2)t(2-t)}{2} + (n+2)\frac{(2-t)^2}{2n}, & \frac{n+2}{n+1} \le t < 2; \\ 0, & t \ge 2. \end{cases}$$

To prove the claim, we may assume that $t \ge 1$. For simplicity, we put $M_{a,b} = (a,b) + (t-1)\mathcal{P}$ for every $(a,b) \in \Gamma$. First suppose that $1 \le t < \frac{n+2}{n+1}$. Then since $1 - n(t-1) > t-1$, $M_{0,0} \cap M_{1,0} = \emptyset$. Similarly, one can easily see that any two $M_{a,b}$ do not intersect each other; see Figure 1. Thus $f(t) = \mathrm{vol}(t\mathcal{P}) - (n+4)\,\mathrm{vol}((t-1)\mathcal{P})$.

Next suppose that $\frac{n+2}{n+1} \le t < 2$. Then $\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 : 0 \le y \le t-1\} = M_{0,0} \cup M_{1,0} \cup T_0$, where T_0 is a triangle with vertex $(t-1, 0)$, $(1,0)$ and $\left(t-1, \frac{2-t}{n}\right)$. Similarly, there exist $(n+1)$-triangles $T_1, \dots, T_{n+1}$ having the same volumes as T_0 such that

$$\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 : 1 \le y \le t\} = M_{-n,1} \cup \cdots \cup M_{1,1} \cup M_{0,1} \cup M_{1,1} \cup T_1 \cup \cdots \cup T_{n+1}$$

and any two T_i's do not intersect each other; see Figure 2. Thus

$$\begin{aligned} f(t) &= \mathrm{vol}(\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 : t-1 \le y \le 1\}) + (n+2)\,\mathrm{vol}(T_0) \\ &= \frac{(n+2)t(2-t)}{2} + (n+2)\frac{(2-t)^2}{2n}. \end{aligned}$$

Finally, suppose that $t \ge 2$. Then since $\mathcal{P}$ is covered by $M_{a,b}$'s, we have $f(t) = 0$, as required.

Using the above claim, let us calculate $e_{\mathrm{HK}}(A)$. Note that $\mathrm{vol}(t\mathcal{P}) = \frac{(n+2)t^2}{2}$.

$$\begin{aligned} e_{\mathrm{HK}}(A) &= \int_0^{\frac{n+2}{n+1}} \frac{(n+2)t^2}{2}\,dt - (n+4)\int_1^{\frac{n+2}{n+1}} \frac{(n+2)(t-1)^2}{2}\,dt \\ &\quad + \int_{\frac{n+2}{n+1}}^2 \frac{(n+2)t(2-t)}{2}\,dt + (n+2)\int_{\frac{n+2}{n+1}}^2 \frac{(2-t)^2}{2n}\,dt \\ &= (n+2)\left[\frac{1}{2} + \frac{1}{6(n+1)}\right], \end{aligned}$$

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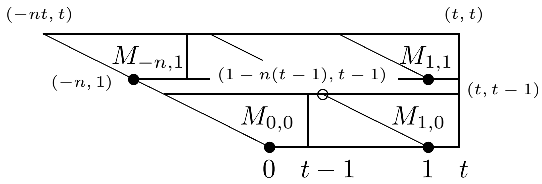

FIGURE 1. The case where $1 \leq t < \frac{n+2}{n+1}$

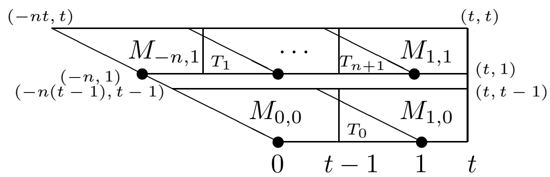

FIGURE 2. The case where $\frac{n+2}{n+1} \leq t < 2$

as required. ∎

DISCUSSION 3.6. Let A be a complete local ring which satisfies (#). Also, suppose that $e = 3$. What is the smallest value of $e_{\mathrm{HK}}(A)$ among such rings?

The function $f_e(s) = 3\left(\frac{s^3}{6} - 5\frac{(s-1)^3}{6}\right)$, which appeared in Eq. (3.1), takes the maximal value

$$f\left(\frac{5+\sqrt{5}}{4}\right) = \frac{15+5\sqrt{5}}{16} = 1.636\cdots$$

in $s \in [1, 2]$. Hence $e_{\mathrm{HK}}(A) \geq 1.636\cdots$. But we believe that this is not best possible.

Suppose that $e_{\mathrm{HK}}(A) < 2$. Then A is F-rational by Lemma 3.4. Thus it is Cohen-Macaulay and $3+1 \leq v = \mathrm{emb}(A) \leq d+e-1 = 3+3-1 = 5$. If $v \neq 5$, then A is a hypersurface and $e_{\mathrm{HK}}(A) \geq \frac{2}{3} \cdot e = 2$ by Fact 2.4. Hence we may assume that $v = 5$, that is, A has maximal embedding dimension. If we write as $A = R/I$, where R is a complete regular local ring with $\dim R = 5$, then height $I = 2$. By Hilbert-Burch's theorem, there exists a 2×3-matrix $\mathbb{M}$ such that $I = I_2(\mathbb{M})$, the ideal generated by all 2-minors of $\mathbb{M}$. In particular, A can be written as $A = B/aB$, where $B = k[X]/I_2(X)$,

X is a generic 2×3-matrix and a is a prime element of B. This implies that

$$e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(B/aB) \geq e_{\mathrm{HK}}(B) = 3\left\{\frac{1}{2} + \frac{1}{4!}\right\} = \frac{13}{8} = 1.625;$$

see [5, Section 3].

For example, if $A = k[[T, xT, xyt, yT, x^{-1}yT]]$ is a rational normal scroll, then $e_{\mathrm{HK}}(A) = \frac{7}{4} = 1.75$ by Example 3.5. Is this the smallest value?

Discussion 3.7. Let A be a complete local ring which satisfies (#). Also, suppose that $e = 4$. What is the smallest value of $e_{\mathrm{HK}}(A)$ among such rings?

As in Discussion 3.6, it suffices to consider F-rational local rings only. For example, let $A = k[[x, y, z]]^{(2)}$ be the Veronese subring. Then A is an F-rational local domain with $e(A) = 4$ and $e_{\mathrm{HK}}(A) = 2$. Also, let A be the completion of the Rees algebra $R(\mathfrak{n})$ over an F-rational double point $(R, \mathfrak{n})$ of dimension 2. Then A is an F-rational local domain with $e(A) = 4$ and $e_{\mathrm{HK}}(A) \geq 2$ (we believe that this inequality is strict).

On the other hand, the function $f_e(s)$ which appeared in Eq. (3.1), takes the maximal value

$$f\left(\frac{6+\sqrt{6}}{5}\right) = \frac{28 + 8\sqrt{6}}{25} = 1.903\cdots$$

in $s \in [1, 2]$. Hence the fact that we can prove now is "$e_{\mathrm{HK}}(A) \geq 1.903\cdots$" only.

Based on Corollary 2.6 and Discussion 3.7, we pose the following conjecture.

Conjecture 3.8. *Let A be a complete local ring which satisfies (#), and let $r \geq 2$ be an integer. If $e(A) = r^2$, then*

$$e_{\mathrm{HK}}(A) \geq \frac{(r+1)(r+2)}{6}.$$

Also, the equality holds if and only if A is isomorphic to $k[[x, y, z]]^{(r)}$.

In the rest of this section, we prove the second statement of Theorem 3.1. Let $(A, \mathfrak{m}, k)$ be a complete local ring which satisfies (#). If

$e_{\mathrm{HK}}(A) = \frac{4}{3}$, then A is an F-rational hypersurface with $e(A) = 2$ by the above observation. Furthermore, suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$. Then we may assume that A can be written as the form $k[[X, Y, Z, W]]/(X^2 - \varphi(Y, Z, W))$. To study Hilbert-Kunz multiplicities for these rings, we prove the improved version of Theorem 2.2.

Proposition 3.9. *Let k be an algebraically closed field of* $\operatorname{char} k \neq 2$, *and let $A = k[[X, Y, Z, W]]/(X^2 - \varphi(Y, Z, W))$ be an F-rational hypersurface local ring. Let a, b, c be integers with $2 \leq a \leq b \leq c$.*

Suppose that there exists a function $\operatorname{ord} : A \to \mathbb{Q} \cup \{\infty\}$ *which satisfies the following conditions*:

(1) $\operatorname{ord}(\alpha) \geq 0$; *and* $\operatorname{ord}(\alpha) = \infty \iff \alpha = 0$.

(2) $\operatorname{ord}(x) = 1/2$, $\operatorname{ord} y = 1/a$, $\operatorname{ord} z = 1/b$, *and* $\operatorname{ord} w = 1/c$.

(3) $\operatorname{ord}(\varphi) \geq 1$.

(4) $\operatorname{ord}(\alpha + \beta) \geq \min\{\operatorname{ord}(\alpha), \operatorname{ord}(\beta)\}$.

(5) $\operatorname{ord}(\alpha\beta) \geq \operatorname{ord}(\alpha) + \operatorname{ord}(\beta)$.

Then we have

$$e_{\mathrm{HK}}(A) \geq 2 - \frac{abc}{12}(N^3 - n^3),$$

where

$$N = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2}, \quad n = \max\left\{0,\ N - \frac{2}{c}\right\}.$$

In particular, if $(a, b, c) \neq (2, 2, 2)$, then $e_{\mathrm{HK}}(A) > \frac{4}{3}$.

Remark 6. The third condition $\operatorname{ord}(\varphi) \geq 1$ is important. For example, if $\varphi \equiv y^2 \mod (z, w)^3$, then one can take $(a, b, c) = (2, 3, 3)$, but $(a, b, c) = (2, 3, 4)$.

Proof. First, we define a filtration $\{F_n\}_{n \in \mathbb{Q}}$ as follows:

$$F_n := \{\alpha \in A : \operatorname{ord}(\alpha) \geq n\}.$$

Then every F_n is an ideal and $F_m F_n \subseteq F_{m+n}$ holds for all m, $n \in \mathbb{Q}$. Using F_n instead of $\mathfrak{m}^n$, we shall estimate $l_A(\mathfrak{m}^{[q]}/J^{[q]})$.

Set $J = (y, z, w)A$ and fix a sufficiently large power $q = p^e$. Put

$$s = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \quad N = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2}.$$

Since J is a minimal reduction of $\mathfrak{m}$ and $xy^{q-1}z^{q-1}w^{q-1}$ generates the socle of $A/J^{[q]}$, we have that $F_{sq} \subseteq J^{[q]}$. Also, since $B = A/J^{[q]}$ is an Artinian Gorenstein local ring, we get

$$F_{\frac{(N+1)q}{2}}B \subseteq 0 :_B F_{\frac{Nq}{2}}B \cong K_{B/F_{\frac{Nq}{2}}B},$$

where K_C denotes a canonical module of a local ring C. Hence, by the Matlis duality theorem, we get

$$l_A\left(\frac{F_{\frac{(N+1)q}{2}} + J^{[q]}}{J^{[q]}}\right) \leq l_B\big(F_{\frac{(N+1)q}{2}}\big) \leq l_B\big(K_{B/F_{\frac{Nq}{2}}B}\big) = l_B\big(B/F_{\frac{Nq}{2}}B\big).$$

On the other hand, since $x^q \in F_{\frac{q}{2}}$ by the assumption, we have

$$x^q F_{\frac{Nq}{2}} \subseteq F_{\frac{(N+1)q}{2}}.$$

Therefore by a similar argument as in the proof of Theorem 2.2, we get

$$\begin{aligned}
l_A(\mathfrak{m}^{[q]}/J^{[q]}) &\leq l_A\left(\frac{Ax^q + J^{[q]} + F_{\frac{(N+1)q}{2}}}{F_{\frac{(N+1)q}{2}} + J^{[q]}}\right) + l_A\left(\frac{F_{\frac{(N+1)q}{2}} + J^{[q]}}{J^{[q]}}\right) \\
&\leq l_A\Big(A/\big(J^{[q]} + F_{\frac{(N+1)q}{2}}\big) : x^q\Big) + l_B\Big(B/F_{\frac{Nq}{2}}B\Big) \\
&\leq 2 \cdot l_A\Big(A/J^{[q]} + F_{\frac{N}{2}q}\Big).
\end{aligned}$$

In fact, since

$$\begin{aligned}
&\lim_{q\to\infty} \frac{1}{q^3}\, l_A\Big(A/J^{[q]} + F_{\frac{Nq}{2}}\Big) \\
&= e(A) \cdot \lim_{q\to\infty} \frac{1}{q^3} \operatorname{vol}\left\{(x,y,z) \in [0,q]^3 : \frac{y}{a} + \frac{z}{b} + \frac{w}{c} \leq \frac{Nq}{2}\right\} \\
&= 2 \cdot \operatorname{vol}\left\{(x,y,z) \in [0,1]^3 : \frac{y}{a} + \frac{z}{b} + \frac{w}{c} \leq \frac{N}{2}\right\} \\
&= \frac{abc}{24}(N^3 - n^3),
\end{aligned}$$

we get

$$e_{\mathrm{HK}}(A) \geq 2 - 2 \cdot \frac{abc}{24}(N^3 - n^3) = 2 - \frac{abc}{12}(N^3 - n^3),$$

as required. ∎

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Example 3.10. Let k be an algebraically closed field of $\operatorname{char} k \neq 2$, and let $(A, \mathfrak{m}, k)$ be a hypersurface. Put $\operatorname{gr}_{\mathfrak{m}}(A) = k[X, Y, Z, W]/(g(X, Y, Z, W))$.

$$g(X,Y,Z,W) = X^2 + Y^3 + Z^3 + W^3 \implies e_{\mathrm{HK}}(A) \geq \frac{55}{32};$$

$$g(X,Y,Z,W) = X^2 + Y^2 + Z^3 + W^3 \implies e_{\mathrm{HK}}(A) \geq \frac{14}{9};$$

$$g(X,Y,Z,W) = X^2 + Y^2 + Z^2 + W^c \implies e_{\mathrm{HK}}(A) \geq \frac{3}{2} - \frac{2}{3c^2}.$$

Proof of Theorem 3.1(2). Put $G = \operatorname{gr}_{\mathfrak{m}}(A)$ and $\mathfrak{M} = \operatorname{gr}_{\mathfrak{m}}(A)_+$. The implication $(a) \Rightarrow (b)$ follows from Proposition 3.9. $(b) \Rightarrow (c)$ is clear. Suppose (c). Then $e_{\mathrm{HK}}(G_{\mathfrak{M}}) = \frac{4}{3}$. Also, by Proposition 1.3 and Theorem 3.1(1), we have that $\frac{4}{3} \leq e_{\mathrm{HK}}(A) \leq e_{\mathrm{HK}}(G_{\mathfrak{M}}) = \frac{4}{3}$. Thus $e_{\mathrm{HK}}(A) = \frac{4}{3}$, as required. □

Also, the following corollary follows from the proof of Proposition 3.9 and Example 3.10.

Corollary 3.11. *Let A be a local ring which satisfies (#). Also, assume that $k = \overline{k}$ and $p \neq 2$. Then the following conditions are equivalent*:

(1) $\frac{4}{3} < e_{\mathrm{HK}}(A) \leq \frac{3}{2}$.

(2) $\operatorname{gr}_{\mathfrak{m}}(A) \cong k[X, Y, Z]/(X^2 + Y^2 + Z^2)$.

(3) *A is isomorphic to a hypersurface $k[[X, Y, Z, W]]/(X^2+Y^2+Z^2+W^c)$ for some integer $c \geq 3$.*

When this is the case, $e_{\mathrm{HK}}(A) \geq \frac{3}{2} - \frac{2}{3c^2}$.

§4. A generalization of the main result to higher dimensional case

In this section, we want to generalize Theorem 3.1 to the case of $\dim A \geq 4$. Let $d \geq 1$ be an integer and $p > 2$ a prime number. If we put

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2),$$

then we can guess that $e_{\mathrm{HK}}(A_{p,d}) = s_{\mathrm{HK}}(p, d)$ holds according to the observations until the previous section. In the following, let us formulate this as a conjecture and prove that it is also true in case of $\dim A = 4$.

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In [10], Han and Monsky gave an algorism to calculate $e_{\mathrm{HK}}(A_{p,d})$, but it is not so easy to represent $e_{\mathrm{HK}}(A_{p,d})$ as a quotient of two polynomials of p for any fixed $d \geq 1$.

d	1	2	3	4
$e_{\mathrm{HK}}(A_{p,d})$	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{29p^2+15}{24p^2+12}$

On the other hand, surprisingly, Monsky proved the following theorem:

THEOREM 4.1. (Monsky [19]) *Under the above notation, we have*

$$\lim_{p \to \infty} e_{\mathrm{HK}}(A_{p,d}) = 1 + \frac{c_d}{d!}, \tag{4.1}$$

where

$$\sec x + \tan x = \sum_{d=0}^{\infty} \frac{c_d}{d!} x^d \quad \left(|x| < \frac{\pi}{2}\right). \tag{4.2}$$

Remark 7. It is known that the Taylor expansion of $\sec x$ (resp. $\tan x$) at origin can be written as follows:

$$\sec x = \sum_{i=0}^{\infty} \frac{E_{2i}}{(2i)!} x^{2i},$$
$$\tan x = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i}(2^{2i}-1)B_{2i}}{(2i)!} x^{2i-1},$$

where E_{2i} (resp. B_{2i}) is said to be Euler number (resp. Bernoulli number).

Also, c_d appeared in Eq. (4.1) is a positive integer since $\cos t$ is an unit element in a ring $\mathcal{H} = \left\{\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} : a_n \in \mathbb{Z} \text{ for all } n \geq 0\right\}$.

Based on the above observation, we pose the following conjecture.

CONJECTURE 4.2. *Let $d \geq 1$ be an integer and $p > 2$ a prime number. Put*

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2).$$

Let $(A, \mathfrak{m}, k)$ be a d-dimensional unmixed local ring with $k = \overline{\mathbb{F}_p}$. Then the following statements hold.

(1) *If A is not regular, then $e_{\mathrm{HK}}(A) \geq e_{\mathrm{HK}}(A_{p,d}) \geq 1 + \frac{c_d}{d!}$. In particular, $s_{\mathrm{HK}}(p,d) = e_{\mathrm{HK}}(A_{p,d})$.*

(2) *If* $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(A_{p,d})$, *then* $\widehat{A} \cong A_{p,d}$ *as local rings.*

In the following, we prove that this is true in case of $\dim A = 4$. Note that

$$\lim_{p\to\infty} e_{\mathrm{HK}}(A_{p,4}) = \lim_{p\to\infty} \frac{29p^2+15}{24p^2+12} = \frac{29}{24} = 1 + \frac{c_4}{4!}.$$

Theorem 4.3. *Let* $(A, \mathfrak{m}, k)$ *be an unmixed local ring of characteristic* $p > 0$ *with* $\dim A = 4$. *If* $e(A) \geq 3$, *then* $e_{\mathrm{HK}}(A) \geq \frac{5}{4} = \frac{30}{24}$.

Suppose that $k = \overline{k}$ *and* $\operatorname{char} k \neq 2$. *Put*

$$A_{p,4} = \overline{\mathbb{F}_p}[[X_0, X_1, \ldots, X_4]]/(X_0^2 + \cdots + X_4^2).$$

Then the following statement holds.

(1) *If* A *is not regular, then*

$$e_{\mathrm{HK}}(A) \geq e_{\mathrm{HK}}(A_{p,4}) = \frac{29p^2+15}{24p^2+12}.$$

(2) *The following conditions are equivalent*:

  (a) *Equality holds in* (1).

  (b) $e_{\mathrm{HK}}(A) < \frac{5}{4}$.

  (c) *The completion of* A *is isomorphic to* $A_{p,4}$.

Proof. Put $e = e(A)$, the multiplicity of A. We may assume that A is complete with $e \geq 2$ and k is infinite. In particular, A is a homomorphic image of a Cohen-Macaulay local ring, and there exists a minimal reduction J of $\mathfrak{m}$. Then $\mu_A(\mathfrak{m}/J^*) \leq e - 1$ by Lemma 2.5. We first show that $e_{\mathrm{HK}}(A) \geq \frac{5}{4}$ if $e \geq 3$.

Claim 1. *If* $3 \leq e \leq 10$, *then* $e_{\mathrm{HK}}(A) \geq \frac{5}{4}$.

Putting $r = e-1$ and $s = 2$ in Theorem 2.2, since $v_2 = \frac{1}{2}$, we have

$$e_{\mathrm{HK}}(A) \geq e\left\{v_2 - \frac{(e-1)1^4}{4!}\right\} = \frac{(13-e)e}{24} \geq \frac{(13-3)\cdot 3}{24} = \frac{30}{24},$$

as required.

Claim 2. *If* $11 \leq e \leq 29$, *then* $e_{\mathrm{HK}}(A) \geq \frac{737}{384} \left(> \frac{5}{4}\right)$.

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By Fact 2.4, we have $v_{3/2} = \frac{1-\beta_{4+1}}{2} = \frac{77}{384}$. Putting $r = e-1$ and $s = \frac{3}{2}$ in Theorem 2.2, we have

$$e_{\mathrm{HK}}(A) \geq e\left\{v_{3/2} - \frac{e-1}{24}\cdot\left(\frac{1}{2}\right)^4\right\} = \frac{(78-e)e}{384} \geq \frac{11(78-11)}{384} = \frac{737}{384},$$

as required.

Claim 3. *If* $e \geq 30$, *then* $e_{\mathrm{HK}}(A) \geq \frac{5}{4}$.

By Proposition 1.4, we have $e_{\mathrm{HK}}(A) \geq \frac{e}{4!} \geq \frac{30}{24}$.

In the following, we assume that $k = \overline{k}$, $\operatorname{char} k \neq 2$ and $e \geq 2$. To see (1), (2), we may assume that $e = 2$ by the above argument. Then since $e_{\mathrm{HK}}(A) = 2$ if A is not F-rational, we may also assume that A is F-rational and thus is a hypersurface. Thus A can be written as the following form:

$$A = k[[X_0, X_1, \dots, X_4]]/(X_0^2 - \varphi(X_1, X_2, X_3, X_4)).$$

If A is isomorphic to $A_{p,4}$, then by [10], it is known that $e_{\mathrm{HK}}(A) = \frac{29p^2+15}{24p^2+12}$. Suppose that A is not isomorphic to $A_{p,4}$. Then one can take a minimal numbers of generators x, y, z, w, u of $\mathfrak{m}$ and one can define a function $\operatorname{ord} : A \to \mathbb{Q} \cup \{\infty\}$ such that

$$\operatorname{ord}(x) = \operatorname{ord}(y) = \operatorname{ord}(z) = \operatorname{ord}(z) = \frac{1}{2}, \quad \operatorname{ord}(u) = \frac{1}{3}.$$

If we put $J = (y, z, w, u)A$ and $F_n = \{\alpha \in A : \operatorname{ord}(\alpha) \geq n\}$, then by a similar argument as in the proof of Proposition 3.9, we have

$$l_A(\mathfrak{m}^{[q]}/J^{[q]}) \leq 2 \cdot l_A(A/J^{[q]} + F_{2q/3}).$$

Divided the both-side by q^d and taking a limit $q \to \infty$, we get

$$e(A) - e_{\mathrm{HK}}(A) \leq 2 \cdot e(A) \cdot \operatorname{vol}\left\{(y, z, w, u) \in [0,1]^4 : \frac{y}{2} + \frac{z}{2} + \frac{w}{2} + \frac{u}{3} \leq \frac{2}{3}\right\}.$$

To calculate the volume in the right-hand side, we put

$$F_u = \begin{cases} \frac{1}{6}\left(\frac{4}{3} - \frac{2}{3}u\right)^3 - 3\cdot\frac{1}{6}\left(\frac{1}{3} - \frac{2}{3}u\right)^3 & (0 \leq u \leq \frac{1}{2}) \\ \frac{1}{6}\left(\frac{4}{3} - \frac{2}{3}u\right)^3 & (\frac{1}{2} \leq u \leq 1) \end{cases}$$

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Then one can easily calculate

$$\text{the above volume} = \int_0^1 F_u \, du = \frac{237}{2^4 3^4}.$$

It follows that

$$e_{\mathrm{HK}}(A) \geq 2 - 4 \times \frac{237}{2^4 3^4} = \frac{411}{324} > \frac{5}{4}.$$

□

The following conjecture also holds if $\dim A \leq 4$.

Conjecture 4.4. *Under the same notation as in Conjecture* 4.2, *if* $e(A) \geq 3$, *then*

$$e_{\mathrm{HK}}(A) \geq 1 + \frac{c_d + 1}{d!}.$$

Discussion 4.5. Let $d \geq 2$ be an integer and fix a prime number $p \gg d$. Assume that Conjectures 4.2 and 4.4 are true. Also, assume that $s_{\mathrm{HK}}(p, d) < s_{\mathrm{HK}}(p, d-1)$ for all $d \geq 3$. Let $A = k[X_0, \ldots, X_v]/I$ be a d-dimensional homogeneous unmixed k-algebra with $\deg X_i = 1$, and let $\mathfrak{m}$ be the unique homogeneous maximal ideal of A. Suppose that k is an algebraically closed field of characteristic $p > 0$. Then $e_{\mathrm{HK}}(A) = s_{\mathrm{HK}}(p, d)$ implies that $\widehat{A_{\mathfrak{m}}} \cong A_{p,d}$.

In fact, if $e_{\mathrm{HK}}(A) = s_{\mathrm{HK}}(p, d)$, then we may assume that $e_{\mathrm{HK}}(A) < 1 + \frac{c_d+1}{d!}$. Thus $e(A_{\mathfrak{m}}) = 2$ if Conjecture 4.4 is true. For any prime ideal $PA_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$ such that $P \neq \mathfrak{m}$, we have $e_{\mathrm{HK}}(A_P) \leq e_{\mathrm{HK}}(A_{\mathfrak{m}}) = s_{\mathrm{HK}}(p, d) < s_{\mathrm{HK}}(p, n)$, where $n = \dim A_P < d$. Since A_P is also unmixed, it is regular. Thus $A_{\mathfrak{m}}$ has an isolated singularity. Hence A is a non-degenerate quadric hypersurface In other words, $\widehat{A_{\mathfrak{m}}}$ is isomorphic to $A_{p,d}$.

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