A NOTE ON CUMULATIVE SUMS OF MARKOVIAN VARIABLES

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Consider a positive regular Markov chain $X_0, X_1, X_2, \cdots$ with s (s finite) number of states $E_1, E_2, \cdots, E_s$, and a transition probability matrix $\boldsymbol{P} = (p_{ij})$ where $p_{ij} = Pr\{X_r = E_i \,|X_{r-1} = E_j\}$ $(r \geqq 1)$, and an initial probability distribution given by the vector $\boldsymbol{p}_0$. Let $\{Z_r\}$ be a sequence of random variables such that

$$Z_r = h_{ij}, \text{ when } X_{r-1} = E_j,\ X_r = E_i,$$

and consider the sum $S_N = Z_1 + Z_2 + \cdots Z_N$. It can easily be shown that (cf. Bartlett [1] p. 37),

$$\Phi_N(t) = E[e^{tS_N}|\boldsymbol{p}_0] = \boldsymbol{t}_1' \sum_1^s \lambda_i^N(t)\boldsymbol{s}_i(t)\boldsymbol{t}_i'(t)\boldsymbol{p}_0 \tag{1}$$

where $\lambda_1(t), \lambda_2(t) \cdots \lambda_s(t)$ are the latent roots of $\boldsymbol{P}(t) \equiv (p_{ij}e^{th_{ij}})$ and $\boldsymbol{s}_i(t)$ and $\boldsymbol{t}_i'(t)$ are the column and row vectors corresponding to $\lambda_i(t)$, and so constructed as to give $\boldsymbol{t}_i'(t)\boldsymbol{s}_i(t) = 1$ and $\boldsymbol{t}_i'(0) = \boldsymbol{t}_i'$, $\boldsymbol{s}_i(0) = \boldsymbol{s}_i$, where $\boldsymbol{t}_i'$ and $\boldsymbol{s}_i$ are the corresponding column and row vectors, considering the matrix $\boldsymbol{P}(0) \equiv \boldsymbol{P}$. Denote $\boldsymbol{t}_i'(t)\boldsymbol{p}_0$ by $\alpha_i(t)$.

We assume that $E[e^{Z_r t}|X_{r-1} = E_j]$ exists for real t in an interval I about zero, and for all E_j. Hence $E[e^{S_N t}]$ exists for t in I and therefore can be differentiated any number of times with respect to t in I. We have, differentiating (1) once, putting $t = 0$, and noting that $\boldsymbol{t}_i'(t)\boldsymbol{s}_j(t) = \delta_{ij}$,

$$E[S_N] = N\lambda_1'(0) + \alpha_1'(0) - \left[\frac{d\boldsymbol{t}_1'(t)}{dt}\right]_0 \boldsymbol{s}_1 - \left[\frac{d\boldsymbol{t}_1'(t)}{dt}\right]_0 \sum_2^s \lambda_i^N \boldsymbol{s}_i \boldsymbol{t}_i' \boldsymbol{p}_0. \tag{2}$$

We denote the third and fourth terms on the R.H.S. of (2) by A and $B(N|\boldsymbol{p}_0)$ respectively. It can be noted that $B(N|\boldsymbol{p}_0) \to 0$ as $N \to \infty$, irrespective of the initial distribution $\boldsymbol{p}_0$.

If the Markov chain is initially stationary, i.e. if the initial distribution $\boldsymbol{p}_0$ is the same as the limiting distribution $\boldsymbol{s}_1$, we have

$$E(S_N) = N\lambda_1'(0).$$

In this case the random variables $\{Z_r\}$ have indentical distributions, and we have $\lambda_1'(0) = E(Z)$, the mean of the common distribution.

THEOREM. *Let $b(< 0)$ and $a(> 0)$ be two fixed numbers and let n be the smallest positive integer such that S_n does not lie in the open interval (b, a). Then,*

$$E[S_n] = E(n)E(Z)+\alpha_1'(0)-E[\beta'(0|X_n)] \tag{3}$$

where $\beta(t|X_n)$ is the jth element of $t_1'(t)$, if $X_n = E_j$.

The above result is the Markovian analogue of Wald's lemma [3]:

$$E[\sum_{i-1}^{n} Z_i] = E(Z)E(n)$$

for the sequence of independent and identical random variables $\{Z_r\}$.

PROOF. Consider two positive integers M and N, with $M > N$. Let P_N and Q_N denote the probability that $n \leqq N$ and $n > N$ respectively. (It can be proved, [Phatarfod [2]] that for all s, $N^s(1-P_N) \to 0$ as $N \to \infty$.) Let E^* and E^{**} denote conditional expectations under conditions $n \leqq N$ and $n > N$ respectively. We then have,

$$E[S_M] = P_N E^*[S_n+(S_M-S_n)]+(1-P_N)E^{**}[S_N+S_M-S_N]. \tag{4}$$

We also have from (2),

$$\begin{aligned} E^*[S_n+S_M-S_n] = E^*[S_n]+E^*[(M-n)E(Z)+\beta'(0|X_n) \\ -A-B(M-n|X_n)], \end{aligned} \tag{5}$$

and

$$\begin{aligned} E^{**}[S_N+S_M-S_N] = E^{**}[S_N]+E^{**}[(M-N)E(Z)+\beta'(0|X_N) \\ -A-B(M-N|X_N)]. \end{aligned} \tag{6}$$

From (2), (4), (5) and (6), we obtain,

$$\begin{aligned} \alpha_1'(0)-B(M|\boldsymbol{p}_0) = P_N E^*[S_n-nE(Z)+\beta'(0|X_n)-B(M-n|X_n)] \\ +(1-P_N)E^{**}[S_N-NE(Z)+\beta'(0|X_N)-B(M-N|X_N)]. \end{aligned} \tag{7}$$

Now, let $M \to \infty$, keeping N fixed. $B(M|\boldsymbol{p}_0)$ and $B(M-N|X_N) \to 0$ as $M \to \infty$. Also $E^*[B(M-n|X_n)] \to 0$, since the expectation is taken under condition $n \leqq N$. Hence, we have

$$\begin{aligned} \alpha_1'(0) = P_N E^*[S_n-nE(Z)+\beta'(0|X_n)] \\ +(1-P_N)E^{**}[S_N-NE(Z)+\beta'(0|X_N)]. \end{aligned}$$

Taking the limit when $N \to \infty$, we have, since $P_N \to 1$, and $N(1-P_N) \to 0$, and $E^{**}[S_N]$, $E^{**}[\beta'(0|X_N)]$ bounded,

$$\alpha_1'(0) = E[S_n] - E(n)E(Z) + E[\beta'(0|X_n)],$$

giving us the required result (3).

References

[1] Bartlett, M. S., An introduction to Stochastic Processes, Cambridge University Press (1955).
[2] Phatarfod, R. M., Sequential Analysis of dependent observations — I (to be published in Biometrika).
[3] Wald, A., Sequential Tests of Statistical Hypotheses, Annals. of Math. Statistics **16** (1945), 115.

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