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RESEARCH ARTICLE

Schubert polynomial expansions revisited

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Abstract
We give an elementary approach utilizing only the divided difference formalism for obtaining expansions of Schubert polynomials that are manifestly nonnegative, by studying solutions to the equation $\sum Y_i \partial_i = \mathrm{id}$ on polynomials with no constant term. This in particular recovers the pipe dream and slide polynomial expansions. We also show that slide polynomials satisfy an analogue of the divided difference formalisms for Schubert polynomials and forest polynomials, which gives a simple method for extracting the coefficients of slide polynomials in the slide polynomial decomposition of an arbitrary polynomial.

Contents

1. Introduction

Let S_∞ denote the set of permutations of $\{1,2,\ldots\}$ with finite support, and let $\ell(w)$ denote the length of a permutation, the length of the smallest word in the simple transpositions $s_i = (i, i+1)$ which equals w. The nil-Coxeter monoid is the right-cancellative partial monoid whose elements are permutations in S_∞, equipped with the partial monoid structure

$$u \circ v = \begin{cases} uv & \text{if } \ell(u)+\ell(v) = \ell(uv) \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{1.1}$$

There is a permutation w/i such that $w = (w/i) \circ i$ if and only if i is in the descent set $\mathrm{Des}(w) = \{j \mid w(j) > w(j+1)\}$, in which case it is unique and given by the formula $w/i = ws_i$. An important representation of the nil-Coxeter monoid is the divided difference representation on integral polynomials, which sends s_i to the i'th divided difference operator ∂_i given by the formula

$$\partial_i(f) = \frac{f - f(x_1,\ldots,x_{i-1},x_{i+1},x_i,\ldots)}{x_i - x_{i+1}}. \tag{1.2}$$

The Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_\infty\}$ of Lascoux–Schützenberger [16, 18] are a family of polynomials indexed by permutations w in S_∞, characterized by the normalization condition $\mathfrak{S}_{\mathrm{id}} = 1$, and the relations

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{w/i} & \text{if } i \in \mathrm{Des}(w) \\ 0 & \text{otherwise.} \end{cases}$$

Despite their relatively simple definition, Schubert polynomials are complicated combinatorial objects. Many combinatorial formulas for Schubert polynomials exist, such as the algorithmic method of Kohnert [2, 12], the pipe dreams of Bergeron–Billey [4] and Fomin–Kirillov [8], the slide expansions of Billey–Jockusch–Stanley [6] and Assaf–Searles [3], the balanced tableaux of Fomin–Greene–Reiner–Shimozono [7], the bumpless pipe dreams of Lam–Lee–Shimozono [15], and the prism tableau model of Weigandt–Yong [27].

Expansions of Schubert polynomials have been almost exclusively studied from a 'top-down' perspective – for $w_{0,n}$ the longest permutation in S_n, one checks the conjectured formula agrees with the Ansatz $\mathfrak{S}_{w_{0,n}} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ and then verifies the conjectured formula transforms correctly under applications of ∂_i. It seems the approaches to studying Schubert formulae that are 'bottom-up' are rather limited. They fall into a broad class of results revolving around Pieri rules [26] (containing Monk's rule [21] as a special case) expanding the product of $\mathfrak{S}_w$ with elementary and complete homogenous symmetric polynomials via the k-Bruhat order [5] to establish relations between Schubert polynomials related by nonadjacent transpositions [17, §3]. Another approach, relying on the geometry of Bott–Samelson varieties, is due to Magyar [19], and it builds Schubert polynomials by interspersing isobaric divided differences with multiplications by terms of the form $x_1\cdots x_i$ (cf. [20] for a generalization to Grothendieck polynomials using combinatorial tools).

In this paper, we develop a new general method for finding combinatorial expansions of Schubert polynomials, *which works from the bottom-up*, by directly reconstructing a Schubert polynomial $\mathfrak{S}_w$ from the collection of Schubert polynomials $\mathfrak{S}_{ws_i}$ for $i \in \mathrm{Des}(w)$.

We demonstrate here our technique on a simpler toy example, where we recover the family of normalized monomials $\{S_c = \frac{x^c}{c!} := \frac{x_1^{c_1}\cdots x_\ell^{c_\ell}}{c_1!\cdots c_\ell!} \mid c = (c_1,\ldots,c_\ell)\}$ using only the indirect information that they are homogeneous with $S_\varnothing = 1$ and satisfy

$$\frac{d}{dx_i} S_c = \begin{cases} S_{c-e_i} & \text{if } c_i \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

where $c - e_i = (c_1, \ldots, c_{i-1}, c_i - 1, c_{i+1}, \ldots, c_\ell)$. Our technique is motivated by Euler's famous theorem

$$\sum_{i=1}^{\infty} x_i \frac{d}{dx_i} f = kf$$

for f a homogeneous polynomial of positive degree k. Iteratively applying this identity shows that

$$\sum_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_k} \frac{d}{dx_{i_1}} \cdots \frac{d}{dx_{i_k}} f = k!\,\mathrm{id}$$

on homogeneous polynomials of degree k, and grouping together terms with the same derivatives applied to f shows that

$$\sum_{(c_1,\ldots,c_\ell)} \frac{\mathsf{x}^c}{c!} \left(\frac{d}{dx_1}\right)^{c_1} \cdots \left(\frac{d}{dx_k}\right)^{c_\ell} = \mathrm{id}.$$

Applying this identity to S_c shows that $S_c = \frac{\mathsf{x}^c}{c!}$, as desired. Notably, this calculation does not use the Ansatz that the family of polynomials we are seeking are monomials.

Let $\mathrm{Pol} \coloneqq \mathbb{Z}[x_1, x_2, \ldots,]$, and let $\mathrm{Pol}^+ \subset \mathrm{Pol}$ denote the ideal of polynomials with no constant term. Our method relies on finding degree 1 'creation operators' $Y_1, Y_2, \ldots$ that solve the equation

$$\sum_{i=1}^{\infty} Y_i \partial_i = \mathrm{id}$$

on Pol^+. Applying this equation to a Schubert polynomial and recursing gives an expansion

$$\sum_{(i_1,\ldots,i_k)\in \mathrm{Red}(w)} Y_{i_k} \cdots Y_{i_1}(1) = \mathfrak{S}_w,$$

where $\mathrm{Red}(w)$ is the set of reduced words for w. In particular, if each Y_i is a monomial nonnegative operator, then this produces a monomial nonnegative expansion of $\mathfrak{S}_w$. Given the simplicity, we now show that Schubert polynomials have a nonnegative monomial expansion using this technique by producing one such family of creation operators (this later appears as §3.1; we will produce an additional family in §5.3). Define the map

$$\mathsf{R}_i(f) = f(x_1, \ldots, x_{i-1}, 0, x_i, x_{i+1}, \ldots).$$

Then

$$\mathrm{id} = \mathsf{R}_1 + (\mathsf{R}_2 - \mathsf{R}_1) + (\mathsf{R}_3 - \mathsf{R}_2) + \cdots = \mathsf{R}_1 + \sum_{i=1}^{\infty} x_i \mathsf{R}_i \partial_i.$$

Here, we use $\mathsf{R}_{i+1} - \mathsf{R}_i = x_i \mathsf{R}_i \partial_i$, which can be seen to hold by noting that $\mathsf{R}_{i+1} = \mathsf{R}_i s_i$, where s_i is the simple transposition swapping x_i and x_{i+1}. Moving R_1 to the other side and noting that $\mathrm{id} - \mathsf{R}_1$ is invertible on polynomials with no constant term with inverse $\mathsf{Z} = \mathrm{id} + \mathsf{R}_1 + \mathsf{R}_1^2 + \cdots$, we conclude that

$$\sum \mathsf{Z} x_i \mathsf{R}_i \partial_i = \mathrm{id}.$$

Applying this to $\mathfrak{S}_w$ immediately gives the following.

Theorem 1.1 (Corollary 3.2). *We have the following monomial positive expansion:*

$$\mathfrak{S}_w = \sum_{(i_1,\ldots,i_k)\in \mathrm{Red}(w)} \mathsf{Z} x_{i_k} \mathsf{R}_{i_k} \cdots \mathsf{Z} x_{i_1} \mathsf{R}_{i_1}(1).$$

Example 3.3 demonstrates how this theorem build Schubert polynomials bottom-up.

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We generalize these ideas to a more general situation (X, M) we call a 'divided difference pair' (dd-pair henceforth), in which the compositions of degree -1 polynomial endomorphisms $X_1, X_2, \ldots$, given by 'shifts' of a fixed endomorphism X, form a representation of a right-cancellable partial graded monoid M generated in degree 1. Writing $\mathrm{Last}(w)$ for the analogue of the descent set of w, we will say that a family of polynomials $\{S_w \mid w \in M\}$ is 'dual' to the dd-pair if it satisfies the normalization condition $S_1 = 1$ and

$$X_i S_w = \begin{cases} S_{w/i} & \text{if } i \in \mathrm{Last}(w) \\ 0 & \text{otherwise.} \end{cases}$$

It is then natural to ask the following.

1. Assuming there is such a family of polynomials $\{S_w \mid w \in M\}$, can we write down a formula for S_w?
2. Does such a family of polynomials exist in the first place?

These questions came up naturally from our previous paper [22] for the operators

$$\mathsf{T}_i^m(f) = \frac{f(x_1, \ldots, x_{i-1}, x_i, 0^m, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i-1}, 0^m, x_i, x_{i+1}, \ldots)}{x_i}$$

called 'm-quasisymmetric divided difference operators'. There we had to essentially guess (via computer assistance) a formula for the family of m-forest polynomials and then through a tedious and unenlightening computation [22, Appendix] show that they interact in the expected way with the T_i^m operators.

The analogue of creation operators Y_i such that $\sum Y_i X_i = \mathrm{id}$ on polynomials with no constant term can be used to solve the first question analogously as for Schubert polynomials, and we find such operators for m-forest polynomials without difficulty.

For the second question, we show that if a dd-pair has creation operators, then surprisingly, the only additional thing that is needed to ensure that the dual family of polynomials exists is a 'code map' $c : M \to \mathsf{Codes}$ from the partial monoid to finitely supported sequences of nonnegative integers, so that the highest index of a nonzero element of $c(m)$ is the maximal element of $\mathrm{Last}(w)$. The Lehmer code of permutations works for the ∂_i formalism, while the m-Dyck path forest code [22, Definition 3.5] works for the T_i^m formalism: this shows directly that Schubert polynomials and m-forest polynomials exist without any Ansatz or combinatorial model.

As a further application, we study the well-known family of polynomials called 'slide polynomials' investigated in detail by Assaf–Searles [3]; this family is also present in earlier works [6, 11] (see [10] for more on the relation to Hivert's foundational work). Forest polynomials and Schubert polynomials decompose nonnegatively in terms of this family (see respectively [23] and [3, 6]). A slide polynomial is determined by a sequence of positive integers $(a_1, a_2, \ldots, a_k)$, and the distinct slide polynomials $\mathfrak{F}_{a_1,\ldots,a_k}$ are indexed by weakly increasing sequences $1 \le a_1 \le a_2 \le \cdots \le a_k$. We construct a dd-pair for the operators

$$\mathsf{D}_i(f) = \frac{f(x_1, \ldots, x_{i-1}, x_i, 0, 0, \ldots) - f(x_1, \ldots, x_{i-1}, 0, x_i, 0, \ldots)}{x_i}$$

whose compositions are governed by the partial monoid whose only relations are that $\mathsf{D}_i \mathsf{D}_j$ is undefined for $i > j$, such that the slide polynomials form the dual family of polynomials. This gives a fast and practical method for directly extracting coefficients of an arbitrary polynomial in the slide basis. Since fundamental quasisymmetric polynomials are a subfamily within slide polynomials, this generalizes [22, Corollary 8.6].

Theorem 1.2 (Corollary 5.8). *The slide expansion of a degree k homogeneous polynomial $f \in \mathrm{Pol}$ is given by*

$$f = \sum_{1 \le i_1 \le \cdots \le i_k} (\mathsf{D}_{i_1} \cdots \mathsf{D}_{i_k} f)\, \mathfrak{F}_{i_1,\ldots,i_k}.$$

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Table 1. *Divided difference formalisms.*

§	Monoid	Divided differences	Dual polynomials	Creation operators
3	Nil-Coxeter monoid S_∞	∂_i	Schubert polynomials $\mathfrak{S}_w$	$\mathsf{Z}x_i\mathsf{R}_i$ and (§5) B_i
4	Thompson monoid ThMon	$\mathsf{T}_i = \mathsf{R}_i\partial_i = \mathsf{R}_{i+1}\partial_i$	Forest polynomials $\mathfrak{P}_F$	$\mathsf{Z}x_i$ and (§5) B_i
	m-Thompson monoid $\mathsf{ThMon}^{\underline{m}}$	$\mathsf{T}_i^{\underline{m}} = \mathsf{T}_i\mathsf{R}_{i+1}^{m-1}$	m-forest polynomials $\mathfrak{P}_F^{\underline{m}}$	$\mathsf{Z}^{\underline{m}}x_i$ and (§5) $\mathsf{B}_i^{\underline{m}}$
5	Weakly increasing monoid WInc	$\mathsf{D}_i = \mathsf{R}_{i+1}^\infty\partial_i = \mathsf{R}_{i+1}^\infty\mathsf{T}_i$	Slide polynomials $\mathfrak{F}_{\mathbf{i}}$	B_i
		$\mathsf{D}_i^{\underline{m}} = \mathsf{R}_{i+1}^\infty\mathsf{T}_i^{\underline{m}}$	m-slide polynomials $\mathfrak{F}_{\mathbf{i}}^{\underline{m}}$	$\mathsf{B}_i^{\underline{m}}$
		$\mathsf{D}_i^{\underline{\infty}} = \mathsf{R}_{i+1}^\infty\mathsf{T}_i^{\underline{\infty}} = \mathsf{T}_i^{\underline{\infty}}$	Monomials $x_{\mathbf{i}}$	$\mathsf{B}_i^{\underline{\infty}}$

Associated to the D_i are a new family of operators we call 'slide creators' B_i that have the property that for any sequence $a_1, \ldots, a_k$ (not necessarily weakly increasing), we have

$$\mathfrak{F}_{a_1,\ldots,a_k} = \mathsf{B}_{a_k}\cdots\mathsf{B}_{a_1}(1),$$

and

$$\sum \mathsf{B}_i\partial_i = \sum \mathsf{B}_i\mathsf{T}_i = \sum \mathsf{B}_i\mathsf{D}_i = \mathrm{id}$$

on Pol^+, i.e., they function as creation operators for Schubert polynomials, forest polynomials, and slide polynomials themselves simultaneously. Using these facts, we obtain the known slide polynomial expansions of Schubert and forest polynomials.

1.1. Outline of the paper

See Table 1 for an overview of where we address each family of polynomials we consider in the paper. In §2, we set up the notion of divided difference pairs and study creation operators and code maps. In §3, we study Schubert polynomials. In §4, we study forest polynomials, including m-forest polynomials. In §5, we study slide polynomials and m-slide polynomials, which include monomials as a limiting case.

2. Divided differences and creation operators

We describe a general framework which encodes the duality between ∂_i and $\mathfrak{S}_w$. In our framework, the pair (∂, S_∞) will be called a divided difference pair (dd-pair for short), and $\{\mathfrak{S}_w \mid w \in S_\infty\}$ will be called a 'dual family of polynomials' to this dd-pair. The two main mathematical insights are as follows.

1. The existence of certain 'creation operators' leads to explicit formulas for the dual polynomials, assuming the dual family of polynomials exist.
2. Creation operators together with a 'code map' show that the dual polynomials exist, without needing to verify any particular Ansatz or combinatorial model that interacts well with the operators.

These considerations are new and interesting even in the case of Schubert polynomials. For example, because we have the $\mathsf{Z}x\mathsf{R}$ creation operators mentioned in the introduction, we will see in §3 that the existence of the Lehmer code on permutations immediately implies that Schubert polynomials exist without any Ansatz or direct verification that the $\mathsf{Z}x\mathsf{R}$ recursion interacts well with the ∂_i operators. In later sections, we will apply this formalism to other families of polynomials.

Remark 2.1. The operators and families of polynomials of interest to us in this paper have integer coefficients, so we will set everything up over $\mathbb{Z}$. This will exclude certain parts of the $\frac{d}{dx_i}$ example from the introduction because of the denominators present in the normalized monomials $S_c = \frac{\mathrm{x}^c}{c!}$. However, all of the theorems we have work equally well over $\mathbb{Q}$, and we will indicate through this section how such modifications apply to this particular example.

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2.1. Partial monoids and polynomial representations

We start by recalling some notions on partial monoids: these will encode the combinatorics of relations between families of operators.

A *partial monoid* M is a set equipped with a partial product map $M \times M \dashrightarrow M$ denoted by concatenation, together with a unit 1, such that $1m = m1 = m$ for all $m \in M$, and $m(m'm'') = (mm')m''$ for any m, m', m'', in the sense that either both products are undefined, or both are defined and equal.

Remark 2.2. We have a monoid when the map is total – that is, when products are always defined. Given a partial monoid M, one forms a monoid on the one-element extension $M \sqcup \{\mathbf{0}\}$ by setting $mm' = \mathbf{0}$ when the product is undefined in M, and if m or m' is $\mathbf{0}$. The notions of partial monoids and monoids with zero are thus essentially equivalent.

A *polynomial representation* of M is a map $\Phi : M \to \mathrm{End}(\mathrm{Pol})$ assigning an endomorphism of Pol to each element of M such that $\Phi(1) = \mathrm{id}$ and such that for $u, v \in M$, we have

$$\Phi(u)\Phi(v) = \begin{cases} \Phi(uv) & \text{if } uv \text{ is defined} \\ 0 & \text{otherwise.} \end{cases}$$

A partial monoid M is *graded* if there is a length function $\ell : M \to \{0, 1, 2, \ldots\}$ such that $\ell(uv) = \ell(u) + \ell(v)$ whenever uv is defined. We write $M_k \subset M$ for those elements of degree k. We always have $M_0 = \{1\}$, and we write $M_1 = \{a_i\}_{i \in I}$ for some indexing set I. If a graded partial monoid is generated in degree 1, then the length $\ell(w)$ for $w \in M$ is the common length k of all expressions $m = a_{i_1} \cdots a_{i_k}$. For such a partial monoid, we write $\mathrm{Fac}(w)$ for the set of $(i_1, \ldots, i_k)$ such that $w = a_{i_1} \cdots a_{i_k}$, and for $w \in M_k$, we write $\mathrm{Last}(w)$ for the set of i such that $w = w'a_i$ for some $w' \in M_{k-1}$. If such a w' is always unique, then we say furthermore that M is *right-cancellative*, and we denote this element by w/i. Finally, we say that such an M has finite factorizations if we always have $|\mathrm{Fac}(w)| < \infty$ (or equivalently if we always have $|\mathrm{Last}(w)| < \infty$).

2.2. Divided difference pairs

We now formalize the relationship between the divided difference operators ∂_i and the partial monoid S_∞ in what we call a ‘divided difference pair’ (dd-pair). It is not our goal to give the most general results possible but to have a formalism that encompasses all examples we want to treat while being possibly useful in other situations.

We fix a polynomial endomorphism $X \in \mathrm{End}(\mathrm{Pol})$ that is of degree -1 (i.e., X takes degree d homogeneous polynomials to degree $d-1$ homogeneous polynomials for all d).

For any $i \geq 1$, we define the shifted operator $X_i \in \mathrm{End}(\mathrm{Pol})$ by the composition

$$X_i : \mathrm{Pol} \cong \mathrm{Pol}_{i-1} \otimes \mathrm{Pol} \to \mathrm{Pol}_{i-1} \otimes \mathrm{Pol} \cong \mathrm{Pol},$$

where the first and last isomorphisms are given by the isomorphism

$$\mathrm{Pol}_{i-1} \otimes \mathrm{Pol} = \mathbb{Z}[x_1, \ldots, x_{i-1}] \otimes \mathbb{Z}[x_i, x_{i+1}, \ldots] \cong \mathrm{Pol},$$

and the middle map is given by $\mathrm{id} \otimes X$. In particular, $X = X_1$ and we always have

$$f \in \mathrm{Pol}_n \implies X_{n+1} f = X_{n+2} f = \cdots = 0, \tag{2.1}$$

since in this case, X acts on constants and thus vanishes as it has degree -1.

Example 2.3. If we set $\partial \in \mathrm{End}(\mathrm{Pol})$ to be the first divided difference

$$\partial(f) = \frac{f(x_1, x_2, x_3, \ldots) - f(x_2, x_1, x_3, \ldots)}{x_1 - x_2}, \tag{2.2}$$

then ∂_i agrees with (1.2).

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Note that ∂ is called the divided difference operator because the formula involves dividing a difference by a linear form. The way in which the various X we consider in later sections arise will also be from taking two degree 0 operators $A, B \in \mathrm{End}(\mathrm{Pol})$ such that $(A - B)f$ is always divisible by a linear form L, and then setting $X = \frac{A-B}{L}$.

Writing dd for divided difference, we call X and the X_i *dd-operators* even if they do not necessarily arise in this way in general.

Definition 2.4. We define a *divided difference pair* (or a dd-pair) to be the data of (X, M), where M is a graded right-cancellative partial monoid, generated in degree 1 by $\{a_i\}_{i\geq 1}$, such that the map $a_i \mapsto X_i$ is a representation of M. For $w \in M$, we write X_w for the associated endomorphism of Pol, and in particular, we have $X_i = X_{a_i}$.

Example 2.5. If we set $M = S_\infty$ with its partial monoid structure given by (1.1), ∂ as in (2.2), then the divided difference representation $s_i \mapsto \partial_i$ makes (∂, S_∞) into a dd-pair.

Example 2.6. For any degree -1 polynomial endomorphism X, we have (X, M) is a dd-pair for M the free monoid on $\{1, 2, \ldots\}$.

Example 2.7. Codes is a monoid via componentwise addition, and we have a representation given by $i \mapsto \frac{d}{dx_i}$ because $\frac{d}{dx_i}\frac{d}{dx_j} = \frac{d}{dx_j}\frac{d}{dx_i}$. Therefore, $(\frac{d}{dx}, \mathsf{Codes})$ is a dd-pair, and for $c = (c_1, \ldots, c_k, 0, \ldots)$, we have $\left(\frac{d}{dx}\right)_c = \left(\frac{d}{dx_1}\right)^{c_1} \cdots \left(\frac{d}{dx_k}\right)^{c_k}$.

We are especially interested in the case where M encodes all additive relations between compositions of the operators X_i. However, this is a hard thing to show in general, so we do not want to assume it from the beginning. It will actually follow from the formalism we now introduce (see Theorem 2.20).

2.3. *Dual families of polynomials to a dd-pair*

We now generalize the relation between $\mathfrak{S}_w$ and the ∂_i to an arbitrary dd-pair (X, M).

Definition 2.8. A family $(S_w)_{w\in M}$ of homogeneous polynomials in Pol is *dual* to a dd-pair (X, M) if $S_1 = 1$, and for each $w \in M$ and $i \in \{1, 2, \ldots\}$, we have

$$X_i S_w = \begin{cases} S_{w/i} & \text{if } i \in \mathrm{Last}(w) \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.9. The Schubert polynomials $\{\mathfrak{S}_w \mid w \in S_\infty\}$ are dual to the dd-pair (∂, S_∞).

Example 2.10. If we had defined everything over $\mathbb{Q}$ instead of $\mathbb{Z}$, then $\{\frac{x^c}{c!} \mid c \in \mathsf{Codes}\}$ would be dual to to the dd-pair $(\frac{d}{dx}, \mathsf{Codes})$.

The terminology is justified by item (4) of the following result.

Proposition 2.11. *If a dd-pair (X, M) has a dual family $\{S_w \mid w \in M\}$, then*

(1) *M has finite factorizations.*
(2) *The polynomials S_w are $\mathbb{Z}$-linearly independent.*
(3) *The representation of $\mathbb{Z}[M]$ is faithful:*

$$\sum c_w X_w = 0 \implies c_w = 0 \text{ for all } w.$$

In particular, M is the partial monoid of compositions generated by the operators X_i.

(4) *Letting $\mathrm{ev}_0 : \mathrm{Pol} \to \mathbb{Z}$ be the map $f \mapsto f(0, 0, \ldots)$, we have $\mathrm{ev}_0 X_v S_w = \delta_{v,w}$. As a consequence, for $f \in \mathbb{Z}\{S_w \mid w \in M\}$, the $\mathbb{Z}$-span of the S_w, we have*

$$f = \sum_{w\in M} (\mathrm{ev}_0 X_w f) S_w. \tag{2.3}$$

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Proof. First, note that for $(i_1, \ldots, i_k) \in \mathrm{Fac}(w)$, we have $X_{i_1} \cdots X_{i_k} S_w = S_1 = 1$. Now we know that for any polynomial f, there are only finitely many X_i such that $X_i f \neq 0$. Applying this repeatedly, we see there are only finitely many sequences $(i_1, \ldots, i_k)$ such that $X_{i_1} \cdots X_{i_k} S_w \neq 0$. Therefore, $|\mathrm{Fac}(w)| < \infty$, and (1) is proved.

The defining relations for S_w imply that $X_v S_w = S_u$ if there exists a $u \in M$ (necessarily unique by right-cancellability) such that $w = vu$, and 0 otherwise. Since S_u is homogeneous of degree $\ell(u)$, we have $\mathrm{ev}_0\, S_u = \delta_{1,u}$, so $\mathrm{ev}_0\, X_v S_w = \delta_{v,w}$, establishing the first part of (4). This implies that the linear functionals $\{\mathrm{ev}_0\, X_v \mid v \in M\}$ are dual to the family of polynomials $\{S_w \mid w \in M\}$, so the polynomials $\{S_w \mid w \in M\}$ are linearly independent and the linear functionals $\{\mathrm{ev}_0\, X_w \mid w \in M\}$ are linearly independent, establishing (2) and (3). Finally, for f in the $\mathbb{Z}$-span of the S_w, if we write $f = \sum b_v S_v$, then applying $\mathrm{ev}_0\, X_w$ to both sides shows $b_w = \mathrm{ev}_0\, X_w f$ which implies the reconstruction formula (2.3). □

Example 2.12. We give an example of a dd-pair whose dual family does not span Pol. Let $\partial' = \partial_2$. For the dd-pair (∂', S_∞) where $s_i \mapsto (\partial')_i = \partial_{i+1}$, for each $\lambda \in \mathbb{Z}$, we can construct a dual family of polynomials $S_w^{(\lambda)} = \mathfrak{S}_w(\lambda x_1 + x_2, \lambda x_1 + x_3, \ldots)$. For no λ does this family of polynomials span Pol since x_1 is not in the span of the linear polynomials.

Example 2.13. The analogue of Proposition 2.11 still holds if we had used $\mathbb{Q}$ instead of $\mathbb{Z}$ in our setup. In this case, the existence of the dual family of monomials $\frac{\mathsf{x}^c}{c!}$ to the dd-pair $(\frac{d}{dx}, \mathsf{Codes})$ shows that the representation of Codes is faithful, and (2.3) recovers the Taylor expansion of any rational polynomial f:

$$f = \sum_c \left(\mathrm{ev}_0 \left(\frac{d}{dx} \right)_c f \right) \frac{\mathsf{x}^c}{c!}.$$

2.4. Creation operators and code maps

Given a dd-pair, an outstanding remaining question is whether they do admit a dual family of polynomials S_w. We give an answer in several cases of interest, using the existence of certain 'creation operators'.

Definition 2.14. We define *creation operators* for the operator X to be a collection of degree 1 polynomial endomorphisms $Y_i \in \mathrm{End}(\mathrm{Pol})$ such that on the ideal $\mathrm{Pol}^+ \subset \mathrm{Pol}$, we have the identity

$$\sum_{i=1}^{\infty} Y_i X_i = \mathrm{id}. \tag{2.4}$$

We will further say that a dd-pair (X, M) has creation operators when the operator X does.

Note that the left-hand side of (2.4) is well defined thanks to (2.1).

Remark 2.15. Note that the left-hand side of (2.4) vanishes on $\mathbb{Z}$, so the identity extends uniquely to Pol by subtracting ev_0 from the right-hand side (i.e., it reads $\sum_{i=1}^{\infty} Y_i X_i = \mathrm{id} - \mathrm{ev}_0$).

Proposition 2.16. *If a dd-pair (X, M) has creation operators Y_i and a family of dual polynomials $\{S_w \mid w \in M\}$, then for $w \in M$, we have*

$$S_w = \sum_{(i_1,\ldots,i_k) \in \mathrm{Fac}(w)} Y_{i_k} \cdots Y_{i_1}(1). \tag{2.5}$$

Proof. M has finite factorizations by Proposition 2.11, so the right-hand side in (2.5) is well defined. To prove it, we induct on the length $k = \ell(w)$. For $k = 0$, this is the identity $S_1 = 1$, and for $k > 0$, we have

$$\begin{aligned} S_w = \sum_{i=1}^{\infty} Y_i X_i S_w = \sum_{i \in \mathrm{Last}(w)} Y_i S_{w/i} &= \sum_{i \in \mathrm{Last}(w)} \sum_{(i_1,\ldots,i_{k-1}) \in \mathrm{Fac}(w/i)} Y_i Y_{i_{k-1}} \cdots Y_{i_1}(1) \\ &= \sum_{(i_1,\ldots,i_k) \in \mathrm{Fac}(w)} Y_{i_k} \cdots Y_{i_1}(1). \end{aligned}$$ □

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An immediate consequence is that if a dd-pair has creation operators, it has at most one dual family of polynomials. The creation operators are not unique in general, and this leads to possibly distinct expansions of S_w as we will see in later sections.

Example 2.17. If we had used $\mathbb{Q}$ instead of $\mathbb{Z}$ in our setup, then for $(\frac{d}{dx}, \mathsf{Codes})$, we can take Y_i to act on homogeneous polynomials of degree k by $Y_i(f) = \frac{1}{k+1} x_i f$ for all k. Then (2.4) holds as it is Euler's famous theorem $\sum x_i \frac{d}{dx_i} = k\,\mathrm{id}$ on homogeneous polynomials of positive degree k. For $c = (c_i)_{i\geq 1} \in \mathsf{Codes}$, we have $\mathrm{Fac}(c) = \{(i_1, \ldots, i_k) \mid c_p = \#\{1 \leq j \leq k \mid i_j = p\}\}$, and (2.5) recovers the formula $S_c = \frac{x^c}{c!}$ for the unique candidate family of polynomials satisfying (1.3).

Let us give an example now to show that the existence of creation operators is not enough to ensure the existence of a dual family of polynomials.

Example 2.18. Define X by linearly extending the assignments $X(x_i) = \delta_{i,1}$ for all $i \geq 1$, and some degree -1 injection Φ on monomials of degree d to monomials of degree $d-1$ for each $d \geq 2$. We can assume that x_1 does not occur in the range of Φ, by applying the shift $x_i \mapsto x_{i+1}$ if necessary. X has the following creation operators Y_i: on the constant polynomials, Y_i is multiplication by x_i. On Pol^+, define $Y_2 = Y_3 = \cdots = 0$ while Y_1 equals Φ^{-1} on monomials in the range of Φ, and 0 on the remaining monomials.

If $(S_w)_{w\in M}$ is dual to some dd-pair (X, M), we have $S_{a_1} = x_1$. Now $a_1 \cdot a_1$ is defined in M since $X_1^2 = X^2$ is nonzero, and we have $X_1(S_{a_1\cdot a_1}) = x_1$. This is not possible by our assumption on Φ, and thus, (X, M) does not have a dual family.

We now give an easily checkable hypothesis on M that ensures that the dual polynomials do in fact exist and, furthermore, form a basis of Pol.

Let Codes denote the set of finitely supported sequences of nonnegative integers $c = (c_1, c_2, \ldots)$. For $c \in \mathsf{Codes}$, write $\mathrm{supp}\, c$ for the set of i such that $c_i \neq 0$, and $|c|$ for the sum of the nonzero entries. Let M be a graded right cancellable monoid.

Definition 2.19. A *code map* for M is an injective map $c : M \to \mathsf{Codes}$ such that $\ell(w) = |c(w)|$ and $\max \mathrm{supp}\, c(w) = \max \mathrm{Last}(w)$ for all $w \in M$. (In particular, M has finite factorizations.)

We note that the existence of a code map is trivially seen to be equivalent to the condition that

$$\#\{w \in M \mid \ell(w) = n \text{ and } \max \mathrm{Last}(w) = k\} \leq \#\{c \in \mathsf{Codes} \mid |c(w)| = n \text{ and } \max \mathrm{supp}\, c(w) = k\},$$

but in practice, verifying code maps exist seems to be more straightforward than checking this inequality by other means.

Theorem 2.20. *Suppose that a dd-pair (X, M) has creation operators and a code map. Then*

(1) *The code map is bijective.*
(2) *There is a unique dual family $(S_w)_{w\in M}$ defined by (2.5). It is a basis of* Pol.
(3) *The subfamily $(S_w)_w$ where* $\max \mathrm{supp}\, c(w) \leq d$ *is a basis of* Pol_d *for any* $d \geq 0$.

Proof. Define recursively $S_1 = 1$ and

$$S_w = \sum_{i\in \mathrm{Last}(w)} Y_i S_{w/i}.$$

By Proposition 2.16, the dual family of polynomials must be equal to $\{S_w \mid w \in M\}$ if it exists.

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We begin by addressing (1). Let

$$M_{k,d} = \{w \in M \mid \ell(w) = k \text{ and } \max\operatorname{supp} c(w) \leq d\}.$$

We claim that for $f \in \mathrm{Pol}_d^{(k)}$, the homogeneous degree k polynomials in Pol_d, we have

$$f = \sum_{w \in M_{k,d}} X_w(f) \mathfrak{S}_w. \tag{2.6}$$

By induction on k, we can show (2.6) but with $w \in M_{k,d}$ replaced with the condition $\ell(w) = k$ since

$$f = \sum_{i=1}^{\infty} Y_i X_i f = \sum_{i=1}^{\infty} Y_i \sum_{\ell(w')=k-1} (X_{w'} X_i f) \mathfrak{S}_{w'} = \sum_{\ell(w)=k} \sum_{i \in \mathrm{Last}(w)} Y_i (X_w(f) \mathfrak{S}_{w/i}) = \sum_{\ell(w)=k} X_w(f) \mathfrak{S}_w.$$

To conclude, it suffices to show that if $\ell(w) = k$ and $w \notin M_{k,d}$, then $X_w f = 0$; this is true because if $i = \max\operatorname{supp} c(w) > d$, then $i \in \mathrm{Last}(w)$ and so $X_w f = X_{w/i} X_i f = 0$.

Writing $\mathsf{Codes}_{k,d} = \{c \in \mathsf{Codes} \mid \max\operatorname{supp} c \leq d \text{ and } |c(w)| = k\}$, the code map induces an injection $M_{k,d} \to \mathsf{Codes}_{k,d}$ so $|M_{k,d}| \leq |\mathsf{Codes}_{k,d}|$. However, (2.6) implies the inclusion

$$\mathrm{Pol}_d^{(k)} \subset \mathbb{Z}\{\mathfrak{S}_w \mid w \in M_{k,d}\}, \tag{2.7}$$

so $|\mathsf{Codes}_{k,d}| = \operatorname{rank} \mathrm{Pol}_d^{(k)} \leq |M_{k,d}|$. We conclude that $|M_{k,d}| = |\mathsf{Codes}_{k,d}| = \operatorname{rank} \mathrm{Pol}_d^{(k)}$, implying (1) and the fact the $\mathfrak{S}_w$ are $\mathbb{Z}$-linearly independent.

Observe that (2.7) is a containment of equal rank free abelian groups. Furthermore, $\mathrm{Pol}_d^{(k)}$ is saturated (i.e., for any $\lambda \in \mathbb{Z}$, we have $\lambda f \in \mathrm{Pol}_d^{(k)}$ implies $f \in \mathrm{Pol}_d^{(k)}$), so the containment (2.7) is in fact an equality, and we conclude that $\{\mathfrak{S}_w \mid w \in M_{k,d}\}$ is a $\mathbb{Z}$-basis of $\mathrm{Pol}_d^{(k)}$. Taking the union of these bases for all k and fixed d shows that $\{\mathfrak{S}_w \mid \max\operatorname{supp} c(w) \leq d\}$ is a basis for Pol_d, which shows (3).

By considering these basis statements and the identity (2.6) for growing d, and using the fact that $\bigcup M_{k,d} = M$, we deduce that $\{\mathfrak{S}_w \mid w \in M\}$ is a basis of Pol, proving the second half of (2). For arbitrary $f \in \mathrm{Pol}$, we have the identity

$$f = \sum_{w \in M} (\mathrm{ev}_0 X_w f) \mathfrak{S}_w.$$

We thus infer that

(a) If $\mathrm{ev}_0 X_w f = 0$ for all $w \in M$, then $f = 0$, and
(b) $\mathfrak{S}_w$ is the unique polynomial such that $\mathrm{ev}_0 X_{w'} \mathfrak{S}_w = \delta_{w',w}$ for all $w' \in M$.

We are now ready to show that $X_i \mathfrak{S}_w = \delta_{i \in \mathrm{Last}(w)} \mathfrak{S}_{w/i}$ for any i and w.[1] If $w' \in M$, we have

$$\mathrm{ev}_0 X_{w'}(X_i \mathfrak{S}_w) = \mathrm{ev}_0 X_{w' \cdot i} \mathfrak{S}_w = \delta_{w, w' \cdot i}.$$

Here, the last two terms are considered as zero if $w' \cdot i$ is not defined. If $i \notin \mathrm{Last}(w)$, then $\delta_{w,w' \cdot i} = 0$ for all $w' \in M$, so we conclude by (a) that $X_i \mathfrak{S}_w = 0$. However, if $i \in \mathrm{Last}(w)$, then $\delta_{w,w' \cdot i} = \delta_{w/i,w'}$, which by (b) implies $X_i \mathfrak{S}_w = \mathfrak{S}_{w/i}$, as desired. □

Example 2.21. For $(\frac{d}{dx}, \mathsf{Codes})$, there is a code map on Codes given by the identity. Therefore, using $\mathbb{Q}$ instead of $\mathbb{Z}$ in our setup, we can conclude that $\{\mathfrak{S}_c = \frac{x^c}{c!} \mid c \in \mathsf{Codes}\}$ found in Example 2.17 is the dual family of polynomials to $(\frac{d}{dx}, \mathsf{Codes})$ without directly verifying the recursion (1.3).

[1]Here, $\delta_{i \in \mathrm{Last}(w)}$ is 1 if $i \in \mathrm{Last}(w)$, and 0 otherwise.

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3. Schubert polynomials

The divided difference $\partial_i \in \mathrm{End}(\mathrm{Pol})$ for $i = 1, 2, \ldots$ is defined as follows:

$$\partial_i f(x_1, x_2, \ldots) = \frac{f - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

The partial monoid M is given by the nil-Coxeter monoid S_∞ of permutations of $\{1, 2, \ldots\}$ fixing all but finitely many elements with partial product $u \circ v = uv$ if $\ell(u) + \ell(v) = \ell(uv)$, undefined otherwise: here, ℓ and uv are the lengths and product in the group S_∞. Denoting the simple transposition $s_i = (i, i+1)$, the corresponding dd-pair (∂, S_∞) comes from the representation $s_i \mapsto \partial_i$.

We have

$$\mathrm{Last}(w) = \mathrm{Des}(w) = \{i \mid w(i) > w(i+1)\},$$

and $\mathrm{Fac}(w) = \mathrm{Red}(w)$, the set of reduced words for w (i.e., the set of sequences $(i_1, \ldots, i_k)$ with $k = \ell(w)$ such that $w = s_{i_1} \cdots s_{i_k}$). The Lehmer code is the bijective map $S_\infty \to \mathsf{Codes}$ defined for $w \in S_\infty$ by $\mathrm{lcode}(w) = (c_1, c_2, \ldots)$, where $c_i = \#\{j > i \mid w(i) > w(j)\}$. Because $\mathrm{Des}(w) = \{i \mid c_i > c_{i+1}\}$, we have $\max \operatorname{supp} \mathrm{lcode}(w) = \max \mathrm{Last}(w)$, so this is a code map as in Definition 2.19.

The Schubert polynomials are the unique family of homogeneous polynomials dual to the dd-pair (∂, S_∞): we have $\mathfrak{S}_{\mathrm{id}} = 1$ and

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{w/i} & \text{if } i \in \mathrm{Des}(w) \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1 shows the application of various divided difference operators starting from $\mathfrak{S}_{1432}$.

The standard way the existence of Schubert polynomials is shown is through the Ansatz $\mathfrak{S}_{w_{0,n}} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ for $w_{0,n}$ the longest permutation in S_n. Because every $u \in S_\infty$ has $u \le w_{0,n}$ for some n, it turns out it suffices to check that $\partial_{w_{0,n-1}^{-1} w_{0,n}} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} = x_1^{n-2} x_2^{n-3} \cdots x_{n-2}$, which is done with direct calculation.

Using our setup, because there is a code map, we can simultaneously avoid the Ansatz and establish an explicit combinatorial formula by exhibiting creation operators for the ∂_i.

3.1. Creation operators for ∂_i

We now describe creation operators for ∂_i, which will give formulas for the Schubert polynomials. We define the *Bergeron–Sottile map* [5]

$$\mathsf{R}_i f(x_1, x_2, \ldots) = f(x_1, \ldots, x_{i-1}, 0, x_i, \ldots).$$

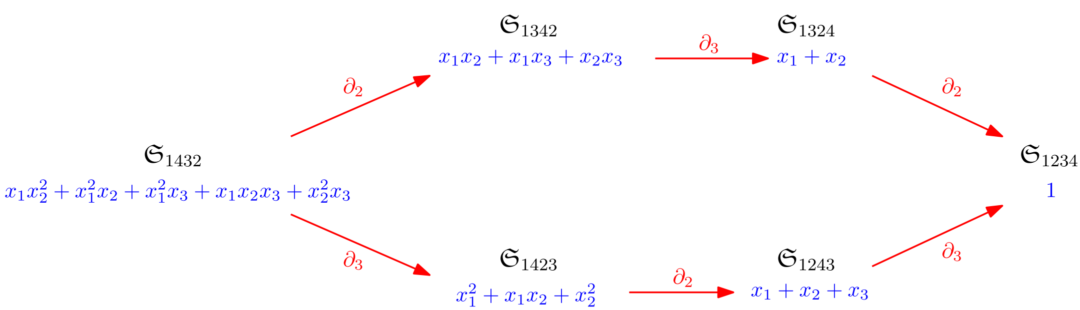

Figure 1. *Sequences of ∂_i applied to a $\mathfrak{S}_w$.*

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Lemma 3.1. *We have*

$$\sum_{i\geq 1} x_i \mathsf{R}_i \partial_i = \mathrm{id} - \mathsf{R}_1.$$

Proof. We sum the relation $x_i \mathsf{R}_i \partial_i = \mathsf{R}_{i+1} - \mathsf{R}_i$ for all $i \geq 1$. □

We define

$$\mathsf{Z} = \mathrm{id} + \mathsf{R}_1 + \mathsf{R}_1^2 + \cdots : \mathrm{Pol}^+ \to \mathrm{Pol}^+.$$

Corollary 3.2. *We have that* $\mathsf{Z} x_i \mathsf{R}_i$ *are creation operators for the dd-pair given by the usual divided differences* ∂_i *and the nil-Coxeter monoid. That is, the identity*

$$\sum_{i\geq 1} \mathsf{Z} x_i \mathsf{R}_i \partial_i = \mathrm{id}$$

holds on Pol^+*. In particular, Schubert polynomials exist, and we have the following monomial positive expansion:*

$$\mathfrak{S}_w = \sum_{(i_1,\ldots,i_k)\in \mathrm{Red}(w)} \mathsf{Z} x_{i_k} \mathsf{R}_{i_k} \cdots \mathsf{Z} x_{i_1} \mathsf{R}_{i_1}(1).$$

Proof. We compute $\mathsf{Z}\sum_{i\geq 1} x_i \mathsf{R}_i \partial_i = Z(\mathrm{id} - \mathsf{R}_1) = (\mathrm{id} - \mathsf{R}_1) + \mathsf{R}_1(\mathrm{id} - \mathsf{R}_1) + \cdots = \mathrm{id}$. □

Example 3.3. Take $w = 14253$ so that $\mathrm{Red}(w) = \{324, 342\}$. Adopting the shorthand $\mathsf{ZxR}_{\mathbf{i}}$ for composite $\mathsf{Z} x_{i_k} \mathsf{R}_{i_k} \cdots \mathsf{Z} x_{i_1} \mathsf{R}_{i_1}$ where $\mathbf{i} = (i_1, \ldots, i_k)$, one gets

$$\begin{aligned}
\mathsf{ZxR}_{(3,2,4)}(1) &= \mathsf{ZxR}_{(2,4)}(x_1 + x_2 + x_3) = \mathsf{ZxR}_{(4)}(x_1x_2 + x_1^2 + x_2^2) = x_1x_2x_4 + x_1^2x_4 + x_1^2x_3 + x_2^2x_4\\
\mathsf{ZxR}_{(3,4,2)}(1) &= \mathsf{ZxR}_{(4,2)}(x_1 + x_2 + x_3) = \mathsf{ZxR}_{(2)}(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)\\
&= x_1x_2^2 + x_1x_2x_3 + x_2^2x_3 + x_1^2x_2.
\end{aligned}$$

On adding the two right-hand sides, one obtains the Schubert polynomial $\mathfrak{S}_{14253}$.

Remark 3.4. The slide expansion of Schubert polynomials [3, 6], reproved in Proposition 5.7, expresses $\mathfrak{S}_w$ as a sum of slide polynomials over $\mathrm{Red}(w)$. Corollary 3.2 also provides an expression where the sum ranges over $\mathrm{Red}(w)$, but these two decompositions are in fact distinct, as the preceding example reveals as neither $\mathsf{ZxR}_{(3,2,4)}(1)$ nor $\mathsf{ZxR}_{(3,4,2)}(1)$ equals a slide polynomial.

3.2. Pipe dream interpretation

We now relate the preceding results to a simple bijection at the level of pipe dreams. Consider the staircase $\mathsf{Stair}_n := (n, n-1, \ldots, 1)$ whose columns are labeled 1 through n left to right. Given $w \in S_n$, a (reduced) pipe dream for w is a tiling of Stair_n using 'cross' and 'elbow' tiles depicted in Figure 2 so that the following conditions hold:

- The tilings form n pipes with the pipe entering in row i exiting via column $w(i)$ for all $1 \leq i \leq n$;
- No two pipes intersect more than once.

Denote the set of pipe dreams for w by $\mathrm{PD}(w)$. Given $D \in \mathrm{PD}(w)$, attach the monomial

$$\mathsf{x}^D := \prod_{\mathrm{crosses}(i,j)\in D} x_i.$$

A famous result of Billey–Jockusch–Stanley [6] (see also [4, 8, 9]) then states that

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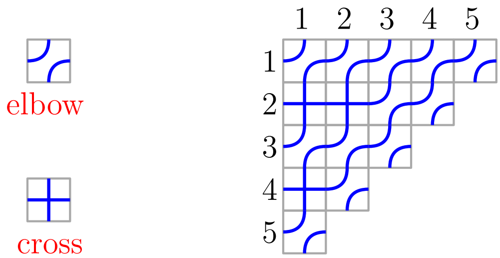

Figure 2. *Elbow and cross tiles (left) and a pipe dream for* $w = 14253$ *(right).*

Theorem 3.5. $\mathfrak{S}_w$ *is the generating polynomial for pipe dreams for* w*:*

$$\mathfrak{S}_w = \sum_{D \in \mathrm{PD}(w)} \mathsf{x}^D.$$

We will give a simple proof, using the recursion

$$\mathfrak{S}_w = \mathsf{R}_1 \mathfrak{S}_w + \sum_{i \in \mathrm{Des}(w)} x_i \mathsf{R}_i \mathfrak{S}_{ws_i}, \tag{3.1}$$

which follows immediately from Lemma 3.1 and the definition of Schubert polynomials.

Proof of Theorem 3.5. We need to show

$$\sum_{D \in \mathrm{PD}(w)} \mathsf{x}^D = \mathsf{R}_1 \sum_{D \in \mathrm{PD}(w)} \mathsf{x}^D + \sum_{i \in \mathrm{Des}(w)} x_i \mathsf{R}_i \sum_{D \in \mathrm{PD}(ws_i)} \mathsf{x}^D.$$

Say that a pipe dream $D \in \mathrm{PD}(w)$ is uncritical if there are no crosses in column 1, and i-critical if the last cross in column 1 is in row i. Denote $\mathrm{PD}(w)^0 \subset \mathrm{PD}(w)$ for the set of uncritical pipe dreams, and $\mathrm{PD}(w)^i \subset \mathrm{PD}(w)$ for the set of i-critical pipe dreams.

Note that if $i \geq 1$ and $\mathrm{PD}(w)^i$ is nonempty, then $i \in \mathrm{Des}(w)$ since pipes i and $i+1$ cross at the location of this last cross in column 1. Because $\mathrm{PD}(w) = \bigsqcup \mathrm{PD}(w)^i$, it suffices to show that

(a) $\sum_{D \in \mathrm{PD}(w)^0} \mathsf{x}^D = \mathsf{R}_1 \sum_{D \in \mathrm{PD}(w)} \mathsf{x}^D$ and
(b) for $i \in \mathrm{Des}(w)$ we have $\sum_{D \in \mathrm{PD}(w)^i} \mathsf{x}^D = x_i \mathsf{R}_i \sum_{D \in \mathrm{PD}(ws_i)} \mathsf{x}^D$.

To see (a), we note there is a weight-preserving bijection

$$\Phi_0 : \mathrm{PD}(w)^0 \to \{D \in \mathrm{PD}(w) \mid D \text{ has no crosses in row } 1\},$$

given by shifting all crosses one unit diagonally southwest. Since $\mathsf{x}^D = \mathsf{R}_1 x^{\Phi_0(D)}$, we have (a).

To see (b), we note there is a bijection

$$\Phi_i : \mathrm{PD}(w)^i \to \{D \in \mathrm{PD}(ws_i) \mid \text{D has no crosses in row } i\}$$

obtained by turning the last cross in column 1 into an elbow and then shifting all crosses in rows i and below one unit diagonally southwest. See Figure 3 for an illustration. As $\mathsf{x}^D = x_i \mathsf{R}_i \mathsf{x}^{\Phi_i(D)}$, we have (b). □

Remark 3.6. Since the image of ∂_i comprises polynomials symmetric in $\{x_i, x_{i+1}\}$, we can replace the $\mathsf{R}_i \partial_i$ in Lemma 3.1 by $\mathsf{R}_{i+1} \partial_i$. The recursion in (3.1) is then equivalent to

$$\mathfrak{S}_w = \mathsf{R}_1 \mathfrak{S}_w + \sum_{i \in \mathrm{Des}(w)} x_i \mathsf{R}_{i+1} \mathfrak{S}_{ws_i}. \tag{3.2}$$

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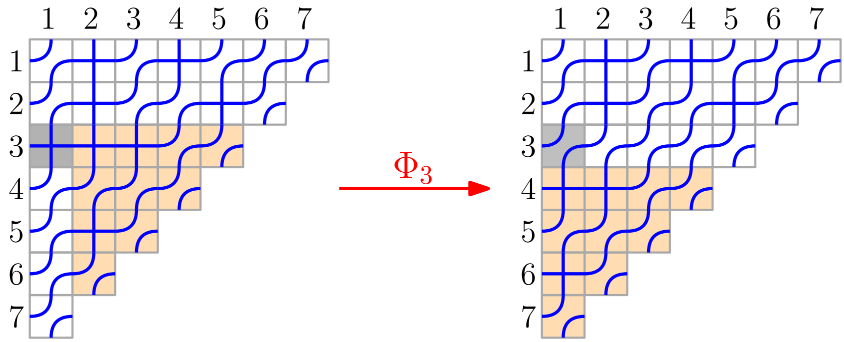

Figure 3. *A* 3*-critical pipe dream* D *for* $w = 1375264$ *(left), and* $\Phi_3(D) \in \mathrm{PD}(ws_3)$.

Dave Anderson has given a representation-theoretic proof [1] of the recursion in (3.2) using Kraśkiewicz–Pragacz modules [13, 14].

4. Forest polynomials

The *quasisymmetric divided difference* [22] is defined as

$$\mathsf{T}_i = \mathsf{R}_i \partial_i = \mathsf{R}_{i+1} \partial_i = \frac{\mathsf{R}_{i+1} - \mathsf{R}_i}{x_i}.$$

The associated dd-pair $(\mathsf{T}, \mathsf{For})$ from [22] comes from the monoid structure on the set For of plane indexed binary forests as we shall briefly recall.

A *rooted plane binary tree* T is a rooted tree with the property that every node has either no child, in which case we call it a *leaf,* or two children, distinguished as the 'left' and 'right' child, in which case we call it an *internal node*. We let $\mathrm{IN}(T)$ denote the set of internal nodes and let $|T| := |\mathrm{IN}(T)|$ be the *size* of T. The unique tree of size 0, whose root node is also its leaf node, is denoted by $*$. We shall call this the *trivial* tree.

An *indexed forest* F is a sequence $(T_i)_{i\geq 1}$ of rooted plane binary trees where all but finitely many T_i are trivial. If all T_i are trivial, then we call F the *empty forest* $\varnothing$. By identifying the leaves with $\mathbb{Z}_{\geq 1}$, going through them from left to right, one can depict an indexed forest as shown in Figure 4. We denote the set of indexed forests by For. Given $F \in \mathsf{For}$, we let $\mathrm{IN}(F) := \sqcup_i \mathrm{IN}(T_i)$ denote its set of internal nodes.

There is a natural monoid structure on For obtained by taking $F \cdot G$ to be the indexed forest where the i'th leaf of F is identified with the i'th root of G for all i. This monoid is generated by the smallest nontrivial forests $\underline{i}$ of size 1 with internal node having left leaf at i, and there is an identification of For with the (right-cancellable) Thompson monoid ThMon given by

$$\mathsf{For} \cong \mathsf{ThMon} = \langle 1, 2, \cdots \mid i \cdot j = j \cdot (i+1) \text{ for } i > j \rangle,$$

by identifying $\underline{i} \mapsto i$.

We may encode $F \in \mathsf{For}$ as elements of Codes as follows. Define $\rho_F : \mathrm{IN}(F) \to \mathbb{Z}_{\geq 1}$ by setting $\rho_F(v)$ equal to the label of the leaf obtained by going down left edges from v. Then the map $\mathsf{c} : \mathsf{For} \to \mathsf{Codes}$ sending $F \mapsto \mathsf{c}(F) = (c_i)_{i\geq 1}$ where $c_i = \{v \mid \rho_F(v) = i\}$ is a bijection [22, Theorem 3.6]. The set $\mathrm{Last}(w)$ is identified with the *left terminal set* of F as

$$\mathrm{LTer}(F) = \{i \mid c_i \neq 0 \text{ and } c_{i+1} = 0\},$$

which in particular immediately implies that $\max \operatorname{supp} c(F) = \max \mathrm{Last}(F)$, so c is a code map. We explain the choice of name. We call $v \in \mathrm{IN}(F)$ *terminal* if both its children are leaves, necessarily i and

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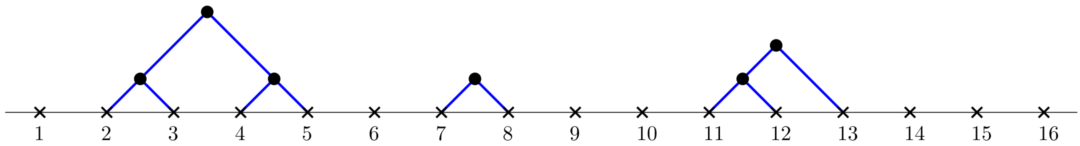

Figure 4. *An indexed forest F with* $\mathsf{c}(F) = (0, 2, 0, 1, 0, 0, 1, 0, 0, 0, 2, 0, 0, \ldots)$.

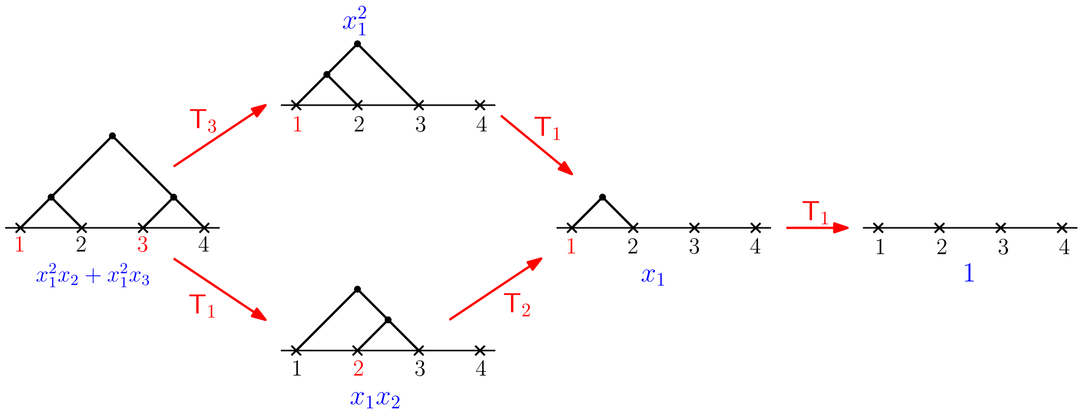

Figure 5. T_i *applied to various* $\mathfrak{P}_F$*, with elements of* $\mathrm{LTer}(F)$ *highlighted in red.*

$i + 1$ where $i := \rho_F(v)$. We then have $c_i \neq 0$ and $c_{i+1} = 0$ (i.e., $i \in \mathrm{LTer}(F)$). Thus, we can record terminal nodes by recording the label of their left leaf, which is what $\mathrm{LTer}(F)$ does.

For $F \in \mathsf{For}$ and $i \in \mathrm{LTer}(F)$, we call $F/i \in \mathsf{For}$ the *trimmed forest* (as in [22, §3.6]), which is obtained by deleting the terminal node v satisfying $\rho_F(v) = i$. The set of factorizations $\mathrm{Fac}(F)$ is then identified with the set of *trimming sequences* [22, Definition 3.8]:

$$\mathrm{Trim}(F) = \{(i_1, \ldots, i_k) \mid (((F/i_k)/i_{k-1})/\cdots)/i_1 = \varnothing\}.$$

Figure 5 (ignoring the polynomials in blue) shows repeated trimming operators applied to the indexed forest F on the left. It follows that $\mathrm{Trim}(F) = \{(1, 1, 3), (1, 2, 1)\}$.

4.1. Creation operators for T_i

We now describe creation operators for T_i.

Theorem 4.1. *We have* $\sum_{i\geq 1} \mathsf{Z}x_i\mathsf{T}_i = \mathrm{id}$ *on* Pol^+*, or in other words,* $\mathsf{Z}x_i$ *are creation operators for* T_i*. In particular, there is a family of 'forest polynomials'* $\mathfrak{P}_F$ *characterized by* $\mathfrak{P}_\varnothing = 1$ *and*

$$T_i\mathfrak{P}_F = \begin{cases} \mathfrak{P}_{F/i} & i \in \mathrm{LTer}(F) \\ 0 & \textit{otherwise,} \end{cases}$$

with the following monomial-positive expansion:

$$\mathfrak{P}_F = \sum_{(i_1,\ldots,i_k)\in\mathrm{Trim}(F)} \mathsf{Z}x_{i_k}\cdots\mathsf{Z}x_{i_1}(1).$$

Proof. Corollary 3.2 already contains this identity in the form $\sum_{i\geq 1} \mathsf{Z}x_i\mathsf{R}_i\partial_i = \mathrm{id}$ on Pol^+. The rest follows from Theorem 2.20. □

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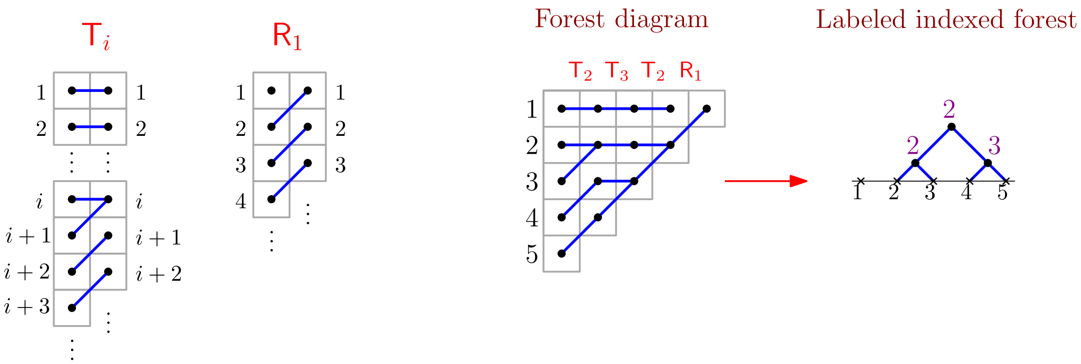

Figure 6. *The graphs corresponding to* T_i *and* R_1 *(left), and a forest diagram with the corresponding labeled indexed forest (right).*

Figure 5 shows the result of applying $\mathsf{T}_1\mathsf{T}_1\mathsf{T}_3$ and $\mathsf{T}_1\mathsf{T}_2\mathsf{T}_1$ to the forest polynomial $\mathfrak{P}_F = x_1^2x_2 + x_1^2x_3$. As per Theorem 4.1, each application of a T trims the indexed forest at that stage.

Example 4.2. We shall consider the indexed forest F whose corresponding forest polynomial $\mathfrak{P}_F$ is computed in [23, Example 3.9]. This happens to be equal to $\mathfrak{S}_{14253}$ from Example 3.3, but as we shall see, the decompositions are different. We have $\mathrm{Trim}(F) = \{(2,2,4),(2,3,2)\}$. Adopting the shorthand $\mathsf{Zx}_{\mathbf{i}}$ for the composite $\mathsf{Zx}_{i_k}\cdots\mathsf{Zx}_{i_1}$ where $\mathbf{i} = (i_1,\ldots,i_k)$, one gets

$$\mathsf{Zx}_{(2,2,4)}(1) = \mathsf{Zx}_{(2,4)}(x_1+x_2) = \mathsf{Zx}_{(4)}(x_1x_2+x_1^2+x_2^2) = x_1x_2x_4 + x_1^2x_4 + x_1^2x_3 + x_2^2x_4$$
$$\mathsf{Zx}_{(2,3,2)}(1) = \mathsf{Zx}_{(3,2)}(x_1+x_2) = \mathsf{Zx}_{(2)}(x_1x_2+x_1x_3+x_2x_3) = x_1x_2^2 + x_1x_2x_3 + x_2^2x_3 + x_1^2x_2.$$

Thus, we find that $\mathfrak{P}_F$ is the sum of the two right-hand sides. Observe that even though two final expressions above align with those computed in Example 3.3, the expressions obtained at the intermediate stages are not the same.

4.2. Diagrammatic Interpretation

We now give a diagrammatic perspective on forest polynomials that evokes the pipe dream perspective on Schubert polynomials. By applying the relation $\mathsf{R}_1 + \sum_{i\geq 1} x_i\mathsf{T}_i = \mathrm{id}$ from Corollary 3.2 to forest polynomials, we obtain

$$\mathsf{R}_1\mathfrak{P}_F + \sum_{i\in\mathrm{LTer}(F)} x_i\mathfrak{P}_{F/i} = \mathfrak{P}_F. \tag{4.1}$$

This identity was previously obtained in [23, Lemma 3.12]. Unwinding this recursion leads to the following combinatorial model similar to the pipe dream expansion of Schubert polynomials, which can be matched up without much difficulty to the combinatorial definitions of forest polynomials in [22, 23].

We will represent each of the operators R_1 and $\mathsf{T}_1,\mathsf{T}_2,\ldots$ as a certain graph on a $(\mathbb{Z}_{\geq 1}\times 2)$-rectangle as shown in Figure 6 on the left. Consider the grid $\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}$ where we adopt matrix notation (i.e. the elements in the grid are $(i,j)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}$, where the first coordinate increases top to bottom and the second coordinate increases left to right).

We define a *forest diagram* to be any graph on vertex set $\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 1}$ such that the subgraph induced on the vertex set $\{(p,q)\mid p\in\mathbb{Z}_{\geq 1}, q\in\{k,k+1\}\}$ either represents T_i for some positive integer i or represents R_1, and such that for p large enough, all such induced subgraphs represent R_1. In particular, we may without loss of information restrict our attention to the finite subgraph on the vertex set $\{(i,j)\mid i+j\leq n+1\}$ for some n. See on the right in Figure 6 for an example. Given any such diagram

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D, we let $\mathrm{nodes}(D)$ denote the set of (i, j), where we have (i, j) directly connected to both $(i, j-1)$ and $(i+1, j-1)$, and associate a monomial

$$\mathsf{x}^D := \prod_{(i,j)\in \mathrm{nodes}(D)} x_i.$$

Note that any such graph is necessarily acyclic and naturally corresponds to an indexed forest, as shown in Figure 6. For $F \in \mathsf{For}$, let $\mathrm{Diag}(F)$ denote the set of diagrams whose underlying forest is F.

Theorem 4.3. *For $F \in \mathsf{For}$, we have the forest diagram formula*

$$\mathfrak{P}_F = \sum_{D\in \mathrm{Diag}(F)} \mathsf{x}^D.$$

Proof. We give a brisk proof sketch that the claimed expansion satisfies (4.1) along the lines of the proof of Theorem 3.5.

Call $D \in \mathrm{Diag}(F)$ *i-critical* if the subgraph induced on $\{(j,1),(j,2) \mid j \geq 1\}$ represents T_i for some positive integer i. Otherwise, we call D *uncritical*, in which case the aforementioned subgraph necessarily represents R_1. Note that if D is i-critical, then $i \in \mathrm{LTer}(F)$.

Denote by $\mathrm{Diag}(F)^0$ the set of uncritical forest diagrams, and by $\mathrm{Diag}(F)^i$ the set of i-critical forest diagrams. Consider the weight-preserving bijection

$$\Phi_0 : \mathrm{Diag}(F)^0 \to \{D \in \mathrm{Diag}(F) \mid \text{no element of nodes(D) is in row 1}\}$$

given by shifting all nodes one unit diagonally southwest. Clearly, $\mathsf{x}^D = \mathsf{R}_1 \mathsf{x}^{\Phi_0(D)}$.

Consider next the bijection

$$\Phi_i : \mathrm{Diag}(F)^i \to \{D \in \mathrm{Diag}(F/i)\}$$

given by taking the subgraph induced on vertices (p,q) with $p \geq 1, q \geq 2$. That is, we ignore vertices of the form $(p, 1)$ as well as all incident edges. It is easily seen that $\mathsf{x}^D = x_i\, \mathsf{x}^{\Phi_i(D)}$. □

4.3. *m-forest polynomials*

We now briefly touch upon the more general family of m-forest polynomials defined combinatorially in [22], where the $m = 1$ case recovers the forest polynomials from earlier. By replacing binary forests with $(m+1)$-ary forests, there is an analogously defined set For^m whose compositional monoid structure is analogously identified with the m-Thompson monoid

$$\mathsf{For}^m \cong \mathsf{ThMon}^{\underline{m}} := \langle \mathsf{T}_1^{\underline{m}}, \mathsf{T}_2^{\underline{m}}, \ldots \mid \mathsf{T}_i^{\underline{m}}\mathsf{T}_j^{\underline{m}} = \mathsf{T}_j^{\underline{m}}\mathsf{T}_{i+m}^{\underline{m}} \text{ for } i > m\rangle.$$

All of the combinatorics and constructions stated specifically for For carry over with minor modifications.

In the terminology of the present paper, the m-forest polynomials $\{\mathfrak{P}_F^{\underline{m}} \mid F \in \mathsf{For}^m\}$ are the unique family of polynomials dual to the dd-pair $(\mathsf{T}^{\underline{m}}, \mathsf{For}^m)$ given by *m-quasisymmetric divided differences*

$$\mathsf{T}_i^{\underline{m}} = \frac{\mathsf{R}_{i+1}^m - \mathsf{R}_i^m}{x_i}.$$

These polynomials were shown to exist in [22] by a laborious explicit computation.

Like before, [22, Definition 3.5] guarantees a code map for $\mathsf{ThMon}^{\underline{m}}$ in the sense of Definition 2.19. Thus, to show that m-forest polynomials exist, it suffices to find creation operators. This is a straightforward adaptation of the proof for $m = 1$. Let's define $\mathsf{Z}^{\underline{m}} = 1 + \mathsf{R}_1^m + \mathsf{R}_1^{2m} + \cdots : \mathrm{Pol}^+ \to \mathrm{Pol}^+$.

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Theorem 4.4. *We have* $\sum_{i\geq 1} \mathsf{Z}^{\underline{m}} x_i \mathsf{T}_i^{\underline{m}} = \mathrm{id}$ *on* Pol^+*, or in other words,* $\mathsf{Z}^{\underline{m}} x_i$ *are creation operators for* $\mathsf{T}_i^{\underline{m}}$*. In particular, there exists a family of 'm-forest polynomials'* $\{\mathfrak{P}_F\}_{F\in\mathsf{For}^m}$ *dual to the dd-pair* $(\mathsf{T}^{\underline{m}}, \mathsf{ThMon}^{\underline{m}})$ *with the following monomial-positive expansion:*

$$\mathfrak{P}_F = \sum_{(i_1,\ldots,i_k)\in \mathrm{Trim}(F)} \mathsf{Z}^{\underline{m}} x_{i_k} \cdots \mathsf{Z}^{\underline{m}} x_{i_1}(1).$$

We will later see an expansion in terms of 'm-slides', a natural generalization of slide polynomials introduced in [22, Section 8].

5. Slide polynomials and Slide expansions

In this section we will show that slide polynomials are dual to a simple dd-pair. We use this to recover the slide polynomial expansions of Schubert polynomials [6, 3] and forest polynomials [23], and to obtain a simple formula for the coefficients of the slide expansion of any $f \in \mathrm{Pol}$.

5.1. Slide polynomials

For a sequence $a = (a_1, \ldots, a_k)$ with $a_i \geq 1$, we define the set of *compatible sequences*

$$\mathcal{C}(a) = \{(i_1 \leq \cdots \leq i_k) : i_j \leq a_j, \text{ and if } a_j < a_{j+1} \text{ then } i_j < i_{j+1}\}. \tag{5.1}$$

Note that this convention is the opposite of what the authors employed in [22]. As we shall soon see, this convention arises naturally from the new dd-pair we will shortly create.

We define the *slide polynomial* to be

$$\mathfrak{F}_a = \sum_{(i_1,\ldots,i_k)\in\mathcal{C}(a)} x_{i_1}\cdots x_{i_k}.$$

Example 5.1. For $\mathfrak{F}_{(1,4,3)}$, we have $\mathcal{C}(a) = \{(1,2,2), (1,2,3), (1,3,3)\}$, so

$$\mathfrak{F}_{(1,4,3)} = x_1x_2^2 + x_1x_2x_3 + x_1x_3^2.$$

Let $\mathsf{WInc} = \{(a_1 \leq \cdots \leq a_k) \mid a_i \geq 1 \text{ for } 1 \leq i \leq k\}$. For a sequence a, we define $\overline{a} \in \mathsf{WInc}$ as the (component-wise) maximal element of $\mathcal{C}(a)$, and undefined if $\mathcal{C}(a)$ is empty. Then it is easily checked that $\mathfrak{F}_a = \mathfrak{F}_{\overline{a}}$ if $\overline{a}$ is defined, and $\mathfrak{F}_a = 0$ otherwise. For instance, note that for $a = (1,4,3)$ in Example 5.1, we have $\overline{a} = (1,3,3)$. The combinatorial construction of $\overline{a}$ from a is already present in [25, Lemma 8]; see also [24]. As shown by Assaf and Searles, the slides $\{\mathfrak{F}_a \mid a \in \mathsf{WInc}\}$ form a basis of Pol [3, Theorem 3.9]. Note that the slides *ibid.* are indexed by $c \in \mathsf{Codes}$, via the bijection with WInc given by letting c_j be the number of indices i such that $a_i = j$.

5.2. Slide extractors and creators

We define a partial monoid structure on WInc by

$$(a_1, \cdots a_k)\cdot(b_1,\ldots,b_\ell) = \begin{cases} (a_1,\ldots,a_k,b_1,\ldots,b_\ell) & \text{if } a_k \leq b_1 \\ \text{undefined} & \text{otherwise.}\end{cases}$$

This makes WInc into a graded right-cancellative monoid with $\mathrm{Last}((b_1,\ldots,b_k)) = \{b_k\}$ and $\mathrm{Fac}((b_1,\ldots,b_k)) = \{(b_1,\ldots,b_k)\}$.

Let R_i^∞ be the *truncation* operator defined by $\mathsf{R}_i^\infty(f) = f(x_1,\ldots,x_{i-1},x_i,0,0,\ldots)$. It is the limit of $\mathsf{R}_i^m(f)$ when m tends to infinity, as these polynomials clearly become stable equal to $\mathsf{R}_i^\infty(f)$.

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Definition 5.2 (Slide extractor). Define the *slide extractor* to be

$$\mathsf{D}_i = \mathsf{R}_{i+1}^{\infty}\partial_i,$$

which for $f \in \mathrm{Pol}$ is given concretely by

$$\mathsf{D}_i f = \frac{f(x_1,\ldots,x_{i-1},x_i,0,0,\ldots) - f(x_1,\ldots,x_{i-1},0,x_i,0,\ldots)}{x_i}.$$

We have $\mathsf{D}_j f \in \mathrm{Pol}_j$, thus $\partial_i \mathsf{D}_j = 0$ if $i > j$, and so $\mathsf{D}_i\mathsf{D}_j = 0$. Thus, the operators D_i give a representation of WInc, and with $\mathsf{D} = \mathsf{D}_1$, we have a dd-pair $(\mathsf{D}, \mathsf{WInc})$.

Theorem 5.3. *Slide polynomials* $(\mathfrak{F}_a)_{a\in\mathsf{WInc}}$ *form the unique dual family of polynomials to the dd-pair* $(\mathsf{D}, \mathsf{WInc})$*. Thus, for* $(b_1 \le \cdots \le b_k) \in \mathsf{WInc}$*, we have*

$$\mathsf{D}_i\,\mathfrak{F}_{b_1,\ldots,b_k} = \delta_{i,b_k}\mathfrak{F}_{b_1,\ldots,b_{k-1}}.$$

Note that the formula above can be checked directly by a simple computation, as we have an explicit expansion for slide polynomials. We will instead use Theorem 2.20, and this will come as a consequence.

Definition 5.4. Define a linear map $\mathsf{B}_i \in \mathrm{End}(\mathrm{Pol})$ as

$$\mathsf{B}_i = \sum_{1\le k\le i} x_k \mathsf{R}_k^{i-k}\mathsf{R}_{i+1}^{\infty}.$$

Explicitly, B_i vanishes outside of Pol_i and is defined on monomials of Pol_i by

$$\mathsf{B}_i(x_1^{p_1}\cdots x_j^{p_j}x_i^{p}) = x_1^{p_1}\cdots x_j^{p_j}\Big(\sum_{j<k\le i} x_k^{p+1}\Big),$$

where $p_j > 0$ or $j = 0$.

Proposition 5.5. *The* B_i *are creation operators for* D_i*: on* Pol^+*, we have*

$$\sum_{i\ge1}\mathsf{B}_i\mathsf{D}_i = \mathrm{id}.$$

Proof. On the one hand, since $\mathsf{R}_1^{\infty} = \mathrm{ev}_0$ vanishes on Pol^+, we obtain by telescoping

$$\sum_{r\ge1}(\mathsf{R}_{r+1}^{\infty} - \mathsf{R}_r^{\infty}) = \mathrm{id}. \tag{5.2}$$

Now, we compute that

$$\begin{aligned}(\mathsf{R}_{r+1}^{\infty} - \mathsf{R}_r^{\infty})f &= f(x_1,\ldots,x_r,0,\ldots) - f(x_1,\ldots,x_{r-1},0,\ldots)\\ &= \sum_{j\ge0} f(x_1,\ldots,x_{r-1},0^j,x_r,0,\ldots) - f(x_1,\ldots,x_{r-1},0^{j+1},x_r,0,\ldots)\\ &= \sum_{j\ge0}\mathsf{R}_r^j\,\mathsf{R}_{r+j+1}^{\infty}(x_{r+j}-x_{r+j+1})\,\partial_{r+j}f\\ &= \sum_{j\ge0}x_r\mathsf{R}_r^j\,\mathsf{R}_{r+j+1}^{\infty}\partial_{r+j}f\\ &= \sum_{j\ge0}(x_r\mathsf{R}_r^j\,\mathsf{R}_{r+j+1}^{\infty})\,\mathsf{D}_{r+j}f.\end{aligned}$$

Summing this over all r, the coefficient of $\mathsf{D}_i f$ is then $\sum_{1\le k\le i} x_k\mathsf{R}_k^{i-k}\mathsf{R}_{i+1}^{\infty} = \mathsf{B}_i$. □

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Our next result, Proposition 5.6, applied to increasing sequences $1 \leq a_1 \leq \cdots \leq a_k$ implies that the slide polynomials are the dual family of polynomials to $(\mathsf{D}, \mathsf{WInc})$. We note that although we could have taken an alternate choice of creation operators such as $\widetilde{\mathsf{B}}_i = \sum_{1\leq k\leq i} x_k \mathsf{R}_k^{i-k}$ (because $\mathsf{R}_{i+1}^{\infty}\mathsf{D}_i = \mathsf{D}_i$), Proposition 5.6 shows surprisingly that composites of the B_i operators construct slide polynomials even for non-decreasing sequences – a property not formally guaranteed by the slide polynomials being the dual family to $(\mathsf{D}, \mathsf{WInc})$. This additional property of B_i will be needed later in Proposition 5.7 to recover the slide expansions of Schubert and forest polynomials.

Proposition 5.6. *For any sequence $(a_1, \ldots, a_k)$ with $a_i \geq 1$, we have*

$$\mathfrak{F}_{a_1,\ldots,a_k} = \mathsf{B}_{a_k} \cdots \mathsf{B}_{a_1}(1).$$

Proof. By induction, it is enough to show that if $a = (a_1, \ldots, a_k)$, then $\mathsf{B}_p \mathfrak{F}_a = \mathfrak{F}_{a_1,\ldots,a_k,p}$ for any $p \geq 1$. In what follows, we write λ^ℓ for the length ℓ sequence $\lambda, \ldots, \lambda$. For $(i_1, \ldots, i_k) \in \mathcal{C}(a)$, we define a set

$$A_{(i_1,\ldots,i_k)} = \begin{cases} \varnothing & \text{if } i_k > p \\ \{(i_1, \ldots, i_\ell, i^{k-\ell+1}) \mid i_\ell < i \leq p\} & \text{if } (i_1, \ldots, i_k) = (i_1, \ldots, i_\ell, p^{k-\ell}) \text{ with } i_\ell < p. \end{cases}$$

Then by definition of B_p and the slide polynomials as generating functions, it suffices to show that

$$\bigsqcup_{(i_1,\ldots,i_k)\in\mathcal{C}(a)} A_{(i_1,\ldots,i_k)} = \mathcal{C}(a_1, \ldots, a_k, p).$$

First, the $A_{(i_1,\ldots,i_k)}$ are obviously disjoint sets, the elements being uniquely determined by the longest initial subsequence of $(i_1, \ldots, i_k)$ strictly less than p, so the union is disjoint as claimed. Next, we show $A_{(i_1,\ldots,i_k)} \subset \mathcal{C}(a_1, \ldots, a_k, p)$. Indeed, since $(i_1, \ldots, i_\ell, p^{k-\ell}) \in \mathcal{C}(a)$,

- ◦ if $\ell < k$, we must have $a_k \geq p$, and so $(i_1, \ldots, i_\ell, p^{k-\ell+1}) \in \mathcal{C}(a_1, \ldots, a_k, p)$, and
- ◦ if $\ell = k$, then because $p > i_\ell$, we also have $(i_1, \ldots, i_\ell, p^{k-\ell+1}) \in \mathcal{C}(a_1, \ldots, a_k, p)$.

The other sequences $(i_1, \ldots, i_\ell, i^{k-\ell+1}) \in A_{(i_1,\ldots,i_k)}$ must lie in $\mathcal{C}(a_1, \ldots, a_k, p)$ as well since it is a smaller sequence with the same indices at which strict ascents occur.

Finally, every sequence in $\mathcal{C}(a_1, \ldots, a_k, p)$ can be written as $(i_1, \ldots, i_\ell, i^{k-\ell+1})$ for some $0 \leq \ell \leq k$ and $i_\ell < i \leq p$, and we claim that $(i_1, \ldots, i_\ell, p^{k-\ell}) \in \mathcal{C}(a)$. Note that because the last $k - \ell + 1$ elements of $(i_1, \ldots, i_\ell, i^{k-\ell+1})$ are equal, we have $a_{k-\ell} \geq a_{k-\ell+1} \geq \cdots \geq a_k \geq p$. Therefore, as $(i_1, \ldots, i_\ell, p^{k-\ell+1})$ has the same indices of strict ascents as $(i_1, \ldots, i_\ell, i^{k-\ell+1})$, we have the sequence $(i_1, \ldots, i_\ell, p^{k-\ell+1}) \in \mathcal{C}(a_1, \ldots, a_k, p)$, which in particular implies that $(i_1, \ldots, i_\ell, p^{k-\ell}) \in \mathcal{C}(a)$. □

We can now prove Theorem 5.3.

Proof of Theorem 5.3. We have the code map $c : \mathsf{WInc} \to \mathsf{Codes}$ given by $c(a_1 \leq \cdots \leq a_k) = (c_1, c_2, \ldots)$, where $c_i = \#\{j \mid a_j = i\}$. It satisfies the conditions of Definition 2.19. The B_i are shown to be creation operators for D in Proposition 5.5. We can thus apply Theorem 2.20, which gives us that the dual family to $(\mathsf{D}, \mathsf{WInc})$ is unique, forms a basis of Pol, and is given explicitly by $\mathsf{B}_{a_k} \cdots \mathsf{B}_{a_1}(1)$ for $(a_1, \ldots, a_k) \in \mathsf{WInc}$. These are precisely the slide polynomials by Proposition 5.6, which concludes the proof. □

5.3. Applications

We first show how to recover the slide expansions of Schubert polynomials and forest polynomials, the first one being the celebrated BJS formula [6].

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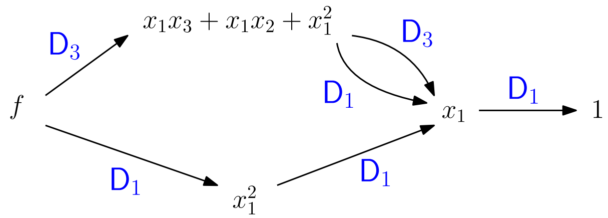

Figure 7. *Repeatedly applying* Ds *to extract slide coefficients for* $f = \mathfrak{S}_{21534}$.

Proposition 5.7. *We have the following expansions for any* $w \in S_\infty$ *and any* $F \in \mathsf{For}$*:*

$$\mathfrak{S}_w = \sum_{(i_1,\ldots,i_k)\in \mathrm{Red}(w)} \mathfrak{F}_{i_1,\ldots,i_k}$$
$$\mathfrak{P}_F = \sum_{(i_1,\ldots,i_k)\in \mathrm{Trim}(F)} \mathfrak{F}_{i_1,\ldots,i_k}.$$

Proof. Note that $\mathsf{B}_i\mathsf{D}_i = \mathsf{B}_i\mathsf{R}_{i+1}^\infty \partial_i$. Because $\mathsf{T}_i = \mathsf{R}_i\partial_i = \mathsf{R}_{i+1}\partial_i$, we can either absorb all or all but one R_{i+1} into B_i to obtain

$$\mathsf{B}_i\mathsf{D}_i = \mathsf{B}_i\mathsf{T}_i = \mathsf{B}_i\partial_i.$$

Then Proposition 5.5 shows that B_i are creation operators for ∂_i and for T_i. We can then use Theorem 2.20 for the corresponding dd-pairs:

$$\mathfrak{S}_w = \sum_{(i_1,\ldots,i_k)\in \mathrm{Red}(w)} \mathsf{B}_{i_k}\cdots\mathsf{B}_{i_1}(1) = \sum_{(i_1,\ldots,i_k)\in \mathrm{Red}(w)} \mathfrak{F}_{i_1,\ldots,i_k}$$
$$\mathfrak{P}_F = \sum_{(i_1,\ldots,i_k)\in \mathrm{Trim}(F)} \mathsf{B}_{i_k}\cdots\mathsf{B}_{i_1}(1) = \sum_{(i_1,\ldots,i_k)\in \mathrm{Trim}(F)} \mathfrak{F}_{i_1,\ldots,i_k}. \qquad \square$$

Because slide polynomials are a basis of Pol, Proposition 2.11 implies the following.

Corollary 5.8. *The slide expansion of a degree k homogeneous polynomial* $f \in \mathrm{Pol}$ *is given by*

$$f = \sum_{(i_1\le\cdots\le i_k)\in \mathsf{WInc}} (\mathsf{D}_{i_1}\cdots\mathsf{D}_{i_k} f)\, \mathfrak{F}_{i_1,\ldots,i_k}.$$

Example 5.9. Consider $f = \mathfrak{S}_{21534} = x_1x_3^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2 + x_1^3$. Figure 7 shows applications of slide extractors in weakly decreasing order of the indices. Corollary 5.8 says

$$\mathfrak{S}_{21534} = \mathfrak{F}_{1,3,3} + \mathfrak{F}_{1,1,3} + \mathfrak{F}_{1,1,1}.$$

As an application, let us reprove the positivity of slide multiplication established combinatorially by Assaf–Searles [3, Theorem 5.1] using the ‘quasi-shuffle product’. In contrast, we use a Leibniz rule for the D_i that makes the positivity manifest. We shall not pursue unwinding our approach to make the combinatorics explicit.

Lemma 5.10. $\mathsf{R}_j\mathfrak{F}_a$ *is a slide polynomial or* 0.

Proof. Assume the result is true for all lower degree slide polynomials. By Theorem 5.3, it suffices to show that $\mathsf{D}_i\mathsf{R}_j\mathfrak{F}_a = 0$ for all i, except at most one for which $\mathsf{D}_i\mathsf{R}_j\mathfrak{F}_a = \mathfrak{F}_b$ for some $b \in \mathsf{WInc}$.

Let $a = (a_1, \ldots, a_k) \in \mathsf{WInc}$, and let $a' = (a_1, \ldots, a_{k-1}) \in \mathsf{WInc}$. The identity

$$\mathsf{D}_i\mathsf{R}_j = \begin{cases} \mathsf{D}_i & \text{if } i \le j-2 \\ \mathsf{D}_i + \mathsf{R}_{j-1}\mathsf{D}_{i+1} & \text{if } i = j-1 \\ \mathsf{R}_j\mathsf{D}_{i+1} & \text{if } i \ge j, \end{cases}$$

together with Theorem 5.3 implies that

$$\mathsf{D}_i\mathsf{R}_j\mathfrak{F}_a = \begin{cases} \delta_{i,a_k}\mathfrak{F}_{a'} & \text{if } a_k \le j-1 \\ \delta_{i,a_k-1}\mathsf{R}_{j-1}\mathfrak{F}_{a'} & \text{if } a_k = j \\ \delta_{i,a_k-1}\mathsf{R}_j\mathfrak{F}_{a'} & \text{if } a_k \ge j+1, \end{cases}$$

and we conclude by the inductive hypothesis. □

Corollary 5.11. *The product of slide polynomials is slide-positive.*

Proof. By Corollary 5.8, it suffices to show that $\mathsf{D}_i(fg)$ is slide positive if each of f, g are slide positive. For $f, g \in \mathsf{Pol}$ we have a 'Leibniz rule' that says,

$$\mathsf{D}_i(fg) = \mathsf{D}_i(f)\mathsf{R}_{i+1}^{\infty}\mathsf{R}_i(g) + \mathsf{R}_{i+1}^{\infty}(f)\mathsf{D}_i(g). \tag{5.3}$$

If f, g are slide polynomials, then by Theorem 5.3, we know that $\mathsf{D}_i(f), \mathsf{D}_i(g)$ are either slide polynomials or 0, so from Lemma 5.10, the slide positivity follows by induction. □

Our second application is to determine the inverse of the 'Slide Kostka' matrix (i.e., express monomials in terms of slide polynomials). This was obtained by the first and third author via involved combinatorial means in [24, Theorem 5.2].

To state the result, fix a sequence $a = (a_1, \ldots, a_k) \in \mathsf{WInc}$. Group equal terms and write $a = (M_1^{m_1}, M_2^{m_2}, \ldots, M_p^{m_p})$, with $M_1 < \cdots < M_p$. Set $M_0 := 0$. For a fixed $i \in \{1, \ldots, p\}$, define $E_i(a) \subset \mathsf{WInc}$ by

$$E_i(a) = \{(b_1, \ldots, b_{m_i}) \mid b_{j+1} - b_j \in \{0,1\} = 0 \text{ and } b_1 > M_{i-1}\},$$

where $b_{m_i+1} := M_i$. Let $n(b) = M_i - b_1$ for $b \in E_i(a)$, which counts the number of j such that $b_{j+1} - b_j = 1$ for $1 \le j \le m_i$. Finally, let

$$E(a) = \{b \in \mathsf{WInc} \mid b = e^1 \cdots e^p \text{ where each } e^i \in E_i(a)\},$$

To $b = e^1 \cdots e^p \in E(a)$, assign the sign $\epsilon(b) = (-1)^{\sum_i n(e^i)}$. For instance, if $a = (2,4,4)$, then $E(a) = \{(2,4,4), (1,4,4), (2,3,4), (1,3,4), (2,3,3), (1,3,3)\}$ with respective signs $1, -1, -1, 1, -1, 1$.

Corollary 5.12 [24, Theorem 5.2]. *The slide expansion of any monomial is signed multiplicity-free. Explicitly, for any $a = (a_1, \ldots, a_k) \in \mathsf{WInc}$, we have*

$$x_{a_1} \cdots x_{a_k} = \sum_{b=(j_1,\ldots,j_k) \in E(a)} \epsilon(b)\mathfrak{F}_b. \tag{5.4}$$

Sketch of the proof. By Corollary 5.8, the coefficient of $\mathfrak{F}_{j_1,\ldots,j_k}$ for $(j_1, \ldots, j_k) \in \mathsf{WInc}$ in (5.4) is given by $\mathsf{D}_{j_1} \ldots \mathsf{D}_{j_k}(x_{a_1} \cdots x_{a_k})$. By Definition 5.2, we can compute

$$\mathsf{D}_{j_k}(x_{a_1} \cdots x_{a_k}) = \begin{cases} x_{a_1} \cdots x_{a_{k-1}} & \text{if } a_k = j_k \\ -x_{a_1} \ldots x_{a_p} x_{j_k}^{k-p-1} & \text{if } a_p < j_k, a_{p+1} = \cdots = a_k = j_k + 1 \text{ for some } p < k \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\mathsf{D}_{j_k}(x_{a_1}\cdots x_{a_k})$ is either 0 or another monomial up to sign, which shows that the expansion is signed multiplicity-free. More precisely, let $E'(a)$ be the set of $b = (j_1,\ldots,j_k)$ such that $\mathfrak{F}_b$ has nonzero coefficient in (5.4). Then it follows that $b \in E'(a)$ either if $j_k = a_k$ and $(j_1,\ldots,j_{k-1}) \in E'(a_1,\ldots,a_{k-1})$, or if $j_k + 1 = a_k$, there exists $p < k$ such that $a_p < j_k, a_{p+1} = \cdots = a_k = j_k + 1$, and $(j_1,\ldots,j_{k-1}) \in E'(a_1,\ldots,a_p, j_k^{k-p-1})$. We let the interested reader show that $E(a)$ satisfies the same recursion, so that $E(a) = E'(a)$ by induction. The sign is then readily checked.

5.4. m-slides interpolating between monomials and slides

To conclude this article, we briefly describe how the results generalize to monomials, m-slide polynomials and m-forest polynomials. The proofs are nearly identical to the case $m = 1$, so we omit them.

For a sequence $a = (a_1,\ldots,a_k)$ with $a_i \geq 1$, we define the set of *m-compatible sequences*

$$\mathcal{C}^m(a) = \{(i_1 \leq \ldots \leq i_k) : i_j \equiv a_j \bmod m,\ i_j \leq a_j, \text{ and if } a_j < a_{j+1} \text{ then } i_j < i_{j+1}\}. \tag{5.5}$$

The *m-slide polynomial* [22, Section 8] is the generating function

$$\mathfrak{F}^m_a = \sum_{(i_1,\ldots,i_k)\in\mathcal{C}^m(a)} x_{i_1}\cdots x_{i_k}. \tag{5.6}$$

For fixed $a = (a_1,\ldots,a_k)$ and m sufficiently large, we have $\mathfrak{F}^m_a = x_{a_1}\cdots x_{a_k}$ if $(a_1,\ldots,a_k) \in \mathsf{WInc}$ and 0 otherwise. So we may consider monomials as ∞-slide polynomials, and the m-slide polynomials as interpolating between slide polynomials and monomials.

Proposition 5.13. *For $i \geq 1$, consider the m-slide extractors $\mathsf{D}^m_i \in \mathrm{End}(\mathrm{Pol})$ defined as $\mathsf{D}^m_i := \mathsf{R}^\infty_{i+1}\mathsf{T}^m_i$. For $(b_1 \leq \cdots \leq b_k) \in \mathsf{WInc}$, we have*

$$\mathsf{D}^m_i\,\mathfrak{F}^m_{b_1,\ldots,b_k} = \delta_{i,b_k}\mathfrak{F}^m_{b_1,\ldots,b_{k-1}}.$$

Consequently, the m-slide expansion of a degree k homogeneous polynomial $f \in \mathrm{Pol}$ is given by

$$f = \sum_{(i_1\leq\cdots\leq i_k)\in\mathsf{WInc}} (\mathsf{D}^m_{i_1}\cdots\mathsf{D}^m_{i_k}f)\,\mathfrak{F}^m_{i_1,\ldots,i_k}.$$

Example 5.14. Taking $f = \mathfrak{S}_{21534} = x_1x_3^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2 + x_1^3$ as in Example 5.9, we see, for instance, that

$$\mathsf{D}^\infty_1\mathsf{D}^\infty_2\mathsf{D}^\infty_2(f) = \mathsf{D}^\infty_1\mathsf{D}^\infty_2(x_1x_2 + x_1^2) = \mathsf{D}^\infty_1(x_1) = 1,$$

which in turn means the coefficient of $x_1x_2^2$ in $\mathfrak{S}_{21534}$ is 1.

Theorem 5.15. *Consider m-slide creation operators $\mathsf{B}^m_a \in \mathrm{End}(\mathrm{Pol})$ that vanish outside of Pol_a, and are defined on monomials of Pol_a by*

$$\mathsf{B}^m_a(x_1^{p_1}\cdots x_j^{p_j}x_a^p) = x_1^{p_1}\cdots x_j^{p_j}\Big(\sum_{a-rm>j} x_{a-rm}^{p+1}\Big),$$

where $j < a$ and $p_j > 0$ (or $j = 0$) and $p \geq 0$. The following hold.

(1) *For $a = (a_1,\ldots,a_k)$ any sequence with $a_i \geq 1$, we have $\mathsf{B}^m_p\mathfrak{F}^m_a = \mathfrak{F}^m_{a_1,\ldots,a_k,p}$. In particular, for any sequence $(b_1,\ldots,b_k)$ with $b_i \geq 1$, we have*

$$\mathfrak{F}^m_{b_1,\ldots,b_k} = \mathsf{B}^m_{b_k}\cdots\mathsf{B}^m_{b_1}(1).$$

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(2) *We have* $\sum_{i=1}^{\infty} \mathsf{B}_i^m \mathsf{D}_i^m = \sum_{i=1}^{\infty} \mathsf{B}_i^m \mathsf{T}_i^m = \mathrm{id}$ *on* Pol^+ *(i.e.,* B_i^m *are creation operators for both m-slides and m-forest polynomials). In particular,*

$$\mathfrak{P}_F^m = \sum_{(i_1,\ldots,i_k)\in \mathrm{Trim}(F)} \mathfrak{F}_{i_1,\ldots,i_k}^m.$$

Remark 5.16. For $m = \infty$, we recover the rather straightforward dd-pair $(\mathsf{D}^\infty, \mathsf{WInc})$ for monomials, where for $a_k > 1$, we have $\mathsf{D}_i^\infty(x_1^{a_1}\cdots x_k^{a_k}) = \delta_{i,k} x_1^{a_1}\cdots x_k^{a_k-1}$, and the creation operators $\mathsf{B}_i^\infty(x_1^{a_1}\cdots x_k^{a_k}) = \delta_{i\geq k} x_i(x_1^{a_1}\cdots x_k^{a_k})$.

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