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CLASSIFICATION THEORY AND STATIONARY LOGIC

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0. Introduction. Stationary logic $L(\mathrm{aa})$ is obtained for $L_{\omega\omega}$ by adding a quantifier aa which ranges over countable sets and is interpreted to mean "for a closed unbounded set of countable subsets". The dual quantifier for aa is stat, i.e., stat $s\varphi(s)$ is equivalent to $\neg \mathrm{aa}\, s \neg \varphi(s)$. In the study of the $L(\mathrm{aa})$-model theory of structures a particular well behaved class was isolated, the finitely determinate structures. These are structures in which the quantifier "stat" can be replaced by the quantifier "aa" without changing the validity of sentences. Many structures such as **R** and all ordinals are finitely determinate. In this paper we will be concerned with *finitely determinate* first order theories, i.e., those theories all of whose models are finitely determinate.

Example 0.1. [**5**] The theory of dense linear orderings is not finitely determinate. Let S be a stationary costationary subset of ω_1 and

$$A = \sum_{\alpha<\omega} \tau_\alpha$$

where

$$\tau_\alpha = \begin{cases} \eta & \text{if } \alpha \in S \\ 1 + \eta & \text{if } \alpha \notin S. \end{cases}$$

Then

$$A \models \text{stat } s \text{ (sup } s \text{ exists)} \wedge \text{stat } s \neg \text{ (sup } s \text{ exists)}.$$

Example 0.2. [**2**] Any theory of modules is finitely determinate.

Combase [**1**] realised that the second example is an instance of a deeper phenomenon. He proved:

THEOREM 0.3. [**1**] *If T is a stable non-multidimensional theory, then T is finitely determinate.*

That there should be a connection between the hypothesis that T is finitely determinate and classification theory is further suggested by the fact that if a theory is finitely determinate some structure is imposed on its

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models. Suppose a structure

$$A = \bigcup_{i \in I} A_i,$$

where I is an ordered set and $i < j$ implies $A_i \subseteq A_j$. Then $(A_i)_{i \in I}$ is *order indiscernible* if for all

$$i_0 < \ldots < i_k < j_{k+1} < \ldots < j_n, \ i_k < l_{k+1} < \ldots < l_n \text{ and } \bar{a} \in A_{i_k},$$
$$(A, A_{i_0}, \ldots, A_{i_k}, A_{j_{k+1}}, \ldots, A_{j_n}, \bar{a})$$
$$\equiv (A, A_{i_0}, \ldots, A_{i_k}, A_{l_{k+1}}, \ldots, A_{l_n}, \bar{a}).$$

THEOREM 0.3. [**2**] *Suppose* $|A| = \omega_1$. *A is finitely determinate if and only if A is the union of an order indiscernible smooth chain of countable submodels.*

If T is finitely determinate, then all models of regular cardinality have the same sort of structure. To make this statement precise for any regular λ interpret aa_λ to mean "for a closed unbounded set of subsets of cardinality $<\lambda$". (This is how aa_λ is defined in [**2**]. This definition conflicts with the one in [**8**].) As was noted in [**2**] the axioms for $L(\mathrm{aa})$ remain valid for $L(\mathrm{aa}_\lambda)$. So if T is finitely determinate every model of T is also finitely determinate in the λ-interpretation (i.e., where aa_λ and stat_λ replace aa and stat). Also the analogue of Theorem 0.3 is true for aa_λ.

THEOREM 0.4. [**2**] *Suppose* $|A| = \lambda$ *and* λ *is regular. A is finitely determinate in the* λ *interpretation if and only if A is the union of an order indiscernible smooth chain of submodels of cardinality* $<\lambda$.

THEOREM 0.5. *Suppose T is finitely determinate, and* $A \models T$, *and* $|A| = \lambda$ *which is regular. Then A is the union of an order indiscernible chain of submodels of cardinality* $<\lambda$.

The division between finitely determinate theories and non-finitely determinate theories is a structure/non-structure division. Such divisions are the concern of classification theory (cf. [**10**]). In this paper we shall see there is a relation between classification theory and stationary logic.

In Section 1 we show

THEOREM 1.2. *If T is finitely determinate, then T is stable.*

The proof is a variant of Shelah's construction of many models [**9**]. Suppose $\aleph_0 < \lambda \leqq \mu$ and λ is regular. If S is a stationary subset of λ, then Shelah constructs a model of cardinality μ which has S (modulo the ideal of non-stationary sets) as an isomorphism invariant. In our version S is essentially $L(\mathrm{aa}_\lambda)$ definable.

In the second section we investigate which (necessarily stable) theories are finitely determinate.

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THEOREM 2.6. *If T is a superstable theory with* NDOP, *then T is finitely determinate.*

The proof uses a transfer theorem for $L(\text{aa})$ which reduces the problem to showing every a-model of some regular cardinality λ is finitely determinate in the λ-interpretation. The decomposition theorem for a-models of a superstable theory with NDOP allows us to link the finite determinacy of the model with the determinacy of a representing labelled tree. However since an element of the model may depend on some finite set of elements of the tree, we need a stronger notion than finite determinacy.

Definition. Suppose $|A| = \lambda$ and λ is a regular cardinal. A is *ω-determinate* in the λ interpretation if A is the union of a smooth chain $(A_\nu)_{\nu<\lambda}$ of submodels of cardinality $<\lambda$ such that for all

$$\nu_0 < \ldots < \nu_n < \tau_0 < \ldots < \tau_m,\ \nu_n < \sigma_0 < \ldots < \sigma_m \text{ and } \bar{a} \in A_{\nu_n},$$
$$(A, A_{\nu_0}, \ldots, A_{\nu_n}, A_{\tau_0}, \ldots, A_{\tau_m}, \bar{a})$$
$$\equiv^{\omega}_{\infty\omega}(A, A_{\nu_0}, \ldots, A_{\nu_n}, A_{\sigma_0}, \ldots, A_{\sigma_m}, \bar{a}).$$

In Section 3, we give a proof that the relevant trees are ω-determinate.

This paper has been written to require only a minimum amount of background from the reader. All the necessary facts about stationary logic will be stated. Section 1 (and Section 3 which is of limited independent interest) should be readable by most logicians. Section 2 involves stability concepts such as non-forking and a-saturation. Here a familiarity with the elementary parts of stability theory is assumed, say the contents of Sections A and B of [7]. Our notation is that of [7]. We also assume the reader is familiar with back and forth (or game theoretic) criteria for elementary equivalence and equivalence in infinitary languages (of [**6**]).

We conclude the introduction with a few remarks and examples. One question which might be asked "why do we restrict ourselves to superstable theories with NDOP?" It is easy to give examples of ω-stable theories with DOP which are not finitely determinate.

Example 0.6. There is an ω-stable theory (of Morley rank 2) which is not finitely determinate. (This example was also noted by Combase.)

Construction. A model of this theory is the disjoint union of infinite unary relations U and V. Also the model has a ternary relation $R \subseteq U \times U \times V$ where

$$(\{z : R(x, y, z)\})_{x,y \in U}$$

partitions V into infinite blocks. By varying the cardinalities of the blocks, any graph on U can be coded (i.e., the adjacency relation is $L(\text{aa})$-

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definable). So this theory has a non-finitely determinate model.

The restriction to superstable theories is a result of ignorance. Certainly there are finitely determinate theories which are stable but not superstable. Any stable not superstable theory of modules is such an example. Also the theory of ω infinitely refining equivalence relations is finitely determinate. (By Theorem 2.6 the reduct to any finite language is finitely determinate.) But this theory is not only unsuperstable but also multidimensional.

A third question asks whether these results can be extended to infinitary versions of determinacy. Combase [**1**] shows that every ω-stable non-multidimensional theory is α-determinate for all ordinals $\alpha < \omega_1$. (In fact, he shows something more.)

Example 0.7. There is a superstable non-multidimensional theory which is not ω-determinate (in the ω_1-interpretation).

Construction. We first define a model of the theory. The model is the disjoint union of two distinguished subsets $U_{\langle\ \rangle}$ and V. Further for every $s \in {}^{<\omega}2$, there is a subset U_s of $U_{\langle\ \rangle}$ and U_s is the disjoint union of $U_{s\langle 0\rangle}$ and $U_{s\langle 1\rangle}$. All the above sets are infinite. Let

$$V = \{ (\eta, \rho) : \eta, \rho \in {}^{\omega}2, \eta(0) = 0 \text{ and } \rho(0) = 1 \}.$$

For every $n > 1$ there is a ternary relation

$$R_n \subseteq U_{\langle 0\rangle} \times U_{\langle 1\rangle} \times V$$

where for all $(\eta, \rho) \in V$,

$$\{ (x, y) : R_n(x, y, (\eta, \rho)) \} = U_{\eta\restriction n} \times U_{\rho\restriction n}.$$

Suppose M is a model of this theory then M is determined by the following cardinal invariants: for all $\eta \in {}^{\omega}2$,

$$| \{ x \in M : \text{for all } n, x \in U_{\eta\restriction n}(M) \} |;$$

and for all $\eta, \rho \in {}^{\omega}2$,

$$| \{ z \in M : \text{for all } n, \{ (x, y) : R_n(x, y, z) \} = U_{\eta\restriction n} \times U_{\rho\restriction n} \} |.$$

So the theory is superstable and non-multimensional. Further with an infinitary formula namely

$$\bigwedge_{s \in {}^{<\omega}2} (U_s(x) \Leftrightarrow U_s(y)),$$

we can define the equivalence relation on $U_{\langle\ \rangle}$ which says x and y realize the same type. Then on the types using another infinitary formula we can choose a model which codes any bipartite graph.

This example can also be given in a finite language. Then the theory is not $\omega + \omega -$ determinate.

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1. Non-stable theories. Fix T a non-stable theory and $<$ a definable anti-symmetric relation witnessing the non-stability of T. (I.e., T has a model M in which $<$ linearly orders an infinite subset of M^n for some n. We can assume $n = 1$.) Expand T by Skolem functions. If I is a linear order, let $M(I)$ be the Ehrenfeucht-Mostowski model of T generated by the order indiscernible $\{a_i : i \in I\}$ where $a_i < a_j$ if $i < j$. (We leave it to the context to make the meaning of $<$ clear.)

Fix S a subset of ω_1 with $0 \in S$. Let

$$I = \sum_{\alpha \leqq \omega_1} I_\alpha,$$

where

$$I_\alpha = \begin{cases} \eta, \text{ if } \alpha \in S \text{ or } \alpha = \omega_1 \\ \eta_1 \cdot \eta, \text{ if } \alpha \notin S \cup \{\omega_1\}. \end{cases}$$

(We will explain how to avoid the use of CH later. Also $\eta_1 \cdot \eta$ denotes η_1 copies of η.) Call a subset $J \subseteq I$ *full* if for some limit ordinal $\alpha < \omega_1$

$$J = \sum_{\beta < \alpha} J_\beta + I_{\omega_1}$$

where $J_\beta = I_\beta$ if $\beta \in S$ and $J_\beta = X \cdot \eta$ for some countable $X \subseteq \eta_1$ otherwise. (Of course $J_\beta \subseteq I_\beta$.) Almost all subsets of $M(I)$ (i.e., a cub of countable subsets) are of the form $M(J)$ for some full J. From now on J will always denote a full set. Further the set

$$\left\{ M(J) : J = \sum_{\beta < \alpha} J_\beta + I_{\omega_1} \text{ and } \alpha \in S \right\}$$

is a stationary co-stationary subset of $\mathscr{P}_{\omega_1}(M(I))$ (providing S is a stationary co-stationary subset of ω_1).

Consider

$$J = \sum_{\beta < \alpha} J_\beta + I_{\omega_1}$$

a full subset of I. We will characterize in $M(I)$ by a formula of $L(\text{aa})$ when $\alpha \in S$. Define

$$[x, s] = \{ y \notin s: \text{ for all } z \in s, z < y \text{ if and only if } z < x \text{ and } z > y \text{ if and only if } z > x \}.$$

Define

$$\text{coin}[x, s] = \omega \text{ if and only if}$$

$$\text{aa } t \, \forall y \, \exists z (y \in [x, s] \rightarrow (t(z) \wedge z \in [x, s] \wedge z < y)).$$

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(Since $[x, s]$ is a definable relation, $\mathrm{coin}[x, s] = \omega$ can be expressed by an $L(\mathrm{aa})$-formula.) Suppose now $i \in I_\alpha$, then

$$\mathrm{coin}[a_i, M(J)] = \omega \text{ if and only if } \alpha \in S.$$

To see this note first that for all $j \in I_\alpha$,

$$a_j \in [a_i, M(J)].$$

By the order indiscernibility of $\{a_l : l \in I\}$, it is easy to see that if $b \in [a_i, M(J)]$ there is $j \in I_\alpha$ so that $a_j < b$. Hence if $\alpha \in S$ and so $|I_\alpha| = \omega$, then I_α witnesses

$$\mathrm{coin}[a_i, M(J)] = \omega.$$

Also if $\alpha \notin S$ then for all countable $B \subseteq [a_i, M(J)]$ there is $j \in I_\alpha$ so that $a_j < B$ (i.e., $a_j < b$ for all $b \in B$).

LEMMA 1.1. *For $M(I)$, $M(J)$ and α as above, $\alpha \in S$ if and only if*

$$\exists x(x \notin M(J) \wedge \mathrm{coin}[x, M(J)] = \omega).$$

Proof. Suppose $\alpha \notin S$ but for some $a \in M(I) \setminus M(J)$

$$\mathrm{coin}[a, M(J)] = \omega.$$

For notational simplicity we will write $\tau(i_0, \ldots, i_n)$ in place of $\tau(a_{i_0}, \ldots, a_{i_n})$. Choose terms

$$\tau_n(\bar{j}_n, \bar{i}_n) \in [a, M(J)] \quad (n \in \omega)$$

so that: for all $b \in [a, M(J)]$,

$$\tau_n(\bar{j}_n, \bar{i}_n) < b \text{ for some } n;$$

for all n, $\bar{j}_n \in J$ and $\bar{i}_n \in I \setminus J$. Consider any term

$$\tau(\bar{j}, l_0, \ldots, l_n) \in [a, M(J)]$$

where $\bar{j} \in J$ and $l_r \notin J$ for all r. Choose $(l_0^\nu, \ldots, l_n^\nu)$ $(\nu < \omega_1)$ from I so that: for all r, l_r and l_r^ν make the same cut in J; if $\nu < \mu$ and for some k l_r makes the same cut in J as l_k, then $l_r^\mu < l_k^\nu$; and for all $i \in I \setminus J$ if l_r and i make the same cut in J, then $l_r^\nu < i$ for some ν. (We assume $l_0, \ldots, l_n$ and $l_0^\nu, \ldots, l_n^\nu$ $(\nu < \omega_1)$ are increasing sequences.) So for all ν,

$$\tau(\bar{j}, l_0^\nu, \ldots, l_n^\nu) \in [a, M(J)].$$

Choose m so that

$$\tau_m(\bar{j}_m, \bar{i}_m) < \tau(\bar{j}, l_0^\nu, \ldots, l_n^\nu)$$

for uncountably many ν. Pick ν so that:

$$\tau_m(\bar{j}_m, \bar{i}_m) < \tau(\bar{j}, l_0^\nu, \ldots, l_m^\nu);$$

and for all $i \in \bar{i}_m$ and $r \leqq n$, if i and l_r^ν make the same cut in J then

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$l_r^\nu < i.$

Choose an increasing sequence $j_0, \ldots, j_n \in J$ so that for all $k < r j_r$ makes the same cut in $\bar{j}_m \cup \bar{i}_n \cup \bar{j}$ as l_r^ν; $j_r < l_r$; and if l_k and l_r make different cuts in J, $l_k < j_r$. (In other words $j_0, \ldots, j_n$ is obtained by shifting $l_0^\nu, \ldots, l_n^\nu$ slightly into J.)

Using indiscernibility, we can conclude that if we have $u_0, \ldots, u_n$, and $v_0, \ldots, v_n$ increasing sequences such that (1) the type of u_r, v_r over $\bar{j}$ is the same as that of l_r; (2) for all r and k if l_r and l_k make the same cut in J, then $u_r < v_k$; and (3) for $k < r$ if l_k makes a different cut in J than l_r (and hence is in a smaller cut), then $v_k < u_r$; then

$$\tau(\bar{j}, u_0, \ldots, u_n) < \tau(\bar{j}, v_0, \ldots, v_n).$$

Now repeat the argument above but this time choose the l_r^ν's so that for all r and k if l_r and l_k make the same cut in J then for all $\nu < \mu$ $l_r^\nu < l_r^\mu$; and for all $i \in I \setminus J$ if l_r and i make the same cut in J then for some ν $l_r^\nu > i$. This time we can conclude that if we have $u_0, \ldots, u_n$, and $v_0, \ldots, v_n$ increasing sequences such that (1) the type of u_r, v_r over $\bar{j}$ is the same as that of l_r; (2) for all r and k if l_r and l_k make the same cut in J, then $v_k < u_r$; and (3) for $k < r$ if l_k makes a different cut in J than l_r (and hence is in a smaller cut), then $u_k < v_r$; then

$$\tau(\bar{j}, u_0, \ldots, u_n) < \tau(\bar{j}, v_0, \ldots, v_n).$$

These two conclusions contradict each other.

THEOREM 1.2. *If T is finitely determinate, then T is stable.*

Proof. Assume T is not stable. Then T is consistent with

$$\text{stat } s\, \exists\, x(\neg(x \in s) \wedge \text{coin}[x, s] = \omega)$$
$$\wedge \text{ stat } s \neg \exists\, x(\neg(x \in s) \wedge \text{coin}[x, s] = \omega).$$

Since this consistency is absolute, the assumption of CH causes no problem.

Remark. In the construction η_1 could be replaced by any ordering $(Y, <)$ (not necessarily of cardinality ω_1) such that: the cofinality and coinitiality of Y is $\geqq \omega_1$; there is $\mathscr{C}$ a cub of subsets of Y of cardinality $< \omega_1$ so that for all $Z \in \mathscr{C}$ and $y \notin Z$ the coinitiality and cofinality of

$$\{u \in Y | u \text{ and } y \text{ realize the same cuts in } Z\}$$

is $\geqq \omega_1$. We would then modify the definition of

$$J = \sum_{\beta < \alpha} J_\beta + I_{\omega_1}$$

being *full* to require $J_\beta = X \cdot \eta$ where $X \in \mathscr{C}$ and $\beta \notin S$. We will comment on the construction of such orders below.

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THEOREM 1.3. **[9]** *For any uncountable regular cardinal κ and cardinal $\mu \geqq \kappa$, if T is unstable then T has 2^κ models of cardinality μ.*

Proof. We first need to show that there is a linear ordering $(Y, <)$ of cardinality μ satisfying the property above where κ replaces ω_1. First we construct such an ordering Z of cardinality κ. Define $Z_\alpha (\alpha < \kappa)$, a chain of linear orders of cardinality $<\kappa$, by induction on α. Let Z_0 be any linear order of cardinality $<\kappa$. At limit ordinals, we take unions. Suppose Z_α has been defined. Choose $Z_{\alpha+1} \supseteq Z_\alpha$ so that: $|Z_{\alpha+1}| < \kappa$; for all $\beta < \alpha$ and $x \notin Z_\alpha - Z_\beta$ there are $y_0, y_1 \in Z_{\alpha+1}$ which make the same cut in Z_β as x but y_0 is less than (y_1 is greater than) any element of Z_α making the same cut in Z_β; and there is y_0, y_1 so that $y_0 < (y_1 >)$ any element of Z_α. Let $Z = \cup Z_\alpha$. It is easy to see Z is the desired linear ordering (and that $\{Z_\alpha : \alpha < \kappa\}$ is the desired cub). Let

$$Y = (\mu + \mu^*) \cdot Z.$$

Here μ^* denotes the reverse ordering of μ.

Now for $S \subseteq \kappa$, let

$$I(S) = \sum_{\alpha \leqq \kappa} I_\alpha$$

where

$$I_\alpha = \begin{cases} \eta, \text{ if } \alpha \notin S \text{ or } \alpha = \kappa \\ Y \cdot \eta \text{ if } \alpha \notin S \cup \{\kappa\}. \end{cases}$$

Just as in Lemma 1.1, S is determined (up to equivalence modulo the non-stationary sets) in $M(I(S))$ by a formula of $L(\mathrm{aa}_\kappa)$. So if S and S' are non-equivalent stationary sets, then $M(I(S))$ is not isomorphic to $M(I(S'))$. Since there are 2^κ pairwise non-equivalent stationary subsets of κ (cf. **[4]**, p. 59), we are done.

Superstable theories with NDOP. In this section it will be shown that superstable theories with NDOP have only finitely determinate models. Essentially the proof involves three ingredients: Shelah's tree decomposition theorem for a-models; the transfer theorem for L(aa); and a result on the infinitary determinacy of labelled trees. The transfer theorem allows us to consider only a-models of some large cardinality. Shelah's tree decomposition theorem says that every a-model (of a superstable theory with NDOP) can be represented as a labelled tree. The elementary structure of the a-model is carried by the quantifier rank ω structure of the tree. Finally it is shown that labelled trees of large enough cardinality have a smooth quantifier rank ω structure.

The following theorem is a simple corollary of the proof of Theorem 1.3 in **[8]**.

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THEOREM 2.1. *Suppose T is a first order theory and $\lambda^{\underset{-}{\lambda}} = \lambda$ for some uncountable λ. Then T is finitely determinate if and only if every* a*-model of cardinality λ^+ is a finitely determinate in the λ^+-interpretation.*

By a *tree* we will mean a poset order-isomorphic to a subposet of the poset of finite sequences of a fixed set ordered by "initial segment of" with a single minimal element, $\langle\ \rangle$. We can assume any tree is closed under subsequences. A *tree of sets* $A = \langle A_\eta \rangle_{\eta \in I}$, indexed by a tree I, is a collection of sets such that $\eta < \nu$ implies $A_\eta \subseteq A_\nu$.

Suppose $A = \langle M_\eta \rangle_{\eta \in I}$ is a tree of subsets of a model. We say A is an independent tree if whenever $J_0 \supseteq J_1 \cap J_2$ and $J_1, J_2 \subseteq I$,

$$M_{J_1} \underset{M_{J_0}}{\perp} M_{J_2}.$$

(We adopt the convention that for $J \subseteq I$,

$$M_J = \bigcup_{\eta \in J} M_\eta.)$$

We have not defined NDOP. For our purposes the conclusion of the following theorem can be taken as a definition of superstable theories with NDOP. However we need to know later that being superstable with NDOP has a definition which is absolute for extensions which add no subsets of $2^{\aleph_0}$. (In fact, Bouscaren has a characterization of being superstable with NDOP which shows this property is absolute.)

DECOMPOSITION THEOREM 2.2. [**11**] or [**3**]. *Suppose T is superstable with* NDOP *and M is an* a*-model of T. Then there is a tree $\langle M_\eta \rangle_{\eta \in I}$ of* a*-submodels of M, such that: for all η*

$$|M_\eta| \leqq 2^{\aleph_0};$$

$\langle M_\eta \rangle_{\eta \in I}$ is an independent tree; and M is a*-prime over $\langle M_\eta \rangle_{\eta \in I}$. Moreover if S is a non-empty subtree of I any model* a*-prime over M_S is in fact* a*-minimal (over M_S).*

Suppose $\langle M_\eta \rangle_{\eta \in I}$ and M are as above. We first explain how to label I. For each η choose a well ordering of M_η so that if $\eta < \nu$ the ordering on M_ν is an end extension of the ordering on M_η. Now partition I into at most $2^{2^{\aleph_0}}$ blocks so that for ν and η in the same block: ν and η have the same length; and if $\nu' \leqq \nu$, $\eta' \leqq \eta$ and ν' has the same length as η' then $M_{\nu'}$ is isomorphic to $M_{\eta'}$ via the map induced by the well orderings. We label I by introducing a unary relation for each block.

We will delay the proof of the following theorem.

THEOREM 3.1. *There is a cardinal μ such that for all $\lambda > \mu$ if I is a labelled tree with at most $2^{2^{\aleph_0}}$ unary relations and $|I| \leqq \lambda^+$, then I is ω-determinate in the λ^+ interpretation. Further in any extension of V which adds no subsets of μ, μ retains the property above.*

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We must also accumulate some facts about a-prime models over independent trees.

LEMMA 2.3. ([7] C12 (ii)) *With M an* a*-model,* $M[C]$ *denoting the* a*-prime model over* $M \cup C$, *we have*:

$$B \underset{M}{\perp} C \text{ implies } B \underset{M}{\perp} M[C].$$

LEMMA 2.4. *Suppose* $\langle M_i \rangle_{i \in I}$ *is an independent tree of* a*-models and for any subtree* $J \subseteq I$ *a model* a*-prime over* M_J *is in fact* a*-minimal. For all subtrees* $J_1, J_2, J_0 = J_1 \cap J_2$ *and* N a*-prime over* M_{J_1}:

$$M_{J_2} \underset{M_{J_0}}{\perp} N;$$

and if N' *is* a*-prime over* $M_{J_2} \cup N$ *then* N' *is* a*-prime over* $M_{J_2} \cup M_{J_1}$.

Proof. We can assume J_2 is finite. The proof is by induction on the number k of maximal elements of J_0. The case $k = 0$ is trivial. Suppose now J_0 has $k + 1$ maximal elements. Write J_0 as $S_3 \cup S_4$ and J_2 as $J_3 \cup J_4$ where $J_3 \cap J_1 = S_3$, $J_4 \cap J_1 = S_4$ and S_3 has 1 maximal element and S_4 has k maximal elements (of course, S_3, S_4, J_3 and J_4 are subtrees). Let N'' be a-prime over $N \cup M_{J_4}$. By the induction hypothesis N'' is a-prime over $M_{J_1 \cup J_4}$. So by Lemma 2.3,

$$M_{J_3} \underset{M_{S_3}}{\perp} N''$$

and by monotonicity

$$M_{J_3} \cup M_{J_4} \underset{M_{J_4} \cup M_{J_0}}{\perp} N.$$

By induction

$$M_{J_4} \underset{M_{S_4}}{\perp} N,$$

and so by monotonicity

$$M_{J_4} \cup M_{J_0} \underset{M_{J_0}}{\perp} N.$$

Hence by transitivity

$$M_{J_2} \underset{M_{J_0}}{\perp} N.$$

Now suppose M is a-prime $M_{J_1 \cup J_2}$ (hence also a-minimal). By Proposition B.11 of [7],

$$M_{J_1} \underset{T-V}{\leq} M_{J_1 \cup J_2}$$

(actually only a special case of this is proved in [7] but the general proof is

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much the same). A consequence of this is that any type over M_{J_1} has a unique non-forking extension to $M_{J_1 \cup J_2}$. In M choose M' a-prime over M_{J_1}. Since

$$M_{J_2} \underset{M_{J_1}}{\perp} M' \quad \text{and} \quad M_{J_1} \underset{T-V}{\leq} M_{J_1 \cup J_2},$$

M' and N realize the same type over $M_{J_1 \cup J_2}$. In other words, there is an $M_{J_1 \cup J_2}$ isomorphism of N with M'. So there is an $M_{J_1 \cup J_2}$ embedding of N' into M. By the a-minimality of M, this embedding is an isomorphism.

LEMMA 2.5. *Assume $\langle M_i \rangle_{i \in I}$ is as above and T is superstable. Suppose*

$$I = I_n \supseteq I_{n-1} \supseteq \ldots \supseteq I_0 \text{ and } N_0 \subseteq N_1 \subseteq \ldots \subseteq N_n$$

with N_k a-prime over I_k for all $k \leqq n$ (also N_{k+1} is a-prime over $N_k \cup I_{k+1}$, by Lemma 2.4). Further suppose $J_n \subseteq I$ is finite and A_n is a-prime over M_{J_n} and for all $k \leqq n$ $A_k = A_n \cap N_k$ is a-prime over M_{J_k} ($J_k = J_n \cap I_k$). Then for all finite $\bar{c} \in N_n$ there is $J_n \subseteq H_n \subseteq I_n$, H_n finite, and B_n a-prime over M_{H_n} so that: $\bar{c} \in B_n$;

$$B_n \cap N_k = B_k \supseteq A_k \quad \textit{for all } k \leqq n;$$

and B_k is a-prime over M_{H_k} where $H_k = H_n \cap I_k$. Further the isomorphism type of $(B_n, B_{n-1}, \ldots, B_0)$ over A_n depends only on the type of $(M_{H_n}, M_{H_{n-1}}, \ldots, M_{H_0})$ over M_{J_n}.

Proof. The proof is by induction on n. For $n = 0$, the a-minimality of N_0 over M_{I_0} implies N_0 is also a-prime over $A_0 \cup M_{I_0}$. So there is a finite $J_0 \subseteq H_0 \subseteq I_0$ so that the type of $\bar{c}$ over $A_0 \cup M_{I_0}$ is a-isolated over $A_0 \cup M_{H_0}$. Now choose $B_0 (\subseteq N_0)$ a-prime over $A_0 \cup M_{H_0}$ so that $\bar{c} \in B_0$. By Lemma 2.4, B_0 is a-prime over M_{H_0}.

The isomorphism type of B_0 over A_0 depends only on the type of M_{H_0} over A_0. But, again by Lemma 2.4,

$$M_{H_0} \underset{M_{J_0}}{\perp} A_0.$$

Now suppose $n = m + 1$. Since N_n is a-prime over $A_n \cup N_m \cup M_{I_n}$, there is a finite $K \subseteq I_n$ and a finite $\bar{b} \in N_m$ so that the type of $\bar{c}$ over $A_n \cup N_m \cup M_{I_n}$ is *a*-isolated over $A_n \cup \bar{b} \cup M_K$. Now by induction choose a finite $H_m \supseteq J_m$ and $B_m \supseteq A_m$ so that: for all $k \leqq m$ $B_m \cap N_k = B_k$ is a-prime over M_{H_k} where

$$H_k = H_m \cap I_k; \quad \bar{b} \in B_m; \quad \text{and} \quad K \cap I_m \subseteq H_m.$$

Let $H_n = H_m \cup K$ and B_n be a-prime over $A_n \cup M_k \cup B_n$ so that $\bar{c} \in B_n$. It remains to show B_n is a-prime over M_{H_n} and that $B_n \cap N_m = B_m$.

For the first of these claims let $C \subseteq B_n$ be a-prime over $A_n \cup B_m$. We first show C is a-prime over $M_{H_m \cup J_n}$. Let D be a-prime over $M_{H_m \cup J_n}$.

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Since

$$A_n \underset{M_{J_m}}{\perp} M_{H_n}$$

there is an $M_{H_m \cup J_n}$ embedding of A_n into D. By monotonicity

$$A_n \underset{A_m}{\perp} M_{H_m},$$

and so by Lemma 2.4

$$A_n \underset{A_m}{\perp} B_m.$$

Hence the embedding extends to C. As before, the a-minimality of D implies this embedding is an isomorphism. By Lemma 2.4 if E is a-prime over $C \cup M_K$, then E is a-prime over

$$M_{K \cup H_m \cup J_m} = M_{H_n}.$$

Since

$$E \supseteq M_K \cup A_n \cup B_m,$$

there is an $M_K \cup A_n \cup B_m$ embedding of B_n into E. The a-minimality of E shows B_n is a-prime over M_{H_n}.

For the second claim, note

$$M_{H_n} \underset{M_{H_m}}{\perp} N_m.$$

So

$$M_{H_n} \underset{B_m}{\perp} N_m.$$

Hence

$$B_n \underset{B_m}{\perp} N_m.$$

So $B_n \cap N_m = B_m$.

Finally the statement about isomorphism types is true, since

$$M_{H_n} \underset{M_{J_n}}{\perp} A_n.$$

We now turn to the proof of the main theorem.

THEOREM 2.6. *If T is superstable with* NDOP, *then T is finitely determinate.*

Proof. By taking a forcing extension if necessary, we can assume

$$\lambda > \mu \quad (>2^{\aleph_0})$$

(where μ is as in 3.1) and $\lambda^{\underline{\lambda}} = \lambda$. Now suppose M is an a-model (of T)

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and $|M| = \lambda^+$. Let I be the labelled tree associated with M and

$$\langle I_\alpha \rangle_{\alpha < \lambda^+}$$

a filtration of I witnessing I is ω-determinate in the λ^+-interpretation. Choose a chain of submodels

$$\langle M_\alpha \rangle_{\alpha < \lambda^+}$$

so that M_α is a-prime over M_{I_α} as follows: let M_0 be a-prime over M_{I_0}; if M_α has been chosen, let $M_{\alpha+1}$ be a-prime over $M_{I_{\alpha+1}} \cup M_\alpha$ (by Lemma 2.4 $M_{\alpha+1}$ is a-prime over $M_{I_{\alpha+1}}$); if β is a limit ordinal let

$$M_\beta = \bigcup_{\alpha < \beta} M_\alpha$$

(since M_β is an a-submodel of a model a-prime over M_{I_β}, M_β is a-prime over M_{I_β}). As usual the a-minimality of M over M_I implies

$$M = \bigcup_{\alpha < \lambda^+} M_\alpha.$$

We claim that

$$\langle M_\alpha \rangle_{\alpha < \lambda^+}$$

witnesses M is finitely determinate in the λ^+-interpretation. Lemma 2.5 is exactly what is required to transfer the back and forth systems demonstrating that the

$$\langle I_\alpha \rangle_{\alpha < \lambda^+}$$

witnesses I is ω-determinate in the λ^+-interpretation to a back and forth system demonstrating

$$\langle M_\alpha \rangle_{\alpha < \lambda^+}$$

is ω-determinate (and so finitely determinate).

3. Labelled trees. In this section we prove:

THEOREM 3.1. *There is a cardinal μ such that for all $\lambda > \mu$ if I is a labelled tree with at most $2^{2^{\aleph_0}}$ unary relations and $|I| \leqq \lambda^+$, then I is ω-determinate in the λ^+-interpretation. Further in any extension of ν which adds no new subsets of μ, μ retains the property above.*

Proof. We calculate μ. Let L_0 be the language with a binary relation $<$ and $2^{2^{\aleph_0}}$ unary relations. Let S_0 be the set of complete $(L_0)^{\omega}_{\infty\omega}$-theories. Form L_1 by adding to L_0 a unary predicate U_K for each $K \subseteq S_0$. In general if L_n has been defined, let S_n be the set of complete $(L_n)^{\omega}_{\infty\omega}$-theories and let L_{n+1} be the language obtained from L_n by adding a unary predicate U_K for each $K \subseteq S_n$. Let $L = \cup L_n$ and let μ be the number of complete $L^{\omega}_{\infty\omega}$-theories.

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Suppose we are given I as in the statement of the theorem. We can assume for each n there is a unary predicate $U_n \in L_0$ so that $I \models U_n(\text{a})$ if and only if the height of a is n. We now inductively define an L structure on I. Assume we have defined an L_n structure on I and $t \in I$. Then for all $K \subseteq S_n$, let $I \models U_K(t)$ if and only if

$$\{\Phi \in S_n : | \{t' \in I : t' \text{ is an immediate successor of } t \text{ and } [t'] \models \Phi\} | = \lambda^+\} = K.$$

(Here $[t']$ denotes $\{s : t' \leqq s\}$.)

Note that for any L-structure A and $X \subseteq A$ there is $B \supseteq X$ so that

$$B \prec^{\omega}_{\infty\omega} A \quad \text{and} \quad |B| = |X| + \mu.$$

We now define a λ^+-filtration

$$(I_\nu)_{\nu<\lambda^+}$$

of I. Choose $(I_\nu)_{\nu<\lambda^+}$ so that: for all ν,

$$I_\nu \prec^{\omega}_{\infty\omega} I;$$

for all $t \in I_\nu$, complete theory $\Phi \subseteq L$ and

$$X = \{s : s \text{ an immediate successor of } t \text{ and } [s] \models \Phi\},$$

if $|X| \leqq \lambda$ then $I_\nu \supseteq X$ and if $|X| = \lambda^+$ then

$$|X \cap I_\nu| = \lambda = |X \cap (I_{\nu+1} \setminus I_\nu)|.$$

Note that if $\nu < \tau$ and $t \in I_\tau \setminus I_\nu$ then

$$(I_\tau \setminus I_\nu) \cap [t] = I_\tau \cap [t] \prec^{\omega}_{\infty\omega} [t].$$

We now show

$$(I_\nu)_{\nu<\lambda^+}$$

is the desired λ^+-filtration. It suffices to show that for any $m < \omega$,

$$\nu < \tau_1 < \ldots < \tau_n, \quad \nu < \sigma_1 < \ldots \sigma_n:$$

Player II has a winning strategy for the game of m rounds where at each turn Player I plays a finite subtree from either $(I, I_{\tau_1}, \ldots, I_{\tau_n})$ or $(I, I_{\sigma_1}, \ldots, I_{\sigma_n})$ and Player II plays a subtree of the same cardinality from the other structure. Player II wins if the submodels constructed are isomorphic over their intersection with I_ν. (I.e., the isomorphism must be the identity on I_ν.) Rather than write down the argument in unpleasant and unreadable detail, we indicate the first step in a particular case. Suppose m and $\nu < \tau_1, \sigma_1$ have been fixed and Player I plays

$$t_0 < t_1 < t_2 < t_3 \in (I, I_{\tau_1})$$

where

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$$t_0 \in I_\nu, \quad t_1, t_2 \in I_{\tau_1} \backslash I_\nu \quad \text{and} \quad t_3 \in I \backslash I_{\tau_1}.$$

Since t_1 is an immediate successor of t_0 and $t_1 \notin I_\nu$, there are λ^+ immediate successors u of t_0 so that $[t_1]$ and $[u]$ have the same $(L_1)_{\infty\omega}$-theory. So we can choose such a $u_1 \in I_{\sigma_1} \backslash I_\nu$. Now take $u_2 \in I_{\sigma_1}$ so that

$$([u_1], u_1, u_2) \equiv {}^{m-1}_{\infty\omega}([t_1], t_1, t_2) \quad (\text{in } L_1).$$

Since u_2 and t_2 belong to the same L_1 unary relations, there is $u_3 \in I \backslash I_{\sigma_1}$ so that u_3 is an immediate successor of u_2; and $[u_3]$ satisfies the same $(L_0)^{\omega}_{\infty\omega}$-theory as $[t_3]$. So Player II plays u_0, u_1, u_2, u_3.

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