

ASYMPTOTIC BEHAVIOUR OF CRITICAL CONTROLLED BRANCHING PROCESSES WITH RANDOM CONTROL FUNCTIONS

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Abstract

In this paper, we investigate the asymptotic behaviour of controlled branching processes with random control functions. In a critical case, we establish sufficient conditions for both their almost-sure extinction and for their nonextinction with a positive probability. For some suitably chosen norming constants, we also determine different kinds of limiting behaviour for this class of processes.

Keywords: Branching process; controlled process; extinction probability; limiting behaviour

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1. Introduction

A controlled branching process with random control function is a discrete-time stochastic model of the form

$$Z_0 = N, \quad Z_{n+1} = \sum_{j=1}^{\phi_n(Z_n)} X_{nj}, \quad n = 0, 1, \dots, \quad (1)$$

where an empty sum is taken to be 0, N is a positive integer, and $\{X_{nj}, n = 0, 1, \dots, j = 1, 2, \dots\}$ and $\{\phi_n(k), n, k = 0, 1, \dots\}$ are independent sets of nonnegative integer-valued random variables on the same probability space. The variables X_{nj} are independent and identically distributed, their common probability law is called the offspring probability distribution, and, for $n = 0, 1, \dots$, the $\{\phi_n(k)\}_{k \geq 0}$ are independent stochastic processes such that $\phi_n(k), n = 0, 1, \dots$, are identically distributed. Intuitively, Z_n represents the size of the n th generation of a population under the influence of a random control mechanism. The $\phi_n(k), n = 0, 1, \dots, k = 0, 1, \dots$, will be referred to as control variables.

Several questions concerning this control model have been investigated in [1], [3], [15], and [19]. Recently (see [4], [5], [6], and [7]), it has been studied in the framework of asymptotically linear growth of the mathematical expectations of the control variables. In this paper, we continue the research into this class of branching models and, in a critical case, we investigate some pertinent questions that have not previously been considered in the literature. In Section 2, some notation and working assumptions are given. Sufficient conditions for the almost-sure

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extinction or for the existence of a positive probability of nonextinction are provided in Section 3, and Section 4 is devoted to investigating different kinds of limiting behaviour for $\{Z_n\}_{n \geq 0}$, when suitably normed. In particular, gamma, normal, and degenerate laws are obtained as asymptotic distributions. In order to allow for a more comprehensible reading, the proofs are relegated to Section 5.

2. Notation and working assumptions

From (1), by considering the conditions established for the variables X_{nj} and $\phi_n(k)$, it is easily verified that $\{Z_n\}_{n \geq 0}$ is a homogeneous Markov chain whose state space is included in the nonnegative integers. For simplicity, we shall assume that the offspring probability distribution, denoted by $\{p_k\}_{k \geq 0}$ with $p_k := P(X_{01} = k)$, and the control variables are such that the positive integers form a class of communicating, aperiodic states. Moreover, from now on we will consider controlled branching processes with random control functions such that $P(\phi_0(0) = 0) = 1$, i.e. 0 is an absorbent state, and at least one of the following conditions holds:

- (i) $p_0 > 0$ or
- (ii) $P(\phi_0(k) = 0) > 0, k = 1, 2, \dots$

Under these assumptions, the classical extinction–explosion duality in branching process theory holds [19], namely $P(Z_n \rightarrow 0) + P(Z_n \rightarrow \infty) = 1$.

Let us write $m := E[X_{01}]$, $\sigma^2 := \text{var}[X_{01}]$, $\varepsilon(k) := E[\phi_0(k)]$, and $v^2(k) := \text{var}[\phi_0(k)]$, $k = 0, 1, \dots$, and define $\tau(k)$ as $E[Z_{n+1}Z_n^{-1} \mid Z_n = k]$, $k = 1, 2, \dots$, which is equivalent to $mk^{-1}\varepsilon(k)$; this is intuitively interpreted as the expected growth rate per individual when, in a certain generation, there are k individuals. In order to obtain conditions for the almost-sure extinction, different possible behaviours for the sequence $\{\tau(k)\}_{k \geq 1}$, relative to 1, have been considered in [4]. In particular, the cases $\limsup_{k \rightarrow \infty} \tau(k) < 1$ and $\liminf_{k \rightarrow \infty} \tau(k) > 1$, respectively called subcritical and supercritical, have been investigated. Here, we shall be concerned with the complementary situation, i.e. when

$$\liminf_{k \rightarrow \infty} \tau(k) \leq 1 \leq \limsup_{k \rightarrow \infty} \tau(k),$$

which is referred to as the critical case.

Note that $\{Z_n\}_{n \geq 0}$ almost surely satisfies the relation

$$Z_{n+1} = Z_n + g(Z_n) + \xi_{n+1}, \quad n = 0, 1, \dots, \tag{2}$$

where

$$Z_0 = N, \quad g(k) := m\varepsilon(k) - k, \quad k = 0, 1, \dots, \quad \text{and} \quad \xi_{n+1} := Z_{n+1} - E[Z_{n+1} \mid Z_n].$$

It is clear that $\{\xi_n\}_{n \geq 1}$ is a zero-mean, square-integrable martingale difference sequence. For simplicity, let us write $\ell_\delta(k) := E[|\xi_{n+1}|^\delta \mid Z_n = k]$, $k = 1, 2, \dots$, $\delta > 0$. The conditional second moments depend only on the instantaneous state of the process. In fact, $\ell_2(k) = v^2(k)m^2 + \varepsilon(k)\sigma^2$, $k = 1, 2, \dots$.

Expression (2) can be rewritten, at least in the event $\{g(Z_n) \neq 0\}$, in the alternative form

$$Z_{n+1} = Z_n + g(Z_n)(1 + \eta_{n+1}), \quad n = 0, 1, \dots, \tag{3}$$

where $\eta_{n+1} := \xi_{n+1}g(Z_n)^{-1}$. Obviously, $\{\eta_n\}_{n \geq 1}$ is also a zero-mean, square-integrable martingale difference sequence.

In order to investigate their asymptotic properties, taking into account (2) or (3), the class of critical controlled branching processes with random control function can be treated using some specific methodologies for stochastic difference equations.

3. Extinction probability

The process becomes extinct when, for some n , $Z_n = 0$. Let us denote by

$$q_N := P(Z_n \rightarrow 0 \mid Z_0 = N)$$

the extinction probability in the case that the process starts with N individuals, $N = 1, 2, \dots$. Focusing our attention on those cases in which the limit of $\{\tau(k)\}_{k \geq 1}$ exists and $\lim_{k \rightarrow \infty} \tau(k) = 1$, we will mainly distinguish two situations. If the convergence of $\{\tau(k)\}_{k \geq 1}$ to 1 is very rapid (with respect to the conditional variance), the whole process will behave as a true critical process, i.e. almost-sure extinction will follow. On the other hand, if the convergence of $\{\tau(k)\}_{k \geq 1}$ is very slow, the process will develop as a regular supercritical process, having a positive probability of nonextinction.

We now provide sufficient conditions for almost-sure extinction (Theorem 1) and for the existence of a positive probability of nonextinction (Theorem 2).

Theorem 1. *Assume that*

- (i) $\tau(k) = 1 + o(1)$ with $\tau(k) \geq 1$,
- (ii) $\ell_{2+\delta}(k) = o(\ell_2(k)k^\delta)$ for some $0 < \delta \leq 1$, and
- (iii) $\limsup_{k \rightarrow \infty} 2kg(k)/\ell_2(k) < 1$.

Then $q_N = 1$, $N = 1, 2, \dots$

Theorem 2. *Assume that*

- (i) $\tau(k) = 1 + o(1)$,
- (ii) $\ell_{2+\delta}(k) = o(g(k)k^{1+\delta}(\log k)^{-s})$ for some $0 < \delta \leq 1$ and $s > 1$, and
- (iii) $\liminf_{k \rightarrow \infty} 2kg(k)/\ell_2(k) > 1$.

Then $q_N < 1$, $N = 1, 2, \dots$

Remark 1. Under assumption (iii) of Theorem 1, a sufficient condition for assumption (ii) to be satisfied is that $\ell_{2+\delta}(k) = o(g(k)k^{1+\delta})$. In fact,

$$\frac{\ell_{2+\delta}(k)}{\ell_2(k)k^\delta} = \frac{\ell_{2+\delta}(k)}{g(k)k^{1+\delta}} \frac{kg(k)}{\ell_2(k)}.$$

Hence, by assumption (iii) and the fact that $\ell_{2+\delta}(k) = o(g(k)k^{1+\delta})$, we deduce that $\ell_{2+\delta}(k) = o(\ell_2(k)k^\delta)$. Analogously, under assumption (iii) of Theorem 2, it can be seen that $\ell_{2+\delta}(k) = o(\ell_2(k)k^\delta(\log k)^{-s})$ is a sufficient condition for assumption (ii) to hold. Finally, the third assumption of each theorem can intuitively be interpreted as a speed condition due to the fact that

$$\frac{kg(k)}{\ell_2(k)} = \frac{\tau(k) - 1}{\ell_2(k)k^{-2}}.$$

The previous theorems provide the answer to the extinction problem for the most typical cases in this class of controlled models. Unfortunately, certain situations are not covered by them, for instance the case in which $\lim_{k \rightarrow \infty} 2kg(k)(\ell_2(k))^{-1} = 1$. In the following theorem, we extend our research in that direction. Some results concerning the extinction time $T := \inf\{n > 0: Z_n = 0\}$ are also established.

Theorem 3. *Assume that $\phi_0(k)$, $k = 1, 2, \dots$, have infinitely divisible probability distributions such that*

- (i) $\tau(k) = 1 + k^{-1}c$, $c > 0$, $k = 1, 2, \dots$,
- (ii) $\ell_2(k) = vk + O(1)$, $v > 0$, and
- (iii) $\sup_{k \geq 1} (g_k^{1/k})'''(1) < \infty$ with $g_k(s) := E[s^{\phi_0(k)}]$, $0 \leq s \leq 1$.

(Here, a prime denotes differentiation.) Then

- (a) if $2cv^{-1} \leq 1$, we have $q_N = 1$, $N = 1, 2, \dots$, and
- (b) if $2cv^{-1} < 1$, we have

$$P(T > n - 1) \sim k_1 n^{2cv^{-1}-1}, \quad n \rightarrow \infty,$$

while if $2cv^{-1} = 1$, we have

$$P(T > n - 1) \sim k_2 (\log n)^{-1}, \quad n \rightarrow \infty,$$

for some positive constants k_i , $i = 1, 2$.

Remark 2. According to Kolmogorov’s classical extinction time result for critical Bienaymé–Galton–Watson processes $\{Y_n\}_{n \geq 0}$, we know that $P(Y_n > 0) \sim 2(\sigma^2 n)^{-1}$. We have proved, for our controlled model $\{Z_n\}_{n \geq 0}$, that $P(Z_n > 0)$ decays to 0 more slowly. This is due to the immigration component: note that, by assumption (i) of Theorem 3, there is expected immigration of individuals in each generation. On the other hand, the results obtained are also in parallel with those ones established by Yanev and Yanev for critical branching processes with random migration stopped at 0. Under a dominating immigration, they proved a speed of convergence analogous to that established in Theorem 3 and, assuming a dominating emigration, obtained, as is logical, a faster decay rate (see [16] and [17] for details).

4. Limiting behaviour

In this section, we shall be concerned with the limiting behaviour of the process when suitably normed, in both the nonextinction situation and the extinction case. First, we consider processes that do not become extinct with a positive probability. We assume that $\tau(k) = 1 + ck^{\alpha-1} + o(k^{\alpha-1})$, where $\alpha < 1$, $c > 0$, and $\tau(k) > 1$. Consequently, we deduce that $g(k) = m\varepsilon(k) - k = ck^\alpha + o(k^\alpha)$. For technical reasons, we shall extend g to a twice-continuously differentiable function on \mathbb{R} :

$$g(x) = \begin{cases} cx^\alpha + o(x^\alpha) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We shall denote by $\{a_n\}_{n \geq 0}$ the solution of the deterministic recursive equation

$$a_0 = 1, \quad a_{n+1} = a_n + g(a_n), \quad n = 0, 1, \dots$$

Theorem 4. Assume that

- (i) $\tau(k) = 1 + ck^{\alpha-1} + o(k^{\alpha-1})$, $\alpha < 1, c > 0$, with $\tau(k) > 1$,
- (ii) $\ell_2(k) = vk^\beta + o(k^\beta)$, $\beta \leq \alpha + 1, v > 0$, and
- (iii) $\ell_{2+\delta}(k) = O((\ell_2(k))^{1+\delta/2})$ for some $\delta > 0$.

The following statements then hold.

- (a) If $\beta = \alpha + 1$ and $2cv^{-1} > 1$ then, for all real numbers z ,

$$\lim_{n \rightarrow \infty} P(n^{-1}Z_n^{1-\alpha} \leq z \mid Z_r \rightarrow \infty) = \Gamma_{a,b}(z),$$

where $\Gamma_{a,b}$ denotes the gamma distribution function with parameters $a = (v - v\alpha)^{-1}(2c - v\alpha)$ and $b = 2^{-1}v(1 - \alpha)^2$.

- (b) If $0 < \alpha < 1$ and $\beta < \alpha + 1$ then, in the event $\{Z_r \rightarrow \infty\}$,

- 1. for $\beta < 3\alpha - 1$ we have

$$a_n^{-1}Z_n \rightarrow 1 \text{ almost surely (a.s.),} \quad g(a_n)^{-1}(Z_n - a_n) \text{ converges a.s.,}$$

as $n \rightarrow \infty$, and

- 2. for $\beta \geq 3\alpha - 1$ we have $a_n^{-1}Z_n \xrightarrow{P} 1$ as $n \rightarrow \infty$ and, for all real numbers z ,

$$\lim_{n \rightarrow \infty} P\left(\Delta_n^{-1/2} \frac{Z_n - a_n}{g(a_n)} \leq z \mid Z_r \rightarrow \infty\right) = \phi^*(z),$$

where ϕ^* is the standard normal distribution function and

$$\Delta_n = \begin{cases} vc^{-3}(1 - \alpha)^{-1} \log n & \text{if } \beta = 3\alpha - 1, \\ v(\beta - 3\alpha + 1)^{-1} c^{(\beta-2)/(1-\alpha)} ((1 - \alpha)n)^{(\beta-3\alpha+1)/(1-\alpha)} & \text{if } \beta > 3\alpha - 1. \end{cases}$$

Remark 3. Theorem 4 makes sense because, under its assumptions, Theorem 2 holds and, therefore, $\{Z_n\}_{n \geq 0}$ does not die out with a positive probability. In fact, from assumption (i) it is clear that $\tau(k) = 1 + o(1)$. Moreover, from (4) and assumption (ii), we can show that

$$\lim_{k \rightarrow \infty} \frac{2kg(k)}{\ell_2(k)} = \begin{cases} 2cv^{-1} & \text{if } \beta = \alpha + 1, \\ \infty & \text{otherwise.} \end{cases}$$

Finally, using assumptions (ii) and (iii), we have, for some positive constant M ,

$$\begin{aligned} \frac{\ell_{2+\delta}(k)}{\ell_2(k)k^\delta(\log k)^{-s}} &\leq M \frac{(\ell_2(k))^{\delta/2}}{k^\delta} (\log k)^s \\ &= M \frac{v^{\delta/2}k^{\beta\delta/2} + o(k^{\beta\delta/2})}{k^\delta} (\log k)^s \\ &= Mv^{\delta/2}k^{\delta(\beta-2)/2}(1 + o(1))(\log k)^s \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

and, consequently (see Remark 1), $\ell_{2+\delta}(k) = o(g(k)k^{1+\delta}(\log k)^{-s})$. Note that, for $\beta > \alpha + 1$ or for $\beta = \alpha + 1$ and $2cv^{-1} < 1$, the process becomes extinct with probability 1 (see Theorem 1).

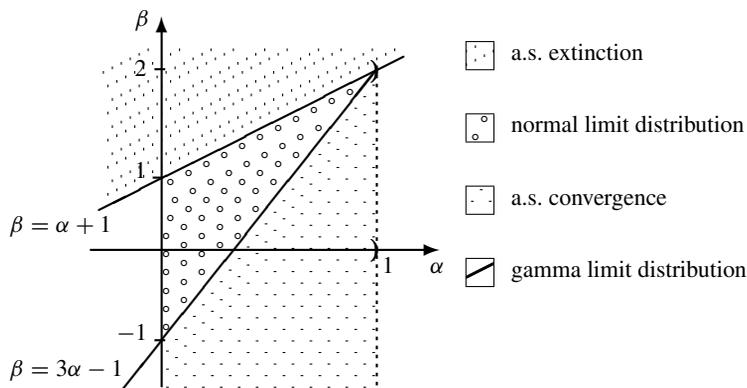


FIGURE 1.

Figure 1 shows, in a plot in the (α, β) -plane, a simplified scheme corresponding to the different kinds of limiting behaviour obtained for $\{Z_n\}_{n \geq 0}$, when suitably normed.

Corollary 1. (a) Under the conditions established in Theorem 4(a), for all real numbers z ,

1. $\lim_{n \rightarrow \infty} P(n^{-1} Z_n^{1-\alpha} \leq z) = q_N \mathbf{1}_{\{z \geq 0\}} + (1 - q_N) \Gamma_{a,b}(z)$,
2. $\lim_{n \rightarrow \infty} P(n^{-1} Z_n^{1-\alpha} \leq z \mid Z_n > 0) = \Gamma_{a,b}(z)$,

with $a = (\nu - \nu\alpha)^{-1}(2c - \nu\alpha)$ and $b = 2^{-1}\nu(1 - \alpha)^2$.

(b) Under the conditions established in Theorem 4(b)2, for all real numbers z ,

$$\lim_{n \rightarrow \infty} P\left(\Delta_n^{-1/2} \frac{Z_n - a_n}{g(a_n)} \leq z \mid Z_n > 0\right) = \phi^*(z).$$

Remark 4. We can show that $a_n \sim (c(1 - \alpha)n)^{(1-\alpha)^{-1}}$ as $n \rightarrow \infty$. In fact, because $\alpha < 1$ and $c > 0$, $\{a_n\}_{n \geq 0}$ converges to infinity and we can therefore suppose that $a_n \neq 0$ for every $n \geq 0$. Taking into account Lemma 1 (in Appendix A) and (4), we have

$$\begin{aligned} a_{n+1}^{1-\alpha} &= a_n^{1-\alpha} \left(1 + \frac{a_{n+1} - a_n}{a_n}\right)^{1-\alpha} \\ &= a_n^{1-\alpha} + (1 - \alpha)a_n^{-\alpha} g(a_n) + O\left(\frac{g(a_n)^2}{a_n^{1+\alpha}}\right) \\ &= a_n^{1-\alpha} + (1 - \alpha)c + o(1). \end{aligned}$$

Hence, we have $a_n^{1-\alpha} = c(1 - \alpha)n + o(n)$, and the result follows.

Remark 5. From a practical point of view, it is interesting to look for easily checked sufficient conditions which guarantee that the hypotheses about the $(2 + \delta)$ th conditional absolute moment of ξ_{n+1} hold. In this sense, we can verify assumption (iii) of Theorem 4 by using both the fact that $|a + b|^r \leq C_r(|a|^r + |b|^r)$, $r > 0$, for some positive constant C_r (called the C_r -inequality)

and the Marcinkiewicz–Zygmund inequality (see Lemma 2 in Appendix A). Doing so, we find that, as $k \rightarrow \infty$,

$$\ell_{2+\delta}(k) \leq C_{2+\delta} \left(\mathbb{E} \left[\left| \sum_{i=1}^{\phi_n(k)} (X_{ni} - m) \right|^{2+\delta} \right] + m^{2+\delta} \mathbb{E}[|\phi_n(k) - \varepsilon(k)|^{2+\delta}] \right)$$

and

$$\mathbb{E} \left[\left| \sum_{i=1}^{\phi_n(k)} (X_{ni} - m) \right|^{2+\delta} \right] = O(\mathbb{E}[(\phi_n(k))^{1+\delta/2}]).$$

Hence, the conditions

$$\mathbb{E}[(\phi_0(k))^{1+\delta/2}] = O(k^{\beta(1+\delta/2)}) \quad \text{and} \quad \mathbb{E}[|\phi_0(k) - \varepsilon(k)|^{2+\delta}] = O(k^{\beta(1+\delta/2)})$$

imply that

$$\ell_{2+\delta}(k) = O(k^{\beta(1+\delta/2)})$$

and, as a consequence, that

$$\ell_{2+\delta}(k) = O((\ell_2(k))^{1+\delta/2}).$$

Under the conditions of Theorem 3, we know that the process becomes extinct with probability 1. In the following result, we investigate the limiting distribution of $n^{-1}Z_n$ conditioned on $\{Z_n > 0\}$.

Theorem 5. *Under the hypotheses of Theorem 3, if $2cv^{-1} \leq 1$ then*

$$\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1}Z_n \leq z \mid Z_n > 0) = \Gamma_{a,b}(z),$$

with $a = 1$ and $b = 2^{-1}v$.

Remark 6. Note that, in this situation, the parameter a does not depend on c or v , unlike the analogous result in the case in which extinction is not almost sure. This result is similar to the result of Kolmogorov and Yaglom concerning the limiting exponential distribution for the critical Bienaymé–Galton–Watson process. Both models have the same exponential limiting distribution on their nonextinction paths, in spite of the fact that the decay rates of their nonextinction probabilities are different. In particular, if $\phi_0(k) := k + Y$, $k = 1, 2, \dots$, where Y is a nonnegative integer-valued random variable independent of $\{X_{nj}, n = 0, 1, \dots, j = 1, 2, \dots\}$ and such that $\mathbb{E}[Y] < \infty$, then $\{Z_n\}_{n \geq 0}$ is a Galton–Watson process with immigration. The classical result about the limiting gamma distribution for this process in the critical case (see [10]) is included in Theorem 4. Analogous results have been also obtained for critical branching processes with random migration, both when immigration dominates emigration and vice versa (see [16], [17], and [18]).

5. Proofs

In the proofs of Theorems 1 and 2 it will be necessary to consider the following result.

Proposition 1. *Let $\{X_n\}_{n \geq 0}$ be a sequence of nonnegative random variables and let $\{\mathcal{F}_n\}_{n \geq 0}$ be a nondecreasing sequence of σ -algebras such that X_n is \mathcal{F}_n -measurable for each n . Let f be a positive function on \mathbb{R}^+ .*

(a) If f is increasing and

$$E[f(X_{n+1}) \mid \mathcal{F}_n] \leq f(X_n) \quad \text{a.s. on } \{X_n > A\}, \quad n = 0, 1, \dots,$$

for some positive constant A , then $P(X_n \rightarrow \infty) = 0$.

(b) Suppose that, for any constant C^* , there exists an $n = 1, 2, \dots$ such that $P(X_n > C^*) > 0$ and, moreover, that $P(X_n \rightarrow 0) + P(X_n \rightarrow \infty) = 1$. If f is decreasing and

$$E[f(X_{n+1}) \mid \mathcal{F}_n] \leq f(X_n) \quad \text{a.s. on } \{X_n > A\}, \quad n = 0, 1, \dots,$$

for some positive constant A , then $P(X_n \rightarrow \infty) > 0$.

Proof. The proposition is proved using reasoning similar to that used by Kersting in [12].

(a) Taking into account the fact that $\{X_n \rightarrow \infty\} = \bigcup_{j=1}^{\infty} \{\inf_{n \geq j} X_n > A\} \cap \{X_n \rightarrow \infty\}$, it will be sufficient to verify that, for every $j \geq 1$,

$$P\left(\left\{\inf_{n \geq j} X_n > A\right\} \cap \{X_n \rightarrow \infty\}\right) = 0. \tag{5}$$

For an arbitrary $j \geq 1$, let us define

$$T(A) := \begin{cases} \inf\{n \geq j : X_n \leq A\} & \text{if } \inf_{n \geq j} X_n \leq A, \\ \infty & \text{otherwise.} \end{cases}$$

We consider the sequence $\{Y_n\}_{n \geq 0}$, where $Y_n := X_{T(A) \wedge (j+n)}$. After some calculations, we find that

$$E[Y_{n+1} \mid \mathcal{F}_{j+n}] \leq Y_n \quad \text{a.s.,} \quad n = 0, 1, \dots$$

Therefore, $\{Y_n\}_{n \geq 0}$ is a nonnegative supermartingale with respect to $\{\mathcal{F}_{j+n}\}_{n \geq 0}$ and, from the martingale convergence theorem (see, e.g. [2, p. 246]), it follows that $\{Y_n\}_{n \geq 0}$ is almost surely convergent to a random variable Y such that $P(0 \leq Y < \infty) = 1$. It now follows that, almost surely,

$$Y = \begin{cases} X_{T(A)} & \text{if } \inf_{n \geq j} X_n \leq A, \\ \lim_{n \rightarrow \infty} X_{j+n} & \text{otherwise,} \end{cases}$$

and we obtain (5).

(b) Let $Y_n^* := f(X_n) \wedge f(A)$, $n = 0, 1, \dots$. It can be shown that

$$E[Y_{n+1}^* \mid \mathcal{F}_n] \leq E[f(X_{n+1}) \mid \mathcal{F}_n] \wedge f(A) \leq Y_n^* \quad \text{a.s.,} \quad n = 0, 1, \dots$$

Hence, $\{Y_n^*\}_{n \geq 0}$ is a nonnegative supermartingale with respect to $\{\mathcal{F}_n\}_{n \geq 0}$ and, again using the martingale convergence theorem, it is almost surely convergent to a finite, nonnegative random variable Y^* . Moreover, because $\{Y_n^*\}_{n \geq 0}$ is bounded, it follows that it is also convergent in L^1 . Suppose that $P(X_n \rightarrow \infty) = 0$; then $P(X_n \rightarrow 0) = 1$ and it follows that $E[Y^*] = f(A)$. Since $\{E[Y_n^*]\}_{n \geq 0}$ is decreasing, $E[Y_n^*] \geq f(A)$, $n = 0, 1, \dots$. Now, $Y_n^* \leq f(A)$, $n = 0, 1, \dots$; hence, it follows, for every n , that $Y_n^* = f(A)$ almost surely and, since f is decreasing, that $X_n \leq A$ almost surely, $n = 0, 1, \dots$. This contradicts the first assumption in Proposition 1(b).

5.1. Proof of Theorem 1

For $x > -1$ and $0 \leq \delta \leq 1$, the following inequality holds:

$$\log(x + 1) \leq x - \frac{1}{2}x^2 + \frac{1}{2}|x|^{2+\delta}. \tag{6}$$

From (2), we have

$$\log(Z_{n+1} + 1) = \log(Z_n + 1) + \log\left(1 + \frac{g(Z_n)}{Z_n + 1} + \frac{\xi_{n+1}}{Z_n + 1}\right).$$

Hence, using (6) with $x = (g(Z_n) + \xi_{n+1})/(Z_n + 1)$, taking expectations, applying the C_r -inequality, and using the fact that, by assumption (i) (of Theorem 1), $g(k) = k(\tau(k) - 1) \geq 0$, for a certain positive constant $C_{2+\delta}$ we obtain

$$\begin{aligned} E[\log(Z_{n+1} + 1) \mid Z_n = k] &\leq \log(k + 1) + \frac{g(k)}{k} - \frac{(g(k))^2}{2(k + 1)^2} - \frac{\ell_2(k)}{2(k + 1)^2} \\ &\quad + \frac{1}{2}C_{2+\delta}\left(\frac{(g(k))^{2+\delta}}{k^{2+\delta}} + \frac{\ell_{2+\delta}(k)}{k^{2+\delta}}\right) \\ &= \log(k + 1) + \frac{g(k)}{k}\Delta_1(k) - \frac{\ell_2(k)}{2k^2}\Delta_2(k), \quad k = 1, 2, \dots, \end{aligned}$$

where

$$\Delta_1(k) = 1 - \frac{g(k)k}{2(k + 1)^2} + \frac{1}{2}C_{2+\delta}\frac{(g(k))^{1+\delta}}{k^{1+\delta}}$$

and

$$\Delta_2(k) = \frac{k^2}{(k + 1)^2} - \frac{1}{2}C_{2+\delta}\frac{(\ell_{2+\delta}(k))^{1+\delta}}{\ell_2(k)k^\delta}.$$

From assumptions (i) and (ii), we deduce that $\lim_{k \rightarrow \infty} \Delta_1(k) = \lim_{k \rightarrow \infty} \Delta_2(k) = 1$, so, for k sufficiently large and $\epsilon > 0$ sufficiently small, we have

$$\frac{g(k)}{k}\Delta_1(k) - \frac{\ell_2(k)}{2k^2}\Delta_2(k) \leq \frac{g(k)}{k}(1 + \epsilon) - \frac{\ell_2(k)}{2k^2}(1 - \epsilon).$$

Now, from assumption (iii), if we choose an ϵ such that $0 < \epsilon < (1 - b)(1 + b)^{-1}$ with $b := \limsup_{k \rightarrow \infty} 2kg(k)/\ell_2(k)$, then the right-hand side of the previous inequality is negative. Therefore, it follows that if $Z_n \geq A$, for some sufficiently large constant A , then

$$E[\log(Z_{n+1} + 1) \mid \mathcal{F}_n] \leq \log(Z_n + 1) \quad \text{a.s.},$$

where \mathcal{F}_n denotes the σ -algebra generated by Z_0, \dots, Z_n . The result then follows by applying Proposition 1(a).

5.2. Proof of Theorem 2

Let $f(x) = (\log x)^{-\alpha}$, $x > 0$, $\alpha \leq s - 1$. It was shown in [12] that, for $x + h \geq 3$, $x \geq 3$, and a sufficiently large constant C^* ,

$$f(x + h) \leq f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + C^*\frac{|h|^{2+\delta}}{(\log x)^{1+\alpha}x^{2+\delta}} + \mathbf{1}_{\{h \leq -x/2\}}.$$

From (2), setting $x = Z_n + 3$ and $h = g(Z_n) + \xi_{n+1}$, applying Chebychev’s conditional equality and the C_r -inequality, and taking expectations, we find that

$$\begin{aligned} E[f(Z_{n+1} + 3) \mid Z_n = k] &\leq f(k + 3) + f'(k + 3)g(k) + \frac{1}{2}f''(k + 3)(g(k))^2 + \ell_2(k) \\ &\quad + C_1 \left(\frac{(g(k))^{2+\delta}}{(\log k)^{1+\alpha}k^{2+\delta}} + \frac{\ell_{2+\delta}(k)}{(\log k)^{1+\alpha}k^{2+\delta}} \right) \\ &\quad + C_2 \frac{\ell_{2+\delta}(k)}{k^{2+\delta}}, \quad k = 1, 2, \dots, \end{aligned}$$

for some positive constants C_i , $i = 1, 2$. By assumptions (i) and (ii) (of Theorem 2), and using reasoning similar to that used in the proof of Theorem 1, we find that, for k sufficiently large and $\epsilon > 0$ sufficiently small,

$$E[f(Z_{n+1} + 3) \mid Z_n = k] \leq f(k + 3) - \frac{\alpha g(k)}{(\log k)^{1+\alpha}k} (1 - \epsilon) + \frac{\alpha \ell_2(k)}{2(\log k)^{1+\alpha}k^2} (1 + \epsilon)$$

and, from assumption (iii), we deduce that if Z_n is sufficiently large, then

$$E[f(Z_{n+1} + 3) \mid \mathcal{F}_n] \leq f(Z_n + 3) \quad \text{a.s.,}$$

where, as in Theorem 1, \mathcal{F}_n is the σ -algebra generated by Z_0, \dots, Z_n . Application of Proposition 1(b) completes the proof.

5.3. Proof of Theorem 3

We will need the following preliminary result.

Proposition 2. *Under the hypotheses of Theorem 3, there exist continuous and nondecreasing functions $A_i, a_i: [0, 1] \rightarrow [0, 1]$, $i = 1, 2$, that have a continuous derivative on a small left-neighbourhood of 1 and satisfy the following conditions.*

- (a) $A_i(1) = a_i(1) = 1$, $A_i(0) > 0$, $A'_i(1) = c > 0$, and $a'_i(1) = 1$, for $i = 1, 2$.
- (b) For $i = 1, 2$, $a_i(s) > s$, $0 \leq s < 1$, and $a_i(s)$ has continuous second derivative on a small left-neighbourhood of 1, with $a''_i(1) = v > 0$.
- (c) There exist nonincreasing functions E_i and e_i , $i = 1, 2$, on $(0, 1)$, such that

$$\left| \frac{a_i(s) - s}{(1 - s)^2} - \frac{v}{2} \right| \leq e_i(s), \quad \left| \frac{1 - A_i(s)}{1 - s} - c \right| \leq E_i(s), \quad i = 1, 2,$$

and

$$\int_{\eta_i}^1 \frac{E_i(s)}{1 - s} ds < \infty, \quad \int_{\eta_i}^1 \frac{e_i(s)}{1 - s} ds < \infty, \quad \text{for some } \eta_i \in (0, 1), \quad i = 1, 2.$$

- (d) $A_1(s)(a_1(s))^k \leq h_k(s) \leq A_2(s)(a_2(s))^k$, $0 \leq s \leq 1, k = 1, 2, \dots$, where $h_k(s) := E[s^{Z_{n+1}} \mid Z_n = k]$.

Proof. Since $\phi_0(k)$ has infinitely divisible probability distribution, it follows that, for each $j \geq 1$, there exists a probability generating function f_{kj} (with associated probability distribution belonging to the same family of probability distributions as that of $\phi_0(k)$), such that

$g_k(s) = (f_{kj}(s))^j, 0 \leq s \leq 1$. Let us consider the sequence of probability generating functions $\{l_k\}_{k \geq 0}$, where $l_k(s) := f_{kk}(f(s)), 0 \leq s \leq 1$, with $f(s) := E[s^{X_{01}}]$. It is easy to verify that $h_k(s) = g_k(f(s)) = (l_k(s))^k$. Let

$$m_k := l'_k(1) \quad \text{and} \quad \pi_k := l''_k(1), \quad k = 1, 2, \dots$$

Taking into account the results of [8] and the existence of the functions A_i and $a_i, i = 1, 2$, parts (a)–(d) are guaranteed to hold if

- (p1) $m_k = 1 + k^{-1}\delta, k = 1, 2, \dots$, for some $\delta > 0$,
- (p2) $|\pi_k - \mu| \leq k^{-1}M, k = 1, 2, \dots$, for some $M > 0$ and $\mu > 0$, and
- (p3) $\sup_{k \geq 1} \sum_{j=0}^{\infty} j^{2+r} \pi_{kj} < \infty$ for some $r > 0$, where $\{\pi_{kj}\}_{j \geq 0}$ denotes the probability distribution associated with l_k .

Now,

$$m_k = k^{-1}\varepsilon(k)m = 1 + k^{-1}c, \quad k = 1, 2, \dots,$$

from which (p1) follows. Using assumptions (i) and (ii) (of Theorem 3), we have

$$\pi_k = k^{-1}\ell_2(k) + (\tau(k))^2 - \tau(k) = v + k^{-1}(O(1) + c(1 + k^{-1}c)), \quad k = 1, 2, \dots,$$

and, consequently, (p2) follows with $\mu = v$. By using assumptions (i) and (ii) again, we deduce that $(v(k))^2 = O(k)$,

Finally, we have

$$l'''_k(1) = 3mf''(1)f''_{kk}(1) + m^3 f'''_{kk}(1) + f'''(1)f'_{kk}(1).$$

Now, by assumptions (i) and (ii), we obtain

$$f'''(1)f'_{kk}(1) = f'''(1)k^{-1}\varepsilon(k) = O(1)$$

and

$$f''_{kk}(1) = k^{-2}(kg''_k(1) - (k - 1)(g'_k(1))^2) = O(1).$$

On the other hand, by assumption (iii), we have $f'''_{kk}(1) = (g_k^{1/k})'''(1) = O(1)$ and, therefore, (p3) holds.

We are now ready to prove Theorem 3. Part (a) follows from Proposition 2 if we apply Theorem 2.1(a) of [8]. To prove part (b), let A_i and $a_i, i = 1, 2$, be the functions provided in Proposition 2 and, for $j = 0, 1, \dots$, let us denote by $a_i^{(j)}$ the j th functional iterate of a_i , i.e. $a_i^{(j)}(s) = a_i(a_i^{(j-1)}(s))$ with $a_i^{(0)}(s) = s$. Under the conditions satisfied by A_i and a_i , by Lemma 3.3 of [8] we have

$$\prod_{j=0}^{n-1} A_i(a_i^{(j)}(0)) \sim n^{-2cv^{-1}} M_i, \quad i = 1, 2,$$

where $M_i, i = 1, 2$, are positive constants. Therefore,

$$\sum_{n=0}^{\infty} \prod_{j=0}^{n-1} A_i(a_i^{(j)}(0)) = \infty, \quad i = 1, 2,$$

and application of Lemmas 2.1 and 2.2 of [9] completes the proof.

5.4. Proof of Theorem 4

Let us introduce the function

$$G(x) := \int_1^x \frac{1}{g(z)} dz, \quad x \geq 1.$$

Using $g(x) = cx^\alpha + o(x^\alpha)$, $x > 0$, we can argue that

$$g'(x) = c\alpha x^{\alpha-1} + o(x^{\alpha-1}) \quad \text{and} \quad G(x) \sim (c(1-\alpha))^{-1}x^{1-\alpha}, \quad x \rightarrow \infty. \quad (7)$$

Proof of part (a). From (7), we have

$$\lim_{x \rightarrow \infty} g'(x)G(x) = (1-\alpha)^{-1} - 1$$

and, by assumption (ii) (of Theorem 4) and (7), we obtain

$$\lim_{k \rightarrow \infty} \frac{\ell_2(k)}{(g(k))^2 G(k)} = c^{-1}v(1-\alpha) > 0.$$

For simplicity, let $\lambda = (1-\alpha)^{-1}$ and $\gamma = c^{-1}v(1-\alpha)$. Taking into account the fact that $2cv^{-1} > 1$, we deduce that $\gamma\lambda < 2$. Therefore, the assumptions of Theorem 1 of [13] are satisfied, and we find that, for all real numbers z ,

$$\lim_{n \rightarrow \infty} P(n^{-1}G(Z_n) \leq z \mid Z_n \rightarrow \infty) = \Gamma_{2/\gamma-\lambda+1, \gamma/2}(z). \quad (8)$$

Now, by (7), in the event $\{Z_n \rightarrow \infty\}$ we have

$$c^{-1}\lambda Z_n^{1-\alpha} G(Z_n)^{-1} \rightarrow 1 \quad \text{a.s., as } n \rightarrow \infty. \quad (9)$$

Finally, from (8) and (9), Slutsky’s theorem, and some properties of the gamma distribution, the result is obtained.

Proof of part (b). In view of relation (3), this part will be proved using Theorem 3 of [11]. Notice that, from assumption (iii),

$$E[|\eta_{n+1}|^{2+\delta} \mid Z_n = k] = O((\varphi^2(k))^{1+\delta/2}), \quad (10)$$

where $\varphi^2(k) := E[\eta_{n+1}^2 \mid Z_n = k] = \ell_2(k)(g(k))^{-2}$.

Let us consider the function

$$\psi(x) := \int_1^x \frac{\varphi^2(z)}{g(z)} dz, \quad x \geq 1.$$

By assumption (ii) and the fact that $g(x) = cx^\alpha + o(x^\alpha)$, it follows that

$$\frac{\varphi^2(x)}{g(x)} \sim vc^{-3}x^{\beta-3\alpha}, \quad x \rightarrow \infty,$$

and, therefore, in the same limit, that

$$\psi(x) \sim \begin{cases} vc^{-3}(\beta - 3\alpha + 1)^{-1}(x^{\beta-3\alpha+1} - 1) & \text{if } \beta \neq 3\alpha - 1, \\ vc^{-3} \log x & \text{if } \beta = 3\alpha - 1. \end{cases} \quad (11)$$

Since $G(x) \sim (c(1 - \alpha))^{-1}x^{1-\alpha}$, we have $G^{-1}(x) \sim (c(1 - \alpha)x)^{(1-\alpha)^{-1}}$, where G^{-1} denotes the inverse function of G . Consequently, if $\widehat{\psi}(x) := \psi(G^{-1}(x))$ then

$$\widehat{\psi}(x) \sim \begin{cases} \nu c^{-3}(\beta - 3\alpha + 1)^{-1}((c(1 - \alpha)x)^{(\beta-3\alpha+1)/(1-\alpha)} - 1) & \text{if } \beta \neq 3\alpha - 1, \\ \nu c^{-3}(1 - \alpha)^{-1} \log x & \text{if } \beta = 3\alpha - 1. \end{cases} \tag{12}$$

The following statements now hold.

(A) The function g is positive, twice differentiable on \mathbb{R}^+ , and ultimately concave, and g' is ultimately convex.

(B) The function φ^2 is positive and continuously differentiable. Furthermore,

$$\int_1^x z^{-2}\varphi^2(G^{-1}(z)) \, dz \sim \int_1^x \nu c^{-2}(c(1 - \alpha))^{(\beta-2\alpha)/(1-\alpha)} z^{(\beta-2\alpha)/(1-\alpha)-2} \, dz$$

and, since $\beta < \alpha + 1$, the integral

$$\nu c^{-2}(c(1 - \alpha))^{(\beta-2\alpha)/(1-\alpha)} \int_1^\infty z^{(\beta-2)/(1-\alpha)} \, dz$$

is finite. Moreover, $\varphi^2(G^{-1}(x)) \sim \nu c^{-2}(c(1 - \alpha)x)^{(\beta-2\alpha)/(1-\alpha)}$, so it is ultimately concave or convex.

(C) If $\beta \geq 3\alpha - 1$ then, from (11), we obtain $\psi(\infty) = \infty$ and we also deduce that

$$|g''(x)g(x)/\varphi^2(x)| = O(x^{4\alpha-\beta-2}),$$

which is ultimately decreasing because $4\alpha - \beta - 2 < 0$. On the other hand, if $\beta < 3\alpha - 1$ then, again using (11), $\psi(\infty) < \infty$ and $|g''(x)g(x)| = O(x^{2(\alpha-1)})$, which is ultimately decreasing since $\alpha - 1 < 0$.

(D) For $\beta \geq 3\alpha - 1$, we have

$$g'(x)(\psi(x))^{1/2} = \begin{cases} O(x^{(\beta-(\alpha+1))/2}) & \text{if } \beta > 3\alpha - 1, \\ O((\log x)^{1/2}x^{\alpha-1}) & \text{if } \beta = 3\alpha - 1, \end{cases}$$

while, for $\beta < 3\alpha - 1$, we have

$$g'(x)(\psi(x))^{1/2} \sim \alpha \left(\frac{\nu(1 - x^{\beta-3\alpha+1})}{c(3\alpha - 1 - \beta)} \right)^{1/2} x^{\alpha-1}.$$

Hence, $g'(x)(\psi(x))^{1/2} \rightarrow 0$ as $x \rightarrow \infty$.

Considering (10), (12), and statements (A)–(D), the assumptions of Theorem 3 of [11] can be verified. Using the second part of this theorem, we obtain the result of part (b)1 of our Theorem 4 and, from its first part, we find that

$$a_n^{-1}Z_n \xrightarrow{P} 1 \quad \text{in the event } \{Z_r \rightarrow \infty\},$$

for $\beta \geq 3\alpha - 1$, and that

$$\lim_{n \rightarrow \infty} P\left(\left(\widehat{\psi}(n) \right)^{-1/2} \frac{Z_n - a_n}{g(a_n)} \leq z \mid Z_r \rightarrow \infty \right) = \phi^*(z),$$

for all real numbers z . (Recall that ϕ^* denotes the standard normal distribution function.) Finally, using Slutsky’s theorem and taking into account the fact that

$$\widehat{\psi}(n) \sim \begin{cases} \frac{\nu}{c^3(1-\alpha)} \log n & \text{if } \beta = 3\alpha - 1, \\ \frac{\nu}{\beta - 3\alpha + 1} c^{(\beta-2)/(1-\alpha)} ((1-\alpha)n)^{(\beta-3\alpha+1)/(1-\alpha)} & \text{if } \beta > 3\alpha - 1, \end{cases}$$

we deduce the result of part (b)2.

5.5. Proof of Corollary 1

(a) For simplicity, let us consider $Y_n = n^{-1}Z_n^{1-\alpha}$ and take $z > 0$. Then

$$P(Y_n \leq z) = P(Y_n \leq z \mid Z_r \rightarrow 0) P(Z_r \rightarrow 0) + P(Y_n \leq z \mid Z_r \rightarrow \infty) P(Z_r \rightarrow \infty)$$

and, using the fact that $\lim_{n \rightarrow \infty} P(Y_n \leq z \mid Z_r \rightarrow 0) = 1$, the fact that $Z_0 = N$, and Theorem 4(a), we deduce that

$$\lim_{n \rightarrow \infty} P(Y_n \leq z) = q_N + \Gamma_{a,b}(z)(1 - q_N).$$

Consequently, part (a)1 of Corollary 1 holds.

On the other hand, we have

$$P(Y_n \leq z, Z_n > 0) = P(Y_n \leq z, Z_n > 0, Z_r \rightarrow \infty) + P(Y_n \leq z, Z_n > 0, Z_r \rightarrow 0).$$

Now,

$$P(Y_n \leq z, Z_n > 0, Z_r \rightarrow \infty) = P(Y_n \leq z, Z_r \rightarrow \infty), \\ P(Y_n \leq z, Z_n > 0, Z_r \rightarrow 0) \leq P(Z_n > 0, Z_r \rightarrow 0),$$

and, since $\lim_{n \rightarrow \infty} P(Z_n > 0) = P(Z_r \rightarrow \infty)$ and $\lim_{n \rightarrow \infty} P(Z_n > 0, Z_r \rightarrow 0) = 0$, by Theorem 4(a), part (a)2 of Corollary 1 is deduced.

(b) The proof of part (b) is similar to that of part (a)2, taking $Y_n = \Delta_n^{-1/2}(g(a_n))^{-1}(Z_n - a_n)$ and using Theorem 4(b).

5.6. Proof of Theorem 5

Theorem 5 is easily proved using Proposition 2 and applying Lemma 2.3 of [9].

Appendix A. Lemmas

Lemma 1. For any $0 \leq k \leq 2$ and $x \geq -1$, we have

$$(1 + x)^k = 1 + kx + O(x^2).$$

Lemma 2. (Marcinkiewicz–Zygmund inequality.) If $\{X_n\}_{n \geq 1}$ is a sequence of independent and identically distributed random variables with $E[X_1] = 0$ and $E[|X_1|^p] < \infty$, $p \geq 2$, then $E[|\sum_{j=1}^n X_j|^p] = O(n^{p/2})$.

We refer the reader to the theorem of [14] for the proof of Lemma 1 and to [2, p. 387] for the proof of Lemma 2.

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