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ABSTRACT DANIELL-LOOMIS SPACES

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In [3] for general integral metric q an integral extension of Lebesgue power was discussed. In this paper we introduce the abstract Daniell-Loomis spaces R_p, p real, $0 < p < \infty$, of q-measurable functions with finite "p-norm", and study their basic properties.

1. INTRODUCTION

Recently in [3] an integral extension proceduce was given which works for general integral metric q. The basic ideas can be traced back to Loomis [9] and Schäfke [10]. One defines the extended functions of class B^q of real-valued functions on a set X with respect to a B^q type seminorm. Using an appropriate local mean convergence we proved convergence theorems; and we introduced q-measurability, which is defined by the property that truncation by integrable functions leads to integrable functions. It allowed us to treat abstract Riemann, that is finitely additive, integration theory, as a fundamental example and applied simultaneously to Loomis's abstract Riemann integration, as well as to the Daniell and Bourbaki integration theories.

In this paper, using the method announced in [3] we shall give a presentation of the abstract Daniell-Loomis spaces R_p, p real, $0 < p < \infty$.

For nonnegative extended real-valued functions f on X, if $p \geqslant 1$, $q_p(f) = [q(f^p)]^{1/p}$ satisfies the requirement of an integral metric, and essentially all the results discussed in [3] are true.

The relevant convergence properties with respect to q or q_p are developed. With weak continuity assumptions on the integral metric q, we prove as a fundamental result that the concepts of q- and q_p-measurability are equivalent (Theorem 1).

This leads us to define the abstract Danniell-Loomis spaces R_p as the class of q-measurable functions with finite $q_p(|.|)$. The simple functions B play the usual role in R_p: $R_p = B^{q_p}$ vector lattice (Theorem 2).

Finally, examples are presented which show that these results make it possible to study R_p-spaces for abstract Riemann or finitely additive, integration theory.

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1. Notation and assumptions.

In what follows we adhere to the notation and results of [**3**], and will be explained whenever necessary in order to make the paper self contained.

We extended the usual $+$ to $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by $r + s := 0$ if $r = -s \in \{\infty, -\infty\}$, $r - s := r + (-s)$. $\overline{\mathbb{R}}_+ := [0, \infty]$, $\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$.

We denote $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$ and $a \cap t := (a \wedge t) \vee (-t)$ if $a, b \in \overline{\mathbb{R}}$, $t \in \overline{\mathbb{R}}_+$.

For an arbitrary nonempty set X let $\overline{\mathbb{R}}^X$ consists of all functions $f\colon X \to \overline{\mathbb{R}}$. All operations and relations between functions are defined pointwise, with $\inf \phi := \infty$.

A functional $q\colon \overline{\mathbb{R}}_+^X \to \overline{\mathbb{R}}_+$ is called an *integral metric on* X if $q(0) = 0$ and $q(f) \leqslant q(g) + q(k)$ if $f \leqslant g + k$, $f, g, k \in \overline{\mathbb{R}}_+^X$.

If $B \subset \overline{\mathbb{R}}^X$, a function $f \in \overline{\mathbb{R}}^X$ is said to be *q-integrable* if it belongs to the closure of B in $\overline{\mathbb{R}}^X$ with respect to q, that is there exists $(h_n) \subset B$ with $q(|f - h_n|) \to 0$ as $n \to \infty$.

B^q denotes the set of all the q-integrable functions.

If additionally an $I\colon B \to \mathbb{R}$ is given which is uniformly continuous on B with respect to q, the unique q-continuous extension of I to B^q will be denoted I^q.

In all of the following, B will be a function vector lattice in $\mathbb{R}^X$, that is a real linear space of functions under pointwise $=, +, \alpha.$, such that $h \in B$ implies $|h| \in B$; then $k \wedge h$, $k \vee h \in B$ for $k, h \in B$. $I\colon B \to \mathbb{R}$ will be assumed linear with $I(h) \geqslant 0$ if $0 \leqslant h \in B$. Then, q-continuity of I in 0 implies uniform q-continuity of I on B.

We collect these assumption in

(1) I, B as above, q is an integral metric on X and I is q-continuous in 0.

With (1), B^q is closed with respect to $+, \alpha., \vee, \wedge, |.|$ and $I^q\colon B^q \to \overline{\mathbb{R}}$ is monotone, linear and q-continuous, (Theorem 1, [**3**]).

A function $f \in \overline{\mathbb{R}}^X$ is said to be *q-measurable* if $f \cap h \in B^q$ for all $0 \leqslant h \in B$.

$M_n(q, B)$ denotes the set of all the q-measurable functions.

For convergence theorems we need a suitable local convergence in the mean of [**3**, p.414].

(2) For $f, f_n \in \overline{\mathbb{R}}^X$, $n \in \mathbb{N}$, $f_n \to f(q, B)$ means that for each $\varepsilon > 0$ and $0 \leqslant h \in B$ there exists $n_0 = n(\varepsilon, h) \in \mathbb{N}$ such that $q(|f - f_n| \wedge h) < \varepsilon$ if $n \geqslant n_0$, (*q-local convergence*).

(3) Lebesgue's convergence theorem, (see Corollary VII, [**3**]): If (1) holds, f_n, $g \in B^q$, $f \in \overline{\mathbb{R}}^X$ is such that $f_n \to f(q, B)$ and $|f_n - f| \leqslant g$, $n \in \mathbb{N}$, then $f \in B^q$ and $q(|f_n - f|) \to 0$.

(4) For any integral metric q and $M \subset \overline{\mathbb{R}}^X$ the corresponding local integral

metric of Schäfke [**10**] (see also [**3**, p.416]) is defined by

$$q_\ell(f) := \sup\{q(f \wedge h);\ 0 \leqslant h \in M\} \text{ for all } f \in \overline{\mathbb{R}}_+^X.$$

With (1), q_ℓ is again an integral metric such that $q_\ell \leqslant q$ and $q_\ell(f) = q(f)$ if $0 \leqslant f \leqslant g$ for some $g \in B^q$. One has $B \subset B^q \subset B^{q_\ell}$ and $I^q = I^{q_\ell}$ on B^q.

For further properties of B^q and B^{q_ℓ} see [**3**].

2. R_p-SPACES

For $q\colon \overline{\mathbb{R}}_+^X \to \overline{\mathbb{R}}_+$, p real, $0 < p < \infty$, with $f^p(t) := (f(t))^p$, $0^p := 0$, $\infty^p := \infty$, we define for all $f \in \overline{\mathbb{R}}_+^X$

$$q_p(f) := \begin{cases} [q(f^p)]^{1/p} & \text{if } p \geqslant 1, \\ q(f^p) & \text{if } 0 < p < 1. \end{cases} \tag{5}$$

Note that the case $p = 1$ was studied in [**3**], and the natural question to consider is to what extent those results can be extended to values of p other than 1.

LEMMA 1. (See Lemma 12, [**3**].) *If $q := \overline{\mathbb{R}}_+^X \to \overline{\mathbb{R}}_+$ is an integral metric with $q(2f) = 2q(f)$, $0 < p < \infty$, then q_p is also an integral metric on X, positive-homogeneous if $p \geqslant 1$.*

PROOF: Observe that $2q(f) \leqslant q(2f)$ implies $q(tf) = tq(f)$, $0 < t < \infty$; also $|f+g|^p \leqslant f^p + g^p$ if $0 < p \leqslant 1$.

If $p > 1$, q_p satisfies Minkowski's inequality for finitely-valued f, g, by Bourbaki [**2**, p.12].

Now, we denote $f_e(x) := f(x)$ if $f(x) \in \mathbb{R}$, $f_e(x) := 0$ else, $f_u(x) := f(x) - f_e(x)$, $f_\infty := f_u \vee 0$.

If $f, g \in \overline{\mathbb{R}}_+^X$ with $q_p(f)$, $q_p(g) < \infty$, we have $[q_p(f+g)]^p \leqslant q[2^p(f^p + g^p)] < \infty$, and $\alpha q_p(f_\infty) = q_p(\alpha f_\infty) \leqslant q_p(f) < \infty$, so that $q_p(f_\infty) = 0$.

Therefore $q_p(f+g) \leqslant [q_p(f+g)_e^p + 0 + 0]^{1/p} \leqslant q_p(f_e + g_e) \leqslant q_p(f_e) + q_p(g_e) \leqslant q_p(f) + q_p(g)$. ∎

For positive-homogeneous integral metric q, Hölder's inequality holds:

(6) Let $1 < r$, $s < \infty$ be a pair of conjugate exponents, for functions $f, g \in \overline{\mathbb{R}}_+^R$ then $q(fg) \leqslant q_r(f) q_s(g)$.

(See for example [**8**, p.64–65], (6) follows with the aid of the expression $uv = \inf\{(1/p)t^r u^r + (1/s)t^{-s} v^s;\ t > 0\}$ for real $u, v \geqslant 0$.)

For positive-homogeneous integral metrics q, Sections 1, 2 of [**3**] hold for B^{q_p} and $B^{(q_p)_\ell}$, and using the q-local convergence of (2) one gets convergence theorems in a form analoguous to the classical ones.

In order to obtain the full results one has to impose certain conditions upon B and q.

(7) Let q be a positive-homogeneous integral metric on $\overline{\mathbb{R}}_+^R$, and $0 < p < \infty$. We assume

$|B|^p = |B|$ with $|B| := \{h;\ 0 \leqslant h \in B\}$.

$C_0(q, B)$: $q(h \wedge t) \to 0$ if $0 < t \to 0$, $0 \leqslant h \in B$, (q continuous at 0).

$C_\infty(q, B)$: $q(h - h \wedge t) \to 0$ if $t \to \infty$, $0 \leqslant h \in B$, (q continuous at ∞).

The above basic assumptions (1) and (7) will be retained in all that follows.

Observe that, with $|B|^p = |B|$, $C_0(q, B)$ implied $C_0(q_p, B)$.

Lemma 2. *Let q be a positive-homogeneous integral metric, then $C_\infty(q_p, B)$ holds, that is, $q_p(h - h \wedge t) \to 0$ if $t \to 0$, $0 \leqslant h \in B$.*

Proof: **Case** $1 \leqslant p < \infty$: Observe that $a^p + b^p \leqslant (a+b)^p$ if $a, b \in \overline{\mathbb{R}}_+$. Thus, $(t - t \wedge s)^p \leqslant t^p \wedge s^p$, $t, s \in \overline{\mathbb{R}}_+$. Therefore $[q_p(h - h \wedge t)]^p := q(h - h \wedge t)^p \leqslant q(h^p - h^p \wedge s^p) \to 0$ if $s \to \infty$.

Case $0 < p < 1$: We have $(h - h \wedge t)^p = ((h - h \wedge t)/\varepsilon)^p \varepsilon^p \leqslant \varepsilon^p((h - h \wedge t)/\varepsilon)$ if $h \geqslant t + \varepsilon$ and $\leqslant \varepsilon \wedge h$ if $h < t + \varepsilon$. So that $(h - h \wedge t)^p \leqslant \varepsilon^{p-1}(h - h \wedge t) + \varepsilon \wedge h$.

Now, if $\varepsilon \to 0$, $\eta > 0$, by $C_0(q, B)$, $q(h \wedge \varepsilon) < \eta/2$, and if $t \to \infty$, to $\eta > 0$, by $C_\infty(q, B)$, $\varepsilon^p q(h - h \wedge t) < \eta/2$. Hence, one has $q_p(h - h \wedge t) = q[(h - h \wedge t)^p] \leqslant q[\varepsilon^{p-1}(h - h \wedge t)] + q(h \wedge \varepsilon) < \eta/2 + \eta/2$, and the proof is complete. ∎

The equivalence between q-convergence and q_p-convergence is made explicit in the following lemmas.

Lemma 3. *Let $f, f_n \in \overline{\mathbb{R}}^X$, then $f_n \to f(q, B)$ implies $f_n \to f(q_p, B)$.*

Proof: One can assume $f = 0$ and $f_n \geqslant 0$. So, by (2) it suffices to show that given any $0 \leqslant h \in B$ if $q(f_n \wedge h) \to 0$ then $q_p(f_n \wedge h) \to 0$.

Case $1 \leqslant p < \infty$: Choose $0 \leqslant h \in B$, $l_n := f_n \wedge h$; by assumption $q(f_n \wedge h) \to 0$.

Now, if $0 < t \in \mathbb{R}$, $[q_p(l_n)]^p := [(q(l_n^p))^{1/p}]^p = q(l_n^p) = q(f_n^p \wedge h^p) \leqslant q[f_n^p \wedge (h^p \wedge t^p)] + q[(f_n^p \wedge h^p) - f_n^p \wedge (h^p \wedge t^p)] \leqslant q[f_n^p \wedge (h^p \wedge t^p)] + q(h^p - h^p \wedge t^p) \leqslant q(f_n^p \wedge (h^p \wedge t^p)) + \varepsilon$, if $t > t_{\varepsilon,h}$, by $C_\infty(q, B)$.

One has, $l_n^p \wedge t^p = (l_n \wedge t)^p = t^p((l_n \wedge t)/t)^p \leqslant t^p((l_n \wedge t)/t)$, since $p \geqslant 1$, $0 \leqslant (l_n \wedge t)/t \leqslant 1$.

Thus, if $t = t_{\varepsilon,h}$, $[q_p(l_n)]^p \leqslant q(l_n^p \wedge t^p) + \varepsilon \leqslant q(t^p((l_n \wedge t)/t)) + \varepsilon = \varepsilon + t^p(1/t)q(l_n \wedge t) = \varepsilon + t^{p-1}q(l_n) = \varepsilon + t^p q(f_n \wedge h) \leqslant 2\varepsilon$, if $n \geqslant n_\varepsilon$.

Hence, $q_p(f_n \wedge h) \to 0$ as $n \to \infty$, for each $0 \leqslant h \in B$, that is $f_n \to 0(q_p, B)$.

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Case $0 < p < 1$: We choose $0 \leqslant h \in B$, $t_{\varepsilon,h} > 0$ as above, and one has

$$q_p(f_n \wedge h) := q[(f_n \wedge h)^p] = q(f_n^p \wedge h^p) \leqslant q(f_n^p \wedge h^p \wedge t) + q(h^p - h^p \wedge t) \leqslant q(f_n^p \wedge h^p \wedge t) + \varepsilon/2,$$

if $t \geqslant t_{\varepsilon,h}$, by $C_\infty(q, B)$.

Hence, $q_p(f_n \wedge h) \leqslant q_p[(f_n \wedge s) \wedge h \wedge s] + \varepsilon/2$, if $s = t^{1/p} \geqslant t_{\varepsilon,h}$.

One can assume $f_n \leqslant s$, $h \leqslant s$, s fixed, $s = t_{\varepsilon,h}$.

If $A_{n,\delta} := \{x \in X;\ f_n(x) \geqslant \delta\}$, one gets $q_p(f_n \wedge h) = q(f_n^p \wedge h^p) \leqslant q\Big[\Big(s^p \chi_{A_{n,\delta}}\Big) \wedge h^p\Big] + q(\delta^p \wedge h^p)$.

Since $0 \leqslant h^p \in B$, $C_0(q, B)$ gives $q(\delta^p \wedge h^p) < \varepsilon/2$ if $\delta^p \leqslant \eta$, $0 < \eta < 1$; hence, $q_p(f_n \wedge h) \leqslant s^p q\Big(\chi_{A_{n,\delta}} \wedge (1/s^p)h^p\Big) + \varepsilon/2$.

Furthermore, $\delta q\Big(\chi_{A_{n,\delta}} \wedge (1/s^p)h^p\Big) = q\Big(\delta\chi_{A_{n,\delta}} \wedge (\delta/s^p)h^p\Big) \leqslant q[\delta\chi_{A_{n,\delta}} \wedge (h/s)^p]$, with δ fixed, $0 < \delta < \min\big(1, \delta^{1/p}\big)$.

Since $0 \leqslant (h/s)^p = (1/s)^p h^p \in B$, there exists $n_0 = n_0(\varepsilon, h, p, s, \delta)$ with $q[f_n \wedge (h/s)^p] < \delta\varepsilon/2\, s^{-p}$ if $n \geqslant n_0$.

Hence, $q_p(f_n \wedge h) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$, hence $f_n \to 0(q_p, B)$. The proof is complete. ▯

We recall that in Lemma 3, if $1 \leqslant p < \infty$ only $C_\infty(q, |B|^p)$ is needed. Also $q(kf) = k^\delta q(f)$ with $0 < \delta < \infty$, δ fixed, independent of $f \in \overline{\mathbb{R}}_+^X$, instead of q positive-homogeneous, is sufficient.

Lemma 4. *Let* $f, f_n \in \overline{\mathbb{R}}^X$, *then* $f_n \to f(q_p, B)$ *implies* $f_n \to f(q, B)$.

Proof: **Case** $1 \leqslant p < \infty$: Use Lemma 3 for $1/p \geqslant 1$, since $(q_p)_{1/p}(f) = [q(f)]^{1/p}$, then q_p is again positive-homogeneous and the assumptions for $1/p$ are fulfilled.

Case $0 < p < 1$: q_p is not positive-homogeneous, one has only $q_p(sf) = s^p q_p(f)$, and the proof of the first part of Lemma 3 works also (with $1/p$ instead of p), only in the last line one has, with $\overline{q} = q_p$ instead q, $t = t_{\varepsilon,h}$ fixed, $\overline{q}_{1/p}(l_n) = q_p\Big(l_n^{1/p}\Big) = q(l_n) \leqslant \varepsilon + \overline{q}\big(t^{1/p}(l_n \wedge t)/t\big) = \big(t^{1/p-1}\big)^p \overline{q}(l_n \wedge t) + \varepsilon \leqslant \varepsilon + t^{1-1/p}\overline{q}(l_n) \leqslant 2\varepsilon$, if $n \geqslant n_{\varepsilon,t}$, or $q_p\Big(l_n^{1/p}\Big) = \overline{q}(l_n) \leqslant 2\varepsilon$, and thus the assertion holds. ▯

Observe that $|B|^p = |B|$ implies $|B|^{1/p} = |B|$, so, this condition is also true in Lemma 2 for $1/p$.

The above results together with the Lebesgue convergence Theorem (3), is the key to proving that the concepts of q- and q_p-measurability are equivalent.

Theorem 1.

$$M_\cap(q, B) = M_\cap(q_p, B)$$

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PROOF: If $f \in M_\cap(q, B)$, by definition, for $0 \leqslant h \in B$, $f \cap h \in B^q$, so there are $h_n \in B$ with $h_n \to f \cap h(q, B)$; then also $h_n \cap h \to f \cap h(q, B)$. By Lemma 3, $h_n \cap h \to f \cap h(q_p, B)$. Since $|h \cap h - h_n \cap h| \leqslant 2h$, the Lebesgue convergence theorem for B^{q_p} (3), gives $f \cap h \in B^{q_p}$, for all $h \in B$, so that $f \in M_\cap(q_p, B)$.

On the other hand, if $f \in M_\cap(q_p, B)$, for all $0 \leqslant h \in B$ then $f \cap h \in B^{q_p}$, there are $h_n \in B$ with $h_n \to f \cap h(q_p, B)$. As above $h_n \cap h \to f \cap h(q, B)$, and the Lebesgue convergence theorem for B^q yields $f \cap h \in B^q$, so $f \in M_\cap(q, B)$. ∎

The class $R_p(B, I)$, or simply R_p, is defined as

$$R_p(B, I) := \{f \in \overline{\mathbb{R}}^X;\ f \text{ is } q\text{-measurable and } q_p(|f|) < \infty\}.$$

Our immediate goal is to show that, with additional weak assumptions on q, R_p is a vector lattice aspace, and the "simple functions" $f \in B$ are dense in the metric $q_p(|\cdot|)$.

For this we need Definition 7 of [3] and the following result concerning the q_ℓ-integrability of q-measurable functions f with $q_\ell(|f|) < \infty$.

An integral metric q is called *B-semiadditive* if one has

$$0 \leqslant h_n \in B,\ \sup\left\{q\left(\sum_{i=1}^{n} h_i\right);\ n \in \mathbb{N}\right\} < \infty \Rightarrow q(h_n) \to 0 \text{ as } n \to \infty,$$

and q is called *B-additive* if $0 \leqslant h$, $k \in B$ imply $q(h+k) = q(h) + q(k)$.

Obviously, q B-additive implies q B-semiadditive.

(8) If q is B-semiadditive and f is q-measurable such that $q_\ell(|f|) < \infty$, then $f \in B^{q_\ell}$ [**3**, Theorem 5].

We recall that by Lemma 1, q_p is an integral metric and $(q_p)_\ell \leqslant q_p$ on $\overline{\mathbb{R}}_+^X$.

THEOREM 2. *Let q be B-semiadditive and $1 \leqslant p < \infty$ or q B-additive and $0 < p < \infty$. Then $R_p := \{f \in M_\cap(q, B);\ q_p(|f|) < \infty\} = B^{q_p}$.*

PROOF: By Theorem 1, $f \in M_\cap(q, B)$ implies $f \in M_\cap(q_p, B)$, and if $q_p(|f|) < \infty$, $q_p B$-semiadditive, by (8), $f \in B^{q_p}$.

Hence, it is enough to show that q_p is B-semiadditive.

Case $1 \leqslant p < \infty$: If q is B-semiadditive, then $q\left(\sum_1^n h_i^p\right) \leqslant \left[q\left(\sum_1^n h_i\right)^p\right] = q_p\left(\sum_1^n h_i\right)^p < k^p$ for all n. Hence, $q(h_n^p) = [q_p(h_n)]^p \to 0$, so that, $q_p(h_n) \to 0$, as $n \to \infty$.

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Case $0 < p < 1$: q is B-additive by assumption. Suppose that q_p is not B-semiadditive, there exist h_n with $q_p(h_n) \geqslant \varepsilon_0$ and $q_p\left(\sum_1^m h_n\right) \leqslant k$, for all $m \in \mathbb{N}$. By Hölder's inequality, with $r = 1/p > 1$, $1/r + 1/s = 1$, $m\varepsilon_0 \leqslant \sum_1^m q_p(h_n) = q\left(\sum_1^m h_n^p.1\right) \leqslant q\left[(\sum h_n^{pr})^{1/r}.\left(\sum_1^m 1^s\right)^{1/s}\right] = q_p\left(\sum_1^m h_n\right)m^{1/s} \leqslant k\ m^{1/r}$ or $m^{1-1/s} \leqslant k/\varepsilon_0$ a contradiction.

Finally, observe that if $|B|^p = |B|$, $f \in B^{q_p}$ implies $q_p(|f|) < \infty$. One has the above equality if $q(h) < \infty$ for each $0 \leqslant h \in B$, and the proof is completed. □

Note that q-semiadditive is not needed in Theorem 1.

Let $N_p = N_p(B, I) := \{f \in \overline{\mathbb{R}}^X;\ q_p(|f|) = 0\}$ (q-nulfunctions).

One has $B \cup N_p \subset R_p$, N_p is closed with respect to $+, -, \alpha., |.|$.

For all $f, g \in \overline{\mathbb{R}}^X$, $f = g(q_p)$ means that $f - g \in N_p$, (see [**3**, p.412–413]).

Since $q_p(|f - g|) = 0$ implies $f = g(q_p)$, strictly speaking, the elements of R_p are equivalence class of functions defined on X.

With Theorem 2 the theory of integration presented in [**3**] is available.

3. Applications and examples (See Section 3 of [3].)

1. With $q(f) = I^-(f) := \inf\{I(g);\ f \leqslant g \in B\}$ for all $f \in \overline{\mathbb{R}}_+^X$, one has $B^q = R_{\text{prop}}(B, I)$ (proper Riemann-I-integrable functions or the "two-sided completion" of Loomis [**9**, p.170]).

If $q_\ell(f) = I_\ell^-(f)$ (of Definition (4)), one gets $R_1(B, I) := B^q =$ closure of B in $\overline{\mathbb{R}}^X$ with respect to the distance $d(f, g) := (I_\ell^-)(|f - g|)$ (abstract Riemann-I-integrable functions of [4]), containing the "one-sided completion" of Loomis [**9**, p.178]).

I^- and I_ℓ^- are positive-homogeneous integral metrics on $\overline{\mathbb{R}}_+^X$, also they are B-additive. Here, $R_p(B, I) = B^{(I_\ell^-)^p}$.

We recall that I_ℓ^- is the "essential upper functional" associated with I^- in the sense of Agner and Portenier [**1**], so that, $R_1(B, I)$ is the set of all the essentially integrable functions (with respect to I^-). Also, in Gould [**6**], Stone's axiom $B \wedge 1 \subset B$ is assumed, so by [**7**] his results are already subsumed by the R_1-space.

2. We consider now B, I arising from finitely-additive set functions μ, with arbitrary set X.

Ω is a semiring of sets from X, $\mu\colon \Omega \to \mathbb{R}_+$ is finitely additive on Ω, $B = B_\Omega =$ real-valued step functions on Ω, and $I = I_\mu = \int .d_\mu$ on B_Ω.

With $q = I_\mu^-$, $q_\ell = (I_\mu^-)_\ell$ one has $B_\Omega^q = R_{\text{prop}}(\mu, \Omega)$ (abstract proper Riemann-

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μ-integrable functions) and $B_\Omega^{\mathfrak{ql}} = R_1(\mu, \Omega)$ (Riemann-μ-integrable functions of [7]), which contains $L(X, \Omega, \mu, \mathbb{R})$ of Dunford-Schwartz [5].

In this situation, I_μ^- is B_Ω-additive and a positive-homogeneous integral metric on X. Also, B_Ω is Stonian, $C_\infty(I_\mu^-, B_\Omega)$ and $C_0(I_\mu^-, B_\Omega)$ of (7) hold.

With (1), if I satisfies Daniell's condition (or I is σ-continuous), that is, $I(h_n) \to 0$ whenever $0 \leqslant h_n \in B$, $h_n \geqslant h_{n+1} \to 0$ pointwise on X, one has that $q = I^\sigma(f) := \inf\left\{\sum_{n=1}^{\infty} I(h_n);\ h_n \in B,\ f \leqslant \sum_{n=1}^{\infty} h_n\right\}$ for all $f \in \overline{\mathbb{R}}_+^X$, is the induced B-additive integral norm with Daniell's $L^1 = B^q$.

Finally, if Ω is a σ-ring and μ is σ-additive, then $R_q(\mu, \Omega) = L^1(\mu, \Omega)$ modulo nulfunctions by [7, p.265].

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