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THE TILING SEMIGROUPS OF ONE-DIMENSIONAL PERIODIC TILINGS

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Abstract

A one-dimensional tiling is a bi-infinite string on a finite alphabet, and its tiling semigroup is an inverse semigroup whose elements are marked finite substrings of the tiling. We characterize the structure of these semigroups in the periodic case, in which the tiling is obtained by repetition of a fixed primitive word.

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1. Introduction

The fundamental algebraic representation of the structure of a tiling $\mathcal{T}$ of $\mathbb{R}^n$ is its *tiling semigroup* $S(\mathcal{T})$, see [2]. $S(\mathcal{T})$ is an inverse semigroup with zero, whose nonzero elements comprise translation classes of connected, finite portions of $\mathcal{T}$ with two distinguished tiles. The product of two such classes in $S(\mathcal{T})$ is the class of the union of translates, matched by using the distinguished tiles. Informally, the tiling semigroup records the structure of $\mathcal{T}$ in terms of its assembly from finite pieces.

In this paper we consider only *one-dimensional* tilings, that is tilings of the real line $\mathbb{R}$. In this case, the theory has special features that merit a separate study. A tiling of $\mathbb{R}$ can be regarded as a bi-infinite string on some alphabet, and the nonempty substrings of the tiling constitute the *language* of the tiling. One-dimensional tilings and their tiling semigroups are studied from a language-theoretic viewpoint in [5, 8]. In this paper we concentrate on the algebraic structure of tiling semigroups and give a complete description of the tiling semigroups of one-dimensional *periodic* tilings.

At the heart of our description is the free monogenic inverse monoid FIM_1, which (with a zero adjoined) is the tiling semigroup of the one-dimensional tiling using a single repeated tile (see [3, Theorem 4.5.4]). The structure of the tiling semigroup $S(\mathcal{T})$ of a general periodic one-dimensional tiling $\mathcal{T}$ is given in our main result

(Theorem 4.1). The structural description has two components, reflecting the coarse and the fine structure of elements of the tiling semigroup. The free monogenic inverse monoid is used to record the coarse structure, and a coordinatization given by subsets of $\mathbb{Z}_m$ (where m is the length of the period of the tiling) is used to record the fine structure that is determined by the combinatorics of the period. Our description also evinces the strongly E^*-unitary property of tiling semigroups (see [7]): that is, we exhibit $S(\mathcal{T})$ as a Rees quotient of an E-unitary inverse semigroup. In detail, we show that $S(\mathcal{T})$ embeds into a Rees quotient of a semidirect product in which FIM_1 acts on the subsets of $\mathbb{Z}_m$, and we characterize which subsemigroups arise in this way.

2. One-dimensional tiling semigroups

We immediately adopt a point of view close to formal language theory, which is adequate for our purposes. For a more general account of tiling semigroups, see [2].

Let Σ be a finite alphabet with cardinality n. A word over Σ is called *primitive* if it is not a proper power in the free semigroup Σ^+. A one-dimensional tiling $\mathcal{T}$ over Σ is a bi-infinite string $\ldots t_{-1}t_0t_1t_2 \ldots$ over Σ, and a *pattern* in $\mathcal{T}$ is a finite string over Σ that occurs as a substring of $\mathcal{T}$. We say that $\mathcal{T}$ is *periodic* if there exists a positive integer k such that $t_i = t_{i+k}$ for all $i \in \mathbb{Z}$. The smallest k satisfying this condition is called the *period* of $\mathcal{T}$. If $\mathcal{T}$ is a periodic tiling of period k, then a pattern of $\mathcal{T}$ of length k is called a *period* of $\mathcal{T}$. Clearly a period of $\mathcal{T}$ is a primitive word.

A *marked pattern* is a pattern with two (not necessarily distinct) distinguished letters in the string, called the *in-tile* and the *out-tile*. When writing marked patterns, we shall mark the in-tile a with a grave accent $\grave{a}$ and the out-tile b with an acute accent $\acute{b}$ – this is the reverse of the conventions in [3, 5] but agrees with those of [8]. If the in-tile and the out-tile coincide then we have a *pointed* string, and we mark the distinguished tile with a check accent $\check{a}$.

Following Lawson [5], we define the tiling semigroup via an intermediate construction. The *Kachel semigroup* $K(\Sigma)$ of Σ is the semigroup with zero whose nonzero elements are all the marked strings over the alphabet Σ (with no reference to any tiling $\mathcal{T}$). The product of two such marked strings u and v is formed as follows. Match the out-tile of u with the in-tile of v and examine the overlap of the strings: if the letters in the overlap all agree, then the product uv is the string equal to the union of u and v with the letters in the overlap identified, marked at the in-tile of u and the out-tile of v. If the letters of the overlap do not agree, then $uv = 0$.

EXAMPLE 1. Take $u = a\grave{b}cb\acute{c}ab$ and $v = \acute{b}\grave{c}abbc$. Match the out-tile of u with the in-tile of v:

$$\begin{array}{cccccccc} a & \grave{b} & c & b & \acute{c} & a & b & \\ & & & \acute{b} & \grave{c} & a & b & b & c. \end{array}$$

Look at the overlap: since the overlapping letters agree then uv is the combined string

$$uv = a\grave{b}c\acute{b}cabbc.$$

However, if $u = a\grave{b}cb\acute{c}ab$ and $v = \acute{b}\grave{c}bcbc$ then $uv = 0$.

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PROPOSITION 2.1 [3, Theorem 9.5.3]. *The Kachel semigroup $K(\Sigma)$ is an inverse semigroup with zero. The semilattice of idempotents $E(K(\Sigma))$ is the set of pointed strings over Σ, with partial order given by $u \leqslant v$ if and only if v is a marked substring of u.*

Now recall our interest in the tiling $\mathcal{T}$. As in [5], let $I(\mathcal{T})$ be the set of marked strings in $K(\Sigma)$ whose underlying string is not a pattern in $\mathcal{T}$, together with 0. Then $I(\mathcal{T})$ is an ideal in $K(\Sigma)$, and the Rees quotient $K(\Sigma)/I(\mathcal{T})$ is the *tiling semigroup* $S(\mathcal{T})$ of $\mathcal{T}$. So $S(\mathcal{T})$ is the semigroup with zero whose nonzero elements are all the marked patterns in $\mathcal{T}$. Two marked patterns u and v are multiplied as follows: uv is the product of u and v in the Kachel semigroup $K(\Sigma)$ if the underlying string of this product occurs in $\mathcal{T}$, and otherwise their product is 0.

We begin with the simplest tiling of all: the one-dimensional tiling using a single tile. This will serve as a first insight into the general structure of tiling semigroups, and as a motivating example for our specific investigation into the structure of the tiling semigroups of one-dimensional periodic tilings.

Let $\mathcal{T}$ be such a tiling, with tile t. A pattern in $\mathcal{T}$ is just a finite string on the singleton alphabet $\{t\}$ and so is determined by its length, and a marked pattern is such a string with two distinguished occurrences of t. A marked pattern in $\mathcal{T}$ is therefore determined by a triple of integers (i, j, k) where $i \leqslant 0$, $j \geqslant 0$ and $i \leqslant k \leqslant j$, which describes a pattern whose tiles are indexed by the integers r with $i \leqslant r \leqslant j$ and with in-tile 0 and out-tile k. The product of two such elements is given by

$$(i, j, k)(a, b, c) = (\min(i, k + a), \max(j, k + b), k + c).$$

This is one of the possible descriptions of the elements of the free monogenic inverse monoid FIM_1, and their multiplication: see [9, Proposition IX.1.1].

PROPOSITION 2.2. *The tiling semigroup of the one-dimensional tiling using a single tile is isomorphic to the free monogenic inverse monoid with a zero adjoined.*

For this tiling, the zero of $S(\mathcal{T})$ is removable, and this is the convention followed in the statement of [3, Theorem 9.5.4]. Our main result (Theorem 4.1) may be seen as a generalization of Proposition 2.2: we describe the structure of the tiling semigroups of any one-dimensional periodic tiling.

3. Power sets, partitions, and cadences

In this section we introduce the notions needed to formulate our structural description of the tiling semigroup of a one-dimensional periodic tiling.

3.1. Power sets and semidirect products Let S be a semigroup and let G be a group with a homomorphism $\lambda : S \to G$. Let $\mathcal{P}(G)$ be the power set of G, and set $\mathcal{P}^*(G) = \mathcal{P}(G) \setminus \{\emptyset\}$. If $Y \subseteq G$ is nonempty and $g \in G$, then we define $Yg = \{yg \mid y \in Y\}$. If $Y = \emptyset$, we set $Yg = \emptyset$. Then S acts on $\mathcal{P}(G)$ on the right as follows: given $Y \in \mathcal{P}(G)$ and $s \in S$ we define $s \cdot Y = Y(s\lambda)^{-1}$. Using this action, we can form

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the semidirect product $\mathcal{P}(G) \rtimes S$ with underlying set $\mathcal{P}(G) \times S$ and multiplication $(X, s)(Y, t) = (X \cap s \cdot Y, st)$. Then $\mathcal{P}(G) \rtimes S$ is a semigroup, and $\{\emptyset\} \times S$ is a two-sided ideal. We denote the Rees quotient $(\mathcal{P}(G) \rtimes S)/(\{\emptyset\} \times S)$ by $\mathcal{P}(G, S, \lambda)$.

The properties of $\mathcal{P}(G, S, \lambda)$ are as follows:

- (K, s) is a (nonzero) idempotent in $\mathcal{P}(G, S, \lambda)$ if and only if s is an idempotent in S;
- if S is regular then so is $\mathcal{P}(G, S, \lambda)$, for if t is an inverse for s in S then $(t \cdot K, t)$ is an inverse for (K, s);
- if S is inverse then so is $\mathcal{P}(G, S, \lambda)$;
- if S is E-unitary then $\mathcal{P}(G, S, \lambda)$ is strongly E^*-unitary;
- if S is F-inverse then $\mathcal{P}(G, S, \lambda)$ is F^*-inverse, for if $s \leqslant \widehat{s}$ with $\widehat{s}$ maximal, then $0 \neq (K, s) \leqslant (G, \widehat{s})$ with $(G, \widehat{s})$ maximal.

We refer to [4, 7] for further information on the class of strongly E^*-unitary semigroups, and its relationships with the algebra of tilings.

In what follows we shall only need to consider $\mathcal{P}(G, S, \lambda)$ when $G = \mathbb{Z}_m$, S is the free monogenic inverse monoid FIM_1, and λ is the composite $FIM_1 \to \mathbb{Z} \to \mathbb{Z}_m$ of the map from FIM_1 to its maximum group image $\mathbb{Z}$ followed by the canonical map $\mathbb{Z} \to \mathbb{Z}_m$. Since $\mathbb{Z}_m$ is written additively, for $u \in FIM_1$ and $B \subseteq \mathbb{Z}_m$ we have $u \cdot B = B - u\lambda$. Hence $\mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$ is a semigroup with zero whose underlying set of nonzero elements is $\mathcal{P}^*(\mathbb{Z}_m) \times FIM_1$, with the product of (A, u) and (B, v) defined by

$$(A, u)(B, v) = \begin{cases} (A \cap (B - u\lambda), uv) & \text{if } A \cap (B - u\lambda) \neq \emptyset, \\ 0 & \text{if } A \cap (B - u\lambda) = \emptyset. \end{cases}$$

Well-known properties of FIM_1 then imply that $\mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$ is a strongly F^*-inverse semigroup.

3.2. Sifting partitions Let Π be a partition of $\mathbb{Z}_m$ with blocks $B_1, \ldots, B_k$. The *sift* $\sigma^1(\Pi)$ of Π is the partition of $\mathbb{Z}_m$ whose blocks are the nonempty subsets of the form $B_i \cap (B_j - 1)$, where $1 \leqslant i, j \leqslant k$. Clearly $\sigma^1(\Pi)$ is a refinement of Π. We define the rth sift σ^r for $r \geqslant 0$ by $\sigma^0(\Pi) = \Pi$ and $\sigma^r(\Pi) = \sigma^1(\sigma^{r-1}(\Pi))$, and we call Π a *sifting partition* if, for some $r \geqslant 0$, the rth sift of Π is the partition of $\mathbb{Z}_m$ into m singleton blocks. Sifting partitions may be characterized as the following result.

PROPOSITION 3.1. *A partition Π of $\mathbb{Z}_m$ is a sifting partition if and only if the only congruence refining Π is the equality congruence on $\mathbb{Z}_m$.*

PROOF. Suppose that some congruence Ξ is a refinement of Π, and let C be a block of Ξ. Then $C + 1 = C'$ for some block C' of Ξ, and so $C = C \cap (C' - 1)$ is contained in some block of the sift $\sigma^1(\Pi)$. It follows that Ξ refines every sift of Π, and so if Π is sifting, then Ξ must be the equality congruence.

On the other hand, if Π is not sifting, then $\sigma^r(\Pi) = \sigma^{r+1}(\Pi)$ for some r, with $\sigma^r(\Pi)$ not the partition of $\mathbb{Z}_m$ into singleton sets. Then for each block B_i of $\sigma^r(\Pi)$, since B_i is also a block of $\sigma^{r+1}(\Pi)$, there exists a block B_j such that $B_i \subseteq B_j - 1$.

Hence if $a, b \in B_i$ then $a + 1, b + 1 \in B_j$ and $\sigma^r(\Pi)$ is a congruence refining Π, and not equal to the equality congruence. □

Since any congruence on $\mathbb{Z}_m$ is given by congruence modulo some subgroup, we may restate Proposition 3.1 as the following result.

COROLLARY 3.2. *A partition Π of $\mathbb{Z}_m$ is not a sifting partition if and only if the block containing 0 also contains a nontrivial subgroup H of $\mathbb{Z}_m$ such that every block is a union of cosets of H.*

3.3. Cadences We shall need a specific construction of a sifting partition associated to any one-dimensional periodic tiling. This is obtained from the cadences of the period, as defined in [6] (generalizing the original definition of cadence in [1]).

Let $\Sigma = \{t_1, t_2, \ldots, t_n\}$ be an alphabet and let w be a word on Σ that involves each letter of Σ. Suppose that $|w| = m$ and write $w = t_{i_0} t_{i_1} \ldots t_{i_{m-1}}$. For each $t_j \in \Sigma$ define

$$C_j(w) = \{r \in \mathbb{Z}_m \mid t_{i_r} = t_j \text{ in } w\}.$$

The set $C_j(w)$ is called the t_j-*cadence* of w, and records the locations of occurrences of t_j in w. The set of cadences $\{C_j(w) \mid 1 \leqslant j \leqslant n\}$ of w is a partition of $\mathbb{Z}_m$, which we call the w-*cadence partition*.

PROPOSITION 3.3. *Let $w \in \Sigma^+$ involve each letter of Σ. Then w is primitive if and only if the w-cadence partition is a sifting partition of $\mathbb{Z}_m$.*

PROOF. Suppose that the w-cadence partition is not a sifting partition. Then by Corollary 3.2 we may suppose that $0 \in C_1(w)$, that $C_1(w)$ contains the subgroup $\langle r \rangle$, $(r \geqslant 1)$ of $\mathbb{Z}_m$ and that each block is a union of cosets of $\langle r \rangle$. Let u be the prefix of w of length r: then $w = u^{|w|/r}$ and so is a proper power.

Conversely, suppose that $w = u^k$ with $|u| = r$ and $k > 1$. Then $i \in C_j(w)$ if and only if $\{i, i + r, \ldots, i + (k - 1)r\} \subseteq C_j(w)$, which implies that the cadence containing 0 also contains the subgroup $\langle r \rangle$ of $\mathbb{Z}_m$, and every cadence is a union of cosets of this subgroup. Hence, by Corollary 3.2, the cadences do not form a sifting partition. □

We also have a useful interpretation of the sifts of the w-cadence partition. Recall that a word $v \in \Sigma^+$ is called w-*periodic* (see [6]) if it is a subword of some power w^q of w. An easy induction on the length r of a w-periodic word v then establishes the following result.

LEMMA 3.4. *A subset $A \subseteq \mathbb{Z}_m$ is a block in the rth sift of the w-cadence partition if and only if it is the set of locations of the initial letter of a w-periodic word v with length $r + 1$.*

4. The main theorem

In this section we establish our characterization of the tiling semigroups of one-dimensional periodic tilings. We give a coordinate system to describe patterns in one-dimensional periodic tilings, so that each pattern is specified uniquely by a certain

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triple of integers and a subset of $\mathbb{Z}_m$, and we show that choosing coordinates gives an embedding of the tiling semigroup into the inverse semigroup $\mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$. Moreover, we specify precisely which inverse subsemigroups of $\mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$ arise as images of such embeddings.

To this end, we make the following definition. An inverse subsemigroup S of $\mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$ is a *sifting subsemigroup* if it satisfies the following conditions:

- the collection of subsets $\Pi(S) = \{B \subseteq \mathbb{Z}_m \mid (B, 0, 0, 0) \in S\}$ is a sifting partition of $\mathbb{Z}_m$;
- $(L, i, j, k) \in S$ if and only if $L + i$ is a block in the $(-i + j)$-sift of $\Pi(S)$.

THEOREM 4.1. *A semigroup S is the tiling semigroup of a one-dimensional periodic tiling with period of length m if and only if it is isomorphic to a sifting subsemigroup of $\mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$.*

PROOF. Let Σ be a finite alphabet and $\mathcal{T}$ be a one-dimensional periodic tiling over Σ. Let $p = p_0 \dots p_{m-1}$ be a period of $\mathcal{T}$. Let $S(\mathcal{T})$ denote the tiling semigroup of $\mathcal{T}$. For $s \in S(\mathcal{T})$ let $\pi(s)$ denote the pattern underlying s. That is, if $s = z_1 \dots \acute{z}_i \dots \grave{z}_j \dots z_k$ or $s = z_1 \dots \grave{z}_i \dots \acute{z}_j \dots z_k$, then $\pi(s) = z_1 \dots z_k$. For $s = z_1 \dots \grave{z}_i \dots \acute{z}_j \dots z_k \in S(\mathcal{T})$, we set $\tau(s) = (1 - i, k - i, j - i)$. The absolute value of the first component of $\tau(s)$ tells us the number of tiles preceding the in-tile of s, the second component tells us the number of tiles succeeding the in-tile of s and the absolute value of the third component tells the distance between the in-tile and the out-tile of s. Clearly $1 - i \leqslant 0 \leqslant k - i$ and $1 - i \leqslant j - i \leqslant k - i$ and so $\tau(s) \in FIM_1$. Given $s \in S$ with $\tau(s) = (i, j, k)$, we set

$$\Omega(s) = \{r \in \mathbb{Z}_m \mid \pi(s) = p_{r-i} \dots p_r \dots p_{r+j}\}.$$

Hence $r \in \Omega(s)$ precisely when $s = p_{r-i} \dots \grave{p}_r \dots \acute{p}_{r+k} \dots p_{r+j}$ or when $s = p_{r-i} \dots \acute{p}_{r+k} \dots \grave{p}_r \dots p_{r+j}$. In the remainder of the proof, we shall assume for notational convenience that $k \geqslant 0$, that is the out-tile is placed to the right of the in-tile.

Define $\varphi : S(\mathcal{T}) \to \mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$ by $s \mapsto (\Omega(s), \tau(s))$, $0 \mapsto 0$. By construction, $s \in S(\mathcal{T})$ is completely determined by $(\Omega(s), \tau(s))$ and so φ is injective.

Next, we show that φ is a homomorphism. Let $s, t \in S(\mathcal{T})$ and assume that $\tau(s) = (i, j, k)$ and $\tau(t) = (a, b, c)$. First we assume that $st \neq 0$. Matching the out-tile of s and the in-tile of t, we find that $\pi(s)$ and $\pi(t)$ agree on their overlap. By glueing together these patterns along the overlap we obtain $\pi(st)$. By the definition of the multiplication in $S(\mathcal{T})$, we know that the absolute value of the number of tiles preceding the in-tile of st will be $\min(i, k + a)$, the number of tiles succeeding the in-tile of st will be $\max(j, k + b)$ and the absolute value of the distance between the in-tile and the out-tile of st will be $k + c$. Hence,

$$\tau(s)\tau(t) = (i, j, k)(a, b, c) = (\min(i, k + a), \max(j, k + b), k + c) = \tau(st).$$

We set $u = \min(i, k + a)$ and $v = \max(j, k + b)$. Let $r \in \Omega(s) \cap (\Omega(t) - k)$. Then

$$s = p_{r+i} \dots \grave{p}_r \dots \acute{p}_{r+k} \dots p_{r+j}$$

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and

$$t = p_{r+k+a} \dots \grave{p}_{r+k} \dots \acute{p}_{r+k+c} \dots p_{r+k+b}.$$

It follows that $st = p_{r+u} \dots \grave{p}_r \dots \acute{p}_{r+k+c} \dots p_{r+v}$ verifying that $r \in \Omega(st)$. Conversely, let $r \in \Omega(st)$. Then $st = p_{r+u} \dots \grave{p}_r \dots \acute{p}_{r+k+c} \dots p_v$ and it follows that

$$s = p_{r+i} \dots \grave{p}_r \dots \acute{p}_{r+k} \dots p_{r+j}$$

and

$$t = p_{r+k+a} \dots \grave{p}_{r+k} \dots \acute{p}_{r+k+c} \dots p_{r+k+b}$$

verifying that $r \in \Omega(s) \cap (\Omega(t) - k)$. Thus we may conclude that if $st \neq 0$, then

$$\begin{aligned}(s\varphi)(t\varphi) &= (\Omega(s), \tau(s))(\Omega(t), \tau(t)) \\ &= (\Omega(s) \cap (\Omega(t) - k), \tau(st)) = (\Omega(st), \tau(st)) = (st)\varphi.\end{aligned}$$

If $st = 0$, then the patterns underlying s and t do not agree on their overlap. It follows that $\Omega(s) \cap (\Omega(t) - k) = \emptyset$, and so

$$(s\varphi)(t\varphi) = (\Omega(s), \tau(s))(\Omega(t), \tau(t)) = (\Omega(s) \cap (\Omega(t) - k), \tau(st)) = 0 = (st)\varphi.$$

This completes the proof that φ is a homomorphism.

To see that the image of $S(\mathcal{T})$ is a sifting semigroup in $\mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$, we make the following observations. Since p is a period of $\mathcal{T}$, it is a primitive word. Clearly the p-cadence partition is $\Pi(S)$, and by Proposition 3.3, $\Pi(S)$ is a sifting partition of $\mathbb{Z}_m$. Now let $s \in S$ with $s\varphi = (L, i, j, k)$. Then $\pi(s)$ has length $-i + j + 1$ and $r \in L$ precisely when $s = p_{r+i} \dots \grave{p}_r \dots \acute{p}_{r+k} \dots p_{r+j}$, and so $L + i$ is the set of locations of the initial letter of $\pi(s)$. By Lemma 3.4 it follows that $L + i$ is a block in the $(-i + j)$th sift of $\Pi(S)$.

Conversely, suppose that S is a sifting subsemigroup of $\mathcal{P}(\mathbb{Z}_m, 1)$. Then $\Pi(S)$ is a sifting partition of $\mathbb{Z}_m$. If $\Pi(S)$ has n blocks $B_1, \dots, B_n$, then we construct a period $p = p_0 p_1 \dots p_{m-1}$ of length m on an alphabet $\{t_1, \dots, t_n\}$ with $p_i = t_j$ if and only if $i \in B_j$. Let $\mathcal{T}$ be the one-dimensional tiling with period w. Then $\Pi(S)$ is the w-cadence partition, and $(L, i, j, k) \in S(\mathcal{T})\varphi$ if and only if $L + i$ is a block in the $(-i + j)$-sift of $\Pi(S)$. It follows that $S(\mathcal{T})\varphi = S$. □

4.1. When the period involves each tile exactly once The description of the structure of the tiling semigroup of a one-dimensional periodic tiling given above simplifies considerably in the case when the period p involves each tile exactly once, so that $n = m$. We write $p = p_0 p_1 \dots p_{m-1}$, and the cadence partition is the partition of $\mathbb{Z}_m$ into m singleton blocks. The sifting subsemigroups of $\mathcal{P}(\mathbb{Z}_m, FIM_1, \lambda)$ corresponding to this sort of tiling are those in which all the elements have the form $(\{x\}, i, j, k)$, and every such subsemigroup is trivially a sifting subsemigroup.

We can simplify the structural description given in Theorem 4.1 as follows. The length of the period is a complete invariant for the tiling, and so determines the

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tiling semigroup. Let P_m be the semigroup with zero with underlying set of nonzero elements $\mathbb{Z}_m \times FIM_1$ and with the product of two nonzero elements given by

$$(x, i, j, k)(y, a, b, c) = \begin{cases} (x, \min(i, a+k), \max(j, b+k), c+k) & \text{if } x+k=y, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 4.2. *Let $\mathcal{T}$ be a one-dimensional periodic tiling for which the period has length m and involves each tile exactly once. Then the tiling semigroup $S(\mathcal{T})$ is isomorphic to P_m.*

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