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ON QUASI-MONOTONE SEQUENCES AND THEIR APPLICATIONS

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In this paper using δ-quasi-monotone sequences a theorem on $\left|\overline{N}, p_n\right|_k$ summability factors of infinite series, which generalises a theorem of Mazhar [7] on $|C, 1|_k$ summability factors of infinite series, is proved. Also we apply the theorem to Fourier series.

1. INTRODUCTION

A sequence (c_n) of positive numbers is said to be quasi-monotone if $n\Delta c_n \geqslant -\alpha c_n$ for some $\alpha > 0$ and it is said to be δ-quasi-monotone, if $c_n \to 0$, $c_n > 0$ ultimately and $\Delta c_n \geqslant -\delta_n$, where (δ_n) is a sequence of positive numbers (see [1]).

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By u_n and t_n we denote the nth $(C, 1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geqslant 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

But since $t_n = n(u_n - u_{n-1})$ (see [6]), condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1.2}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \ (P_{-i} = p_{-i} = 0, \ i \geqslant 1). \tag{1.3}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

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defines the sequence (w_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [5]). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geqslant 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |w_n - w_{n-1}|^k < \infty. \tag{1.5}$$

In the special case when $p_n = 1$ for all values of n (respectively $k = 1$), then $|\overline{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (respectively $|\overline{N}, p_n|$) summability. If we write

$$X_n = \sum_{v=0}^{n} p_v/P_v, \tag{1.6}$$

then (X_n) is a positive increasing sequence tending to infinity with n.

2.

Mazhar [7] has proved the following theorem for $|C, 1|_k$ summability factors by using δ-quasi-monotone sequences.

THEOREM A. *Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) which it is δ-quasi-monotone with $\sum n\delta_n \log n < \infty$, $\sum A_n \log n$ is convergent and $|\Delta\lambda_n| \leqslant |A_n|$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(\log m) \text{ as } m \to \infty, \tag{2.1}$$

then the series $\sum a_n \lambda_n$ is summable $|C, 1|_k$, $k \geqslant 1$.

3.

The aim of this paper is to generalise Theorem A for $|\overline{N}, p_n|_k$ summability. Now we shall prove the following theorem.

THEOREM 1. *Let $\lambda_n \to 0$ as $n \to \infty$ and let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \text{ as } n \to \infty. \tag{3.1}$$

Suppose that there exists a sequence of numbers (A_n) which is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta\lambda_n| \leqslant |A_n|$ for all n. If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \text{ as } m \to \infty, \tag{3.2}$$

then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geqslant 1$.

REMARK. It should be noted that if we take $p_n = 1$ for all values fo n (in this case $X_n \sim \log n$) in Theorem 1, then we get Theorem A.

4.

We need the following lemmas for the proof of Theorem 1.

LEMMA. *Under the conditions of Theorem 1, we have that*

$$|\lambda_n| X_n = O(1) \text{ as } n \to \infty. \tag{4.1}$$

PROOF: Since $\lambda_n \to 0$ as $n \to \infty$.

$$\begin{aligned} |\lambda_n| X_n = X_n \left| \sum_{v=n}^{\infty} \Delta\lambda_v \right| &\leqslant X_n \sum_{v=n}^{\infty} |\Delta\lambda_v| \\ &\leqslant \sum_{v=0}^{\infty} X_v \, |\Delta\lambda_v| \leqslant \sum_{v=0}^{\infty} X_v \, |A_v| < \infty. \end{aligned}$$

Hence $|\lambda_n| X_n = O(1)$ as $n \to \infty$. □

LEMMA 2. *If (A_n) is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$ and $\sum A_nX_n$ is convergent, then*

$$mX_mA_m = O(1) \text{ as } m \to \infty, \tag{4.2}$$

$$\sum_{n=1}^{\infty} n\, X_n \, |\Delta A_n| < \infty. \tag{4.3}$$

The proof of Lemma 2 is similar to the proof of Theorems 1 and 2 of Boas [**1**, case $\gamma = 1$] and hence is omitted.

5.

PROOF OF THEOREM 1: Let (T_n) be the sequence of $(\overline{N}, p_n)$ means of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v. \tag{5.1}$$

Then, for $n \geqslant 1$, we get

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v. \tag{5.2}$$

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Applying Abel's transformation to the right hand side of (5.2), we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\left(\frac{P_{v-1}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{p_n \lambda_n}{n P_n} \sum_{v=1}^{n} v a_v$$

$$= \frac{(n+1) p_n t_n \lambda_n}{n P_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v t_v \frac{v+1}{v}$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} t_v \frac{1}{v} = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}$$

To complete the proof of Theorem 1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_{n,r}|^k < \infty, \text{ for } r = 1, 2, 3, 4. \tag{5.3}$$

First, we have

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,1}|^k = O(1) \sum_{n=1}^{m} \frac{|\lambda_n| p_n |t_n|^k}{P_n}$$

$$= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \frac{p_v}{P_v} |t_v|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m = O(1) \sum_{n=1}^{m-1} |A_n| X_n + O(1) |\lambda_m| X_m = O(1)$$

as $m \to \infty$, by virtue of the hypotheses and Lemma 1.

Now applying Hölder's inequality, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,2}|^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v| p_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v| \frac{p_v}{P_v} |t_v|^k = O(1)$$

as $m \to \infty$.

Again, using the fact that $P_v = O(vp_v)$, by (3.1), we get

$$\begin{aligned}
\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v||t_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}vp_v|A_v||t_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}(v|A_v|)^kp_v|t_v|^k\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}(v|A_v|)^{k-1}v|A_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\
&= O(1)\sum_{v=1}^{m}v|A_v|\frac{p_v}{P_v}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|A_v|)\sum_{r=1}^{v}\frac{p_r}{P_r}|t_r|^k+O(1)m|A_m|\sum_{v=1}^{m}\frac{p_v}{P_v}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|A_v|)|X_v+O(1)m|A_m|X_m \\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta A_v|+O(1)\sum_{v=1}^{m-1}|A_{v+1}|X_v+O(1)m|A_m|X_m \\
&= O(1) \text{ as } m\to\infty,
\end{aligned}$$

by virtue of the hypotheses and Lemma 2.

Finally, using the fact that $P_v = O(vp_v)$, by (3.1), as in $T_{n,1}$ we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}|T_{n,4}|^k &\leqslant \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{n+1}||t_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}|\lambda_{v+1}|p_v|t_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}|\lambda_{v+1}|^kp_v|t_v|^k\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{p_v}{P_v}|t_v|^k = O(1)
\end{aligned}$$

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as $m \to \infty$. Therefore, we get

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,r}|^k = O(1) \text{ as } m \to \infty, \quad \text{for } r = 1, 2, 3, 4.$$

This completes the proof of Theorem 1. □

6.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let

$$f(x) \simeq \sum_{n=0}^{\infty} A_n(x), \ \phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\} \text{ and } \phi_1(t) = \frac{1}{t}\int_0^t \phi(u)du.$$

It is well known that if $\phi_1(t) \in BV(0, \pi)$, $t_n(x) = O(1)$, where $t_n(x)$ is the nth $(C, 1)$ mean of the sequence $(nA_n(x))$ (see [3]). Hence, using this fact, we get the following result for Fourier series.

THEOREM 2. *If $\phi_1(t) \in BV(0, \pi)$ and the sequences (p_n), (λ_n) and (X_n) satisfy the conditions of Theorem 1, then the series $\sum A_n(x)\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geqslant 1$.*

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