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# Commutativity properties of Quinn spectra

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We give a simple sufficient condition for Quinn's 'bordism-type' spectra to be weakly equivalent to commutative symmetric ring spectra. We also show that the symmetric signature is (up to weak equivalence) a monoidal transformation between symmetric monoidal functors, which implies that the Sullivan–Ranicki orientation of topological bundles is represented by a ring map between commutative symmetric ring spectra. In the course of proving these statements, we give a new description of symmetric L theory which may be of independent interest.

*Keywords:* L-theory; bordism theories; ad theories; highly structured ring spectra; $E_\infty$-spectra

## 1. Introduction

In [19], Frank Quinn gave a general machine for constructing spectra from 'bordism-type theories'. In our article [11], we gave axioms for a structure we call an *ad theory* and showed that when these axioms are satisfied (as they are for all of the standard examples) the Quinn machine can be improved to give a symmetric spectrum $\mathbf{M}$. We also showed that when the ad theory is multiplicative (that is, when its 'target category' is graded monoidal) the symmetric spectrum $\mathbf{M}$ is a symmetric ring spectrum. Finally, we showed that there are monoidal functors to the category of symmetric spectra which represent Poincaré bordism over $B\pi$ (considered as a functor of $\pi$) and symmetric L-theory (considered as a functor of a ring $R$ with involution).

In this article, we consider commutativity properties. It relies on the conventions and results of the article [11]; the relevant sections are 3, 6, 7, 9, 10, 17, 18, and 19.

A 'commutative ad theory' is (essentially) an ad theory whose target category is graded symmetric monoidal (the precise definition is given in §3). Our first main result is

THEOREM 1.1 *Let* $\mathbf{M}$ *be the symmetric ring spectrum associated with a commutative ad theory. There is a commutative symmetric ring spectrum* $\mathbf{M}^{\mathrm{comm}}$ *which is weakly equivalent in the category of symmetric ring spectra to* $\mathbf{M}$ *and depends on it in a natural way.*

More precisely, we construct a symmetric spectrum $\mathbf{M}$ together with an action of a specific $E_\infty$-operad which naturally depends on the underlying ad theory. Then we use the bar construction to obtain a zig-zag of weak equivalences to the commutative symmetric ring spectrum $\mathbf{M}^{\mathrm{comm}}$. We only claim the naturality for strict morphisms. A refined statement of naturally and what happens with 2-morphisms under these functors is worth being further investigated but it is not part of the article.

As a consequence, theorem 1.1 shows that the L-theory spectrum of a commutative ring can be realized as a commutative symmetric ring spectrum.[1.]

For the ad theory $\mathrm{ad}_{\mathrm{STop}}$ of oriented topological bordism [11, §6], we showed in [11, §17 and appendix B] that the underlying spectrum of $\mathbf{M}_{\mathrm{STop}}$ is weakly equivalent to the usual Thom spectrum $M\mathrm{STop}$. It is well-known that $M\mathrm{STop}$ is a commutative symmetric ring spectrum, and we have

THEOREM 1.2 $(\mathbf{M}_{\mathrm{STop}})^{\mathrm{comm}}$ *and* $M\mathrm{STop}$ *are weakly equivalent in the category of commutative symmetric ring spectra.*

The proof gives a specific chain of weak equivalences between them.

We also prove a multiplicative property of the symmetric signature. The symmetric signature is a basic tool in surgery theory. In its simplest form, it assigns to an oriented Poincaré complex $X$ an element of the symmetric L-theory of $\pi_1(X)$; this element determines the surgery obstruction up to 8-torsion. Ranicki proved that the symmetric signature of a Cartesian product is the product of the symmetric signatures [21, proposition 8.1(i)]. The symmetric signature gives a map of spectra from Poincaré bordism to L-theory [10, proposition 7.10], and we showed in [11] that it gives a map of symmetric spectra. In order to investigate the multiplicativity of this map, we give a new (but equivalent) description of the L-spectrum, using 'relaxed' algebraic Poincaré complexes (the relation between these and the usual algebraic Poincaré complexes is similar to the relation between Γ-spaces and $E_\infty$ spaces). For a ring with involution $R$, there is an ad theory $\mathrm{ad}^R_{\mathrm{rel}}$, and the associated spectrum $\mathbf{M}^R_{\mathrm{rel}}$ is equivalent to the usual L-spectrum. The symmetric signature gives a map $\mathrm{sig}_{\mathrm{rel}}$ from the Poincaré bordism spectrum (which we denote by $\mathbf{M}_{e,*,1}$; see [11, §7]) to $\mathbf{M}^R_{\mathrm{rel}}$. We prove that this map is weakly equivalent to a ring map between commutative symmetric ring spectra:

THEOREM 1.3 *There are symmetric ring spectra* $\mathbf{A}$ *and* $\mathbf{B}$*, commutative symmetric ring spectra* $\mathbf{C}$ *and* $\mathbf{D}$*, and a commutative diagram*

[1.] Lurie [13] has explained another way to prove that the L-theory spectrum of a commutative ring can be realized as a commutative symmetric ring spectrum. The method used in [13] does not include other examples we consider such as Poincaré bordism and (in [3]) Witt and IP bordism.

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$$\begin{array}{ccccc} \mathbf{M}_{e,*,1} & \longleftarrow & \mathbf{A} & \longrightarrow & \mathbf{C} \\ {\scriptstyle \mathrm{sig}_{\mathrm{rel}}}\big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbf{M}^{\mathbb{Z}}_{\mathrm{rel}} & \longleftarrow & \mathbf{B} & \longrightarrow & \mathbf{D} \end{array}$$

*in which the horizontal arrows and the right vertical arrow are ring maps and the horizontal arrows are weak equivalences.*

In fact, $\mathbf{C}$ is $(\mathbf{M}_{e,*,1})^{\mathrm{comm}}$, and $\mathbf{D}$ is weakly equivalent to $(\mathbf{M}^{\mathbb{Z}}_{\mathrm{rel}})^{\mathrm{comm}}$ in the category of commutative symmetric ring spectra (see remark 18.3).

As far as we are aware, there is no previous result in the literature showing multiplicativity of the symmetric signature at the spectrum level.

In §19, we prove a stronger statement, that up to weak equivalence the symmetric signature is a monoidal transformation between symmetric monoidal functors. In [3], we proved the analogous statement about the symmetric signature for Witt and IP bordism, using the methods of the present article.

Remark 1.4. The *Sullivan–Ranicki orientation* for topological bundles ([25], [15], [22, remark 16.3], [10, §13.5]) is the following composite in the homotopy category of spectra

$$M\mathrm{STop} \simeq \mathbf{Q}_{\mathrm{STop}} \xrightarrow{\mathrm{sig}} L^{\mathbb{Z}},$$

where $\mathbf{Q}_{\mathrm{STop}}$ denotes the Quinn spectrum of oriented topological bordism (which was shown to be equivalent to $M\mathrm{STop}$ in [11, appendix B]). Combining theorems 1.3 and 1.2 shows that the Sullivan–Ranicki orientation is represented by a ring map of commutative symmetric ring spectra. In [11, §8], a zig-zag between the ad theories $\mathrm{ad}_{STop}$ and $\mathrm{ad}_{e,*,1}$ is constructed where all maps are multiplicative. It follows from the results below that the map from topological to Poincaré bordism can be refined to an $E_\infty$-map.

The results of the present work were already used amongst others in [2] in connection with singularities of Baas–Sullivan type.

Here is an outline of the article. In §2 and 3, we give the definition of commutative ad theory. The proof of theorem 1.1 occupies §4–10. We begin in §4 and 5 by giving a multisemisimplicial analogue $\mathcal{S}p_{mss}$ of the category of symmetric spectra. We observe that an ad theory gives rise to an object $\mathbf{R}$ of $\mathcal{S}p_{mss}$ whose realization is the symmetric spectrum $\mathbf{M}$ mentioned above. Section 6 explains the key idea of the proof, which is to interpolate between the various permutations of the multiplication map by allowing a different order of multiplication for each cell. In §7 and 8, we use this idea to create a monad in the category $\mathcal{S}p_{mss}$ which acts on $\mathbf{R}$, and in §10 (after a brief technical interlude in §9) we use a standard rectification argument (as in [17]) to convert $\mathbf{R}$ with this action to a strictly commutative object of $\mathcal{S}p_{mss}$; passage to geometric realization gives $\mathbf{M}^{\mathrm{comm}}$. Next we turn to the proof of theorem 1.3. In §12–14, we introduce the relaxed symmetric Poincaré ad theory and the corresponding version of the symmetric signature. In §15–17, we create a monad in the category $\mathcal{S}p_{mss} \times \mathcal{S}p_{mss}$ which acts on the pair $(\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}})$, and in §18 we

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adapt the argument of §10 to prove theorem 1.3. Section 19 gives the statement of the stronger version of theorem 1.3 mentioned above, and §20 gives the proof. Section 11 gives the proof of theorem 1.2. There is a property of the smash product in symmetric spectra which is needed to prove the main results of the article. It is taken care of in Appendix A. Appendix C investigates the functorial properties of the ad theories of relaxed quasi-symmetric complexes.

### 1.1. Notations

We adopt the notations of the articles ([11], [3], [2]): the ambient categories, ad theories, symmetric spectra, and Quinn spectra are called $\mathcal{A}_x$, $\mathrm{ad}_x$, $\mathbf{M}_x$, and $\mathbf{Q}_x$, respectively, with $x$ some parameter depending on the objects we work with. Note that the ambient category does not determine the ad theory. The latter does determine the associated spectra in a functorial way but we write $\mathbf{M}_x$ instead of $\mathbf{M}(ad_x)$ to keep the notation simpler.

## 2. Some redefinitions

One of the ingredients in the definition of ad theory in [11] is the target '$\mathbb{Z}$-graded category' $\mathcal{A}$ (see [11, definitions 3.3 and 3.10]). For the purposes of that article, there was no reason to allow morphisms in $\mathcal{A}$ between objects of the same dimension (except for identity maps). For the present article, we do need such morphisms (see definition 3.1 and the proof of theorem 1.1). We therefore begin by giving modified versions of some of the definitions of [11].

DEFINITION 2.1. *(cf. definition 3.3 of [11]) Let $\mathbb{Z}$ be the poset of integers regarded as a category with one morphism for each relation. By an involution, we mean an endofunctor $i$ which strictly satisfies $i^2 = id$. Give $\mathbb{Z}$ the trivial involution. A $\mathbb{Z}$-graded category is a small category $\mathcal{A}$ with involution, together with involution-preserving functors $d : \mathcal{A} \longrightarrow \mathbb{Z}$ (called the dimension function) and $\emptyset : \mathbb{Z} \longrightarrow \mathcal{A}$ such that $d\emptyset$ is equal to the identity functor. We will use the notation $|a|$ instead of $da$. Note that if $|a| > |b|$ then there are no morphisms from $a$ to $b$.*

The definition of a strict monoidal structure on a $\mathbb{Z}$-graded category [11, definition 18.1] needs no change, provided that one uses the new definition of $\mathbb{Z}$-graded category.

Next we explain how to modify the specific examples of target categories in [11] by adding morphisms which preserve dimension.

For the category $\mathcal{A}_{\mathrm{STop}}$ [11, example 3.5], the morphisms between objects of the same dimension are the orientation-preserving homeomorphisms.

For the category $\mathcal{A}_{\pi,Z,w}$ [11, definition 7.3], the morphisms between objects $(X, f, \xi)$ and $(X', f', \xi')$ of the same dimension are the maps $g : X \to X'$ such that $f' \circ g = f$ and $g_*(\xi) = \xi'$.

We do not need the analogous modification for the category $\mathcal{A}^R$ [11, definition 9.5] because we will be using the version in §12.

The definition of an ad theory [11, definition 3.10] stays the same. In §14, however, we will restrict our attention to ad theories which are only defined on *strict* ball complexes. A ball complex is called strict if each component of the intersection of

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two cells is a single cell. This assumption does not affect the constructions or results of [11].

## 3. Commutative ad theories

DEFINITION 3.1. *Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category. A permutative structure on $\mathcal{A}$ is a strict monoidal structure $(\boxtimes, \varepsilon)$ [11, definition 18.1] together with a natural isomorphism*

$$\gamma_{x,y} : x \boxtimes y \to i^{|x||y|} y \boxtimes x$$

*such that*

*(a) $i\gamma_{x,y} = \gamma_{ix,y} = \gamma_{x,iy}$,*
*(b) each of the maps $\gamma_{\emptyset,y}$, $\gamma_{x,\emptyset}$ is the identity map of $\emptyset$,*
*(c) the composite*

$$x \boxtimes y \xrightarrow{\gamma_{x,y}} i^{|x||y|} y \boxtimes x \xrightarrow{i^{|x||y|}(\gamma_{y,x})} x \boxtimes y$$

*is the identity.*
*(d) $\gamma_{x,\varepsilon}$ is the identity, and*
*(e) the diagram*

$$\begin{array}{ccc} & x \boxtimes y \boxtimes z & \\ {}^{1\boxtimes\gamma}\swarrow & & \searrow^{\gamma} \\ i^{|y||z|} x \boxtimes z \boxtimes y & \xrightarrow{i^{|y||z|}\gamma\boxtimes 1} & i^{|z|(|x|+|y|)} z \boxtimes x \boxtimes y \end{array}$$

*commutes. A strict map of $\mathbb{Z}$-graded categories with permutative structures is a map $f : \mathcal{A} \to \mathcal{B}$ of $\mathbb{Z}$-graded categories for which $f(x \boxtimes y) = f(x) \boxtimes f(y)$ holds for objects and morphisms. Moreover, such a functor takes $\epsilon$ to $\epsilon$ and the diagram*

$$\begin{array}{ccc} f(x \boxtimes y) & \xrightarrow{f(\gamma)} & f(i^{|x||y|} y \boxtimes x) \\ \downarrow = & & \downarrow = \\ f(x) \boxtimes f(y) & \xrightarrow{\gamma} & i^{|fx||fy|} f(y) \boxtimes f(x) \end{array}$$

*commutes.*

REMARK 3.2. The analogue of the coherence theorem for symmetric monoidal categories [14] holds in this context with essentially the same proof.

DEFINITION 3.3. *A commutative ad theory is a multiplicative ad theory [11, definitions 3.10 and 18.4], with the extra property that every pre $K$-ad which is isomorphic to a $K$-ad is a $K$-ad, together with a permutative structure on the target category $\mathcal{A}$. A strict map of commutative ad theories is a strict map of the ambient categories which takes ads to ads.*

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Examples are $\mathrm{ad}_C$ when $C$ is a commutative DGA (see [11, example 3.12]), $\mathrm{ad}_{\mathrm{STop}}$ (see [11, §6]), $\mathrm{ad}_{e,*,1}$ (see [11, §7]), $\mathrm{ad}_{\mathrm{STopFun}}$ (see [11, end of §8]), $\mathrm{ad}^R_{\mathrm{rel}}$ when $R$ is commutative (see §12), $\mathrm{ad}_{\mathrm{IP}}$ [3, §4], and $\mathrm{ad}_{\mathrm{IPFun}}$ [3, §6.1].

Remark 3.4. The extra property in definition 3.3 is used in the proof of lemma 6.3.

For later use, we record some notation for iterated products.

Definition 3.5. *(i) For a permutation $\eta \in \Sigma_j$, let $\epsilon(\eta)$ denote 0 if $\eta$ is even and 1 if $\eta$ is odd.*

*(ii) Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category with a permutative structure. Let $\eta \in \Sigma_j$. Define a functor*

$$\eta_\star : \mathcal{A}^{\times j} \to \mathcal{A}$$

*(where $\mathcal{A}^{\times j}$ is the $j$-fold Cartesian product) by*

$$\eta_\star(x_1, \ldots, x_j) = i^{\epsilon(\bar\eta)}(x_{\eta^{-1}(1)} \boxtimes \cdots \boxtimes x_{\eta^{-1}(j)})$$

*where $\bar\eta$ is the block permutation that takes blocks $\mathbf{b}_1, \ldots, \mathbf{b}_j$ of size $|x_1|, \ldots, |x_j|$ into the order $\mathbf{b}_{\eta^{-1}(1)}, \ldots, \mathbf{b}_{\eta^{-1}(j)}$.*

Remark 3.6. Note that, by remark 3.2, $\eta_\star(x_1, \ldots, x_j)$ is canonically isomorphic to $x_1 \boxtimes \cdots \boxtimes x_j$.

## 4. Multisemisimplicial symmetric spectra

In this section, we define a category $\mathcal{S}p_{mss}$ (the ss stands for 'semisimplicial') which is a multisemisimplicial version of the category $\mathcal{S}p$ of symmetric spectra. The motivation for the definition is that the sequence $R_k$ in [11, definition 17.2] should give an object of $\mathcal{S}p_{mss}$.

Recall that we write $\Delta_{\mathrm{inj}}$ for the category whose objects are the sets $\{0, \ldots, n\}$ and whose morphisms are the monotonically increasing injections.

A based $k$-fold *multisemisimplicial set* is a contravariant functor from the Cartesian product $(\Delta_{\mathrm{inj}})^{\times k}$ to the category $\mathcal{S}_*$ of based sets. In particular, a based 0-fold multisemisimplicial set is just a based set.

Next note that given a category $\mathcal{C}$ with a left action of a group $G$ one can define a category $G \ltimes \mathcal{C}$ whose objects are those of $\mathcal{C}$ and whose morphisms are pairs $(\alpha, f)$ with $\alpha \in G$ and $f$ a morphism of $\mathcal{C}$; the domain of $(\alpha, f)$ is the domain of $f$ and the target is $\alpha^{-1}$ applied to the target of $f$. Composition is defined by

$$(\alpha, f) \circ (\beta, g) = (\alpha\beta, \beta^{-1}(f) \circ g).$$

Remark 4.1. (i) $\mathcal{C}$ is imbedded in $G \ltimes \mathcal{C}$ by taking the morphism $f$ of $\mathcal{C}$ to the morphism $(e, f)$ of $G \ltimes \mathcal{C}$, where $e$ is the identity element of $G$.

(ii) The morphism $(\alpha, f)$ is the composite $(\alpha, \mathrm{id}) \circ (e, f)$.

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Definition 4.2. *Let $\Sigma_k$ act on $(\Delta^{op}_{\mathrm{inj}})^{\times k}$ by permuting the factors (when $k=0$, $\sum_0$ is the trivial group). For each subgroup $H$ of $\Sigma_k$, let $Hss\mathcal{S}_k$ be the category of functors from $H \ltimes (\Delta^{op}_{\mathrm{inj}})^{\times k}$ to $\mathcal{S}_*$.*

By remark 4.1, an object of $Hss\mathcal{S}_k$ can be thought of as a based $k$-fold multisemisimplicial set with a left 'action' of $H$ in which $H$ also acts on the multidegrees.

Definition 4.3. *(i) A multisemisimplicial symmetric sequence $\mathbf{X}$ is a sequence $X_k$, $k \geq 0$, such that $X_k$ is an object of $\Sigma_k ss\mathcal{S}_k$.*

*(ii) A morphism of multisemisimplicial symmetric sequences from $\mathbf{X}$ to $\mathbf{Y}$ is a sequence of morphisms $f_k : X_k \to Y_k$ in $\Sigma_k ss\mathcal{S}_k$.*

The category of multisemisimplicial symmetric sequences will be denoted by $\Sigma ss\mathcal{S}$.

For our next definition, recall [11, definitions 17.2 and 17.3].

Definition 4.4. *(i) For each $k \geq 0$ extend the object $R_k = \mathrm{ad}^k(\Delta^\bullet)$ of $ss\mathcal{S}_k$ to an object of $\Sigma_k ss\mathcal{S}_k$ by letting*

$$(\alpha, \mathrm{id})_*(F) = i^{\epsilon(\alpha)} \circ F \circ \alpha_\#$$

*(where $\alpha \in \Sigma_k$ and $F \in \mathrm{ad}^k(\Delta^{\mathbf{n}})$).*

*(ii) Let $\mathbf{R}$ denote the object of $\Sigma ss\mathcal{S}$ whose $k$-th term is $R_k$.*

Next we assemble the ingredients needed to define a symmetric monoidal structure on $\Sigma ss\mathcal{S}$.

Definition 4.5. *Given $A \in \Sigma_k ss\mathcal{S}_k$ and $B \in \Sigma_l ss\mathcal{S}_l$, define the object $A \wedge B \in (\Sigma_k \times \Sigma_l) ss\mathcal{S}_{k+l}$ by*

$$(A \wedge B)_{\mathbf{m},\mathbf{n}} = A_{\mathbf{m}} \wedge B_{\mathbf{n}}$$

*(where $\mathbf{m}$ is a $k$-fold multi-index and $\mathbf{n}$ is an $l$-fold multi-index).*

Definition 4.6. *Given $H \subset G \subset \Sigma_k$, define a functor*

$$I_H^G : Hss\mathcal{S}_k \to Gss\mathcal{S}_k$$

*by letting $I_H^G A$ be the left Kan extension of $A$ along $H \ltimes (\Delta^{op}_{\mathrm{inj}})^{\times k} \to G \ltimes (\Delta^{op}_{\mathrm{inj}})^{\times k}$.*

Remark 4.7. For later use, we give an explicit description of $I_H^G A$. For each multi-index $\mathbf{n}$, we have

$$(I_H^G A)_{\mathbf{n}} = \Big( \bigvee_{\alpha \in G} A_{\alpha^{-1}(\mathbf{n})} \Big) / H$$

where the action of $H$ is defined as follows: if $\beta \in H$ and $x$ is an element in the $\alpha$-summand then $\beta$ takes $x$ to the element $(\beta, \mathrm{id})_*(x)$ in the $\alpha\beta^{-1}$-summand.

NOTATION 4.8. We denote the equivalence class of an element $x$ in the $\alpha$-summand of $I_H^G A$ by $[\alpha, x]$; note that $[\alpha, x] = (\alpha, \mathrm{id})_*[e, x]$.

DEFINITION 4.9. *Given $\mathbf{X}, \mathbf{Y} \in \Sigma ss\mathcal{S}$, define $\mathbf{X} \otimes \mathbf{Y} \in \Sigma ss\mathcal{S}$ by*

$$(\mathbf{X} \otimes \mathbf{Y})_k = \bigvee_{j_1+j_2=k} I^{\Sigma_k}_{\Sigma_{j_1}\times\Sigma_{j_2}} \left(\mathbf{X}_{j_1} \wedge \mathbf{Y}_{j_2}\right).$$

The proof that $\otimes$ is a symmetric monoidal product is essentially the same as the corresponding proof in [9, §2.1]. The symmetry map

$$\tau : \mathbf{X} \otimes \mathbf{Y} \to \mathbf{Y} \otimes \mathbf{X}$$

is given (as in [9]) by

$$\tau([\alpha, x \wedge y]) = [\alpha\rho_{l,k}, y \wedge x] \tag{4.1}$$

where $x \in (X_k)_{\mathbf{m}}$, $y \in (Y_l)_{\mathbf{n}}$, and $\rho_{l,k}$ is the permutation of $\{1, \ldots, k+l\}$ which moves the first $l$ elements to the end and the last $k$ elements to the front.

Next we will give the definition of the category $\mathcal{S}p_{mss}$ and its symmetric monoidal product. First we need a sphere object.

DEFINITION 4.10. *(i) Let $S^1$ be the based semisimplicial set that consists of the base point together with a 1-simplex $s$.*

*(ii) Let $S^k$ be the object of $\Sigma_k ss\mathcal{S}_k$ obtained from $(S^1)^{\wedge k}$ by letting*

$$(\alpha, \mathrm{id})_*(s \wedge \cdots \wedge s) = s \wedge \cdots \wedge s$$

*(iii) Let $\mathbf{S}$ be the object of $\Sigma ss\mathcal{S}$ whose $k$-th term is $S^k$.*

It is easy to check that $\mathbf{S}$ is a commutative monoid in $\Sigma ss\mathcal{S}$.

DEFINITION 4.11. *$\mathcal{S}p_{mss}$ is the category of modules over $\mathbf{S}$.*

REMARK 4.12. One can give a more explicit version of this definition: an object of $\mathcal{S}p_{mss}$ consists of an object $\mathbf{X}$ of $\Sigma ss\mathcal{S}$ together with suspension maps

$$\omega : S^1 \wedge X_k \to X_{k+1}$$

for each $k$, such that the iterates of the $\omega$'s satisfy appropriate equivariance conditions.

EXAMPLE 4.13. The object $\mathbf{R}$ of definition 4.4 can be given suspension maps as follows: with the notation of [11, definition 17.4(i)], define

$$\omega : S^1 \wedge R_k \to R_{k+1}$$

by

$$\omega(s \wedge F) = \lambda^*(F)$$

(where $s$ is the 1-simplex of $S^1$ and $F \in \mathrm{ad}^k(\Delta^{\mathbf{n}})$). The resulting object of $\mathcal{S}p_{mss}$ will also be denoted $\mathbf{R}$.

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Definition 4.14. *(cf. [9, definition 2.2.3])*

*For $\mathbf{X}, \mathbf{Y} \in \mathcal{S}p_{mss}$, define the smash product $\mathbf{X} \wedge \mathbf{Y}$ to be the coequalizer of the diagram*

$$\mathbf{X} \otimes \mathbf{S} \otimes \mathbf{Y} \rightrightarrows \mathbf{X} \otimes \mathbf{Y}$$

*where the right action of $\mathbf{S}$ on $\mathbf{X}$ is the composite*

$$\mathbf{X} \otimes \mathbf{S} \to \mathbf{S} \otimes \mathbf{X} \to \mathbf{X}.$$

The proof that $\wedge$ is a symmetric monoidal product is essentially the same as the corresponding proof in [9, §2.2].

## 5. Geometric realization

Let $G$ be a subgroup of $\Sigma_k$. By remark 4.1(i), an object of $Gss\mathcal{S}_k$ has an underlying $k$-fold multisemisimplicial set.

Definition 5.1. *The geometric realization $|A|$ of an object $A \in Gss\mathcal{S}_k$ is the geometric realization of its underlying k-fold multisemisimplicial set where, additionally, the realization of the base points is collapsed to a single point.*

Definition 5.2. *(i) A map in $Gss\mathcal{S}_k$ is a weak equivalence if it induces a weak equivalence of realizations.*

*(ii) A map $\mathbf{X} \to \mathbf{Y}$ in $\Sigma ss\mathcal{S}$ or in $\mathcal{S}p_{mss}$ is a weak equivalence if each map $X_k \to Y_k$ is a weak equivalence.*

Proposition 5.3. *For $A \in \Sigma_k ss\mathcal{S}_k$, the following formula gives a natural left $\Sigma_k$ action on $|A|$:*

$$\alpha([u_1, \ldots, u_k, a]) = [u_{\alpha^{-1}(1)}, \ldots, u_{\alpha^{-1}(k)}, (\alpha, \mathrm{id})_*(a)];$$

*here $(u_1, \ldots, u_k) \in \Delta^{\mathbf{n}}$, $a \in A_{\mathbf{n}}$, and $[u_1, \ldots, u_k, a]$ denotes the class of $(u_1, \ldots, u_k, a)$ in $|A|$.*

Proposition 5.4. *For $H \subset G \subset \Sigma_k$ and $A \in Hss\mathcal{S}_k$, there is a natural isomorphism of based G-spaces*

$$|I_H^G A| \cong G_+ \wedge_H |A|.$$

*Proof.* The proof is easy, using remark 4.7. □

Corollary 5.5. *Geometric realization induces a symmetric monoidal functor from $\mathcal{S}p_{mss}$ to the category of symmetric spectra $\mathcal{S}p$; in particular, the realization of a (commutative) monoid in $\mathcal{S}p_{mss}$ is a (commutative) monoid in $\mathcal{S}p$.*

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## 6. A family of multiplication maps

In this section, we begin the proof of theorem 1.1. We shall construct a monad $\mathbb{P}$ together with maps

$$\mathbb{A} = \mathbb{A}ssoc \to \mathbb{P} \to \mathbb{C}omm = \mathbb{P}'$$

such that (a) the functor $\mathbf{R}$ factors over $\mathbb{P}$-algebras, (b) the transformation $\mathbb{P} \to \mathbb{P}'$ is a weak equivalence in a suitable sense so $\mathbf{R}$ is weakly equivalent as a functor to $\mathbb{P}$-algebras to another functor that factors over $\mathbb{P}'$-algebras.

From now until the end of §10, we fix a $\mathbb{Z}$-graded permutative category $\mathcal{A}$ and a commutative ad theory with values in $\mathcal{A}$. Let $\mathbf{R}$ be the object of $\mathcal{S}p_{mss}$ constructed from this ad theory as in example 4.13.

Let $\mathbf{M}$ be the symmetric ring spectrum associated with the ad theory [11, proposition 17.5 and theorem 18.5]. By definition, $M_k = |R_k|$. The multiplication of $\mathbf{M}$ is induced by the collection of maps

$$\mu : (R_k)_{\mathbf{m}} \wedge (R_l)_{\mathbf{n}} \to (R_{k+l})_{\mathbf{m},\mathbf{n}}$$

defined by

$$\mu(F \wedge G)(\sigma_1 \times \sigma_2, o_1 \times o_2) = i^{l \dim \sigma_1} F(\sigma_1, o_1) \boxtimes G(\sigma_2, o_2) \tag{6.1}$$

(this is well-defined because, by [11, definition 18.1(b)], reversing the orientations $o_1$ and $o_2$ does not change the right-hand side). These maps give $\mathbf{R}$ the structure of a monoid in $\mathcal{S}p_{mss}$ (the proof is essentially the same as for [11, theorem 18.5]).

In general, even though the ad theory is commutative, $\mathbf{R}$ is not a commutative monoid (this would require the product in the target category $\mathcal{A}$ to be *strictly* graded commutative). Instead we have the following. Recall definition 3.5 and notation 4.8.

LEMMA 6.1. *Let $m : \mathbf{R}\wedge\mathbf{R} \to \mathbf{R}$ be the product and let $\eta \in \Sigma_j$. Then the composite*

$$m_\eta : \mathbf{R}^{\wedge j} \xrightarrow{\eta} \mathbf{R}^{\wedge j} \xrightarrow{m} \mathbf{R}$$

*is determined by the formula*

$$\begin{aligned} &m_\eta([e, F_1 \wedge \cdots \wedge F_j])(\sigma_1 \times \cdots \times \sigma_j, o_1 \times \cdots \times o_j) \\ &\quad = i^{\epsilon(\zeta)} \eta_\star (F_1(\sigma_1, o_1), \ldots, F_j(\sigma_j, o_j)), \end{aligned}$$

*where $e$ is the identity element of the relevant symmetric group and $\zeta$ is the block permutation that takes blocks $\mathbf{b}_1, \ldots, \mathbf{b}_j, \mathbf{c}_1, \ldots, \mathbf{c}_j$ of size $\deg F_1, \ldots, \deg F_j, \dim \sigma_1, \ldots, \dim \sigma_j$ into the order $\mathbf{b}_1, \mathbf{c}_1, \ldots, \mathbf{b}_j, \mathbf{c}_j$.*

*Proof.* It suffices to prove this when $\eta$ is a transposition, and in this case the proof is an easy calculation using Eqs. (4.1) and (6.1) and [11, definition 17.3]. □

The key idea in the proof of theorem 1.1 is that there is a family of operations which can be used to interpolate between the various $m_\eta$. To construct this family,

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we allow a different permutation of the factors for each cell of $\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}$, as explained in our next definition. We begin by defining the operations for pre-ads (see [11, definitions 3.8(i) and 3.10(ii)]).

Definition 6.2. *(i) Given a ball complex $K$, let $U(K)$ denote the set of all cells of $K$.*

*(ii) Let $k_1, \ldots, k_j$ be non-negative integers and let $\mathbf{n}_i$ be a $k_i$-fold multi-index for $1 \leq i \leq j$. For any map*

$$a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to \Sigma_j$$

*define a map on preads (i.e., functors $F$ from the oriented cells $(\sigma, o)$ to the ambient category)*

$$a_* : \mathrm{pre}^{k_1}(\Delta^{\mathbf{n}_1}) \times \cdots \times \mathrm{pre}^{k_j}(\Delta^{\mathbf{n}_j}) \to \mathrm{pre}^{k_1+\cdots+k_j}(\Delta^{(\mathbf{n}_1,\ldots,\mathbf{n}_j)})$$

*by*

$$\begin{aligned} &a_*(F_1, \ldots, F_j)(\sigma_1 \times \cdots \times \sigma_j, o_1 \times \cdots \times o_j)) \\ &\quad = i^{\epsilon(\zeta)}(a(\sigma_1 \times \cdots \times \sigma_j))_{\bigstar}(F_1(\sigma_1, o_1), \ldots, F_j(\sigma_j, o_j)), \end{aligned}$$

*where $\zeta$ is the block permutation that takes blocks $\mathbf{b}_1$, …, $\mathbf{b}_j$, $\mathbf{c}_1$, …, $\mathbf{c}_j$ of size $k_1$, …, $k_j$, $\dim \sigma_1$, …, $\dim \sigma_j$ into the order $\mathbf{b}_1$, $\mathbf{c}_1$, …, $\mathbf{b}_j$, $\mathbf{c}_j$.*

Lemma 6.3. *If $F_i \in \mathrm{ad}^{k_i}(\Delta^{\mathbf{n}_i})$ for $1 \leq i \leq j$ then $a_*(F_1, \ldots, F_j) \in \mathrm{ad}^{k_1+\cdots+k_j}(\Delta^{(\mathbf{n}_1,\ldots,\mathbf{n}_j)})$.*

*Proof.* This is immediate from the extra property in definition 3.3, remark 3.6, and [11, definition 18.4(b)]. □

Recalling that $(R_k)_{\mathbf{n}} = \mathrm{ad}^k(\Delta^{\mathbf{n}})$ with basepoint at the trivial ad (see [11, definitions 3.8(ii), 3.10(b) and 18.1(c)]), we have now constructed an operation

$$a_* : (R_{k_1})_{\mathbf{n}_1} \wedge \cdots \wedge (R_{k_j})_{\mathbf{n}_j} \to (R_{k_1+\cdots+k_j})_{\mathbf{n}_1,\ldots,\mathbf{n}_j}$$

for each

$$a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to \Sigma_j.$$

For later use, we give the relation between $a_*$ and the suspension map $\omega : S^1 \wedge R_k \to R_{k+1}$.

Definition 6.4. *For a ball complex $K$, let*

$$\Pi : U(\Delta^1 \times K) \to U(K)$$

*be the map which takes $\sigma \times \tau$ to $\tau$, where $\sigma$ is a simplex of $\Delta^1$ and $\tau$ is a simplex of $K$.*

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Lemma 6.5. *Let $s$ be the 1-simplex of $S^1$, let $F_i \in (R_{k_i})_{\mathbf{n}_i}$ for $1 \le i \le j$, and let $a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to \Sigma_j$. Then*

$$\omega(s \wedge a_*(F_1 \wedge \cdots \wedge F_j)) = (a \circ \Pi)_*(\omega(s \wedge F_1) \wedge \cdots \wedge F_j).$$

*Proof.* This follows from lemma 6.1 (because the permuted multiplication commutes with suspension). It can also be proved by a straightforward calculation using definitions 3.5 and 4.13 and [11, definitions 17.4 and 3.7(ii)]. □

In the remainder of this section, we show that the action of the operations $a_*$ can be described in a way that begins to resemble the action of an operad; this resemblance will be developed further in the next two sections.

Definition 6.6. *(i) For $j, k \ge 0$ define an object $\mathcal{O}(j)_k$ of $\Sigma_k ss\mathcal{S}_k$ by*

$$(\mathcal{O}(j)_k)_{\mathbf{n}} = \mathrm{Map}(U(\Delta^{\mathbf{n}}), \Sigma_j)_+$$

*(where the $+$ denotes a disjoint basepoint); the morphisms in $(\Delta_{\mathrm{inj}}^{op})^{\times k}$ act in the evident way, and the morphisms of the form $(\alpha, \mathrm{id})$ with $\alpha \in \Sigma_k$ act by permuting the factors in $\Delta^{\mathbf{n}}$.*

*(ii) For $j \ge 0$ define $\mathcal{O}(j)$ to be the object of $\Sigma ss\mathcal{S}$ with $k$-th term $\mathcal{O}(j)_k$.*

Definition 6.7. *(i) For $A, B \in \Sigma_k ss\mathcal{S}_k$, define the degreewise smash product*

$$A \barwedge B \in \Sigma_k ss\mathcal{S}_k$$

*by*

$$(A \barwedge B)_{\mathbf{n}} = A_{\mathbf{n}} \wedge B_{\mathbf{n}},$$

*with the diagonal action of $\Sigma_k$.*

*(ii) For $\mathbf{X}, \mathbf{Y} \in \Sigma ss\mathcal{S}$, define $\mathbf{X} \barwedge \mathbf{Y} \in \Sigma ss\mathcal{S}$ by*

$$(\mathbf{X} \barwedge \mathbf{Y})_k = X_k \barwedge Y_k.$$

Remark 6.8. The difference between the degreewise smash product $A \barwedge B$ and the previously defined smash product $A \wedge B$ is that the former is only defined when $A$ and $B$ are $k$-fold multisemisimplicial sets for the same $k$, and the result is again a $k$-fold multisemisimplicial set, whereas $A \wedge B$ is defined when $A$ is $k$-fold and $B$ is $l$-fold, and the result is $(k+l)$-fold.

Our next definition assembles the operations $a_*$ for a given $j$ into a single map.

Definition 6.9. *Let $j \ge 0$. Define a map*

$$\phi_j : \mathcal{O}(j) \barwedge \mathbf{R}^{\otimes j} \to \mathbf{R}$$

*in $\Sigma ss\mathcal{S}$ by the formulas*

$$\phi_j(a \wedge [e, F_1 \wedge \cdots \wedge F_j]) = a_*(F_1 \wedge \cdots \wedge F_j)$$

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*(where $e$ denotes the identity element of the relevant symmetric group) and*

$$\phi_j(a \wedge [\alpha, F_1 \wedge \cdots \wedge F_j]) = (\alpha, \mathrm{id})_* \phi_j((\alpha^{-1}, \mathrm{id})_* a \wedge [e, F_1 \wedge \cdots \wedge F_j]).$$

LEMMA 6.10. *The map $\phi_j$ induces a map*

$$\psi_j : \mathcal{O}(j) \,\overline{\wedge}\, \mathbf{R}^{\wedge j} \to \mathbf{R}$$

*in $\Sigma ss\mathcal{S}$.*

*Proof.* This is a straightforward calculation using example 4.13, definition 4.14, and [11, definition 17.3 and lemma 18.7]. □

## 7. A monad in $\Sigma ss\mathcal{S}$

In the next section, we will show that there is a monad $\mathbb{P}$ in $\mathcal{S}p_{mss}$ with the property that the maps $\psi_j$ constructed in lemma 6.10 give an action of $\mathbb{P}$ on $\mathbf{R}$. As preparation, in this section, we prove the analogous result in $\Sigma ss\mathcal{S}$; that is, we show that there is a monad $\mathbb{O}$ in $\Sigma ss\mathcal{S}$ for which the maps $\phi_j$ of definition 6.9 give an action of $\mathbb{O}$ on $\mathbf{R}$.

First we observe that the collection of objects $\mathcal{O}(j)$ has a composition map analogous to that of an operad. Recall that May defines an operad $\mathcal{M}$ in the category of sets with $\mathcal{M}(j) = \Sigma_j$ [17, definition 3.1(i)]. Let $\gamma_{\mathcal{M}}$ denote the composition operation in $\mathcal{M}$. Also recall definition 4.9 and notation 4.8.

DEFINITION 7.1. *Given $j_1, \ldots, j_i \geq 0$ define a map*

$$\gamma : \mathcal{O}(i) \,\overline{\wedge}\, \big(\mathcal{O}(j_1) \otimes \cdots \otimes O(j_i)\big) \to \mathcal{O}(j_1 + \cdots + j_i)$$

*in $\Sigma ss\mathcal{S}$ by the formulas*

$$\gamma(a \wedge [e, b_1 \wedge \cdots \wedge b_i])(\sigma_1 \times \cdots \times \sigma_i) = \gamma_{\mathcal{M}}(a(\sigma_1 \times \cdots \times \sigma_i), b_1(\sigma_1), \ldots, b_i(\sigma_i))$$

*(where $e$ is the identity element of the relevant symmetric group) and*

$$\gamma(a \wedge [\alpha, b_1 \wedge \ldots \wedge b_i]) = (\alpha, \mathrm{id})_* \gamma((\alpha^{-1}, \mathrm{id})_* a \wedge [e, b_1 \wedge \ldots \wedge b_i]).$$

In order to formulate the associativity property of $\gamma$, we note that for $\mathbf{X}_1, \ldots, \mathbf{X}_i, \mathbf{Y}_1, \ldots, \mathbf{Y}_i \in \Sigma ss\mathcal{S}$ there is a natural map

$$\chi : (\mathbf{X}_1 \,\overline{\wedge}\, \mathbf{Y}_1) \otimes \cdots \otimes (\mathbf{X}_i \,\overline{\wedge}\, \mathbf{Y}_i) \to (\mathbf{X}_1 \otimes \cdots \otimes \mathbf{X}_i) \,\overline{\wedge}\, (\mathbf{Y}_1 \otimes \cdots \otimes \mathbf{Y}_i)$$

given by

$$\chi([\alpha, x_1 \wedge y_1 \wedge \cdots \wedge x_i \wedge y_i]) = [\alpha, x_1 \wedge \cdots \wedge x_i] \wedge [\alpha, y_1 \wedge \cdots \wedge y_i].$$

Lemma 7.2. *The operation $\gamma$ has the following associativity property: the composite*

$$
\begin{aligned}
&\mathcal{O}(i) \barwedge \Big( \big(\mathcal{O}(j_1) \barwedge (\mathcal{O}(l_{11}) \otimes \cdots \otimes \mathcal{O}(l_{1j_1}))\big) \otimes \cdots \\
&\quad \otimes \big(\mathcal{O}(j_i) \barwedge (\mathcal{O}(l_{i1}) \otimes \cdots \otimes \mathcal{O}(l_{ij_i}))\big)\Big) \\
&\xrightarrow{1 \barwedge (\gamma \otimes \cdots \gamma)} \mathcal{O}(i) \barwedge \big(\mathcal{O}(l_{11} + \cdots + l_{1j_1}) \otimes \cdots \otimes \mathcal{O}(l_{i1} + \cdots + l_{ij_i})\big) \\
&\xrightarrow{\gamma} \mathcal{O}(l_{11} + \cdots + l_{ij_i})
\end{aligned}
$$

*is the same as the composite*

$$
\begin{aligned}
&\mathcal{O}(i) \barwedge \Big( \big(\mathcal{O}(j_1) \barwedge (\mathcal{O}(l_{11}) \otimes \cdots \otimes \mathcal{O}(l_{1j_1}))\big) \otimes \cdots \\
&\quad \otimes \big(\mathcal{O}(j_i) \barwedge (\mathcal{O}(l_{i1}) \otimes \cdots \otimes \mathcal{O}(l_{ij_i}))\big)\Big) \\
&\xrightarrow{1 \barwedge \chi} \mathcal{O}(i) \barwedge \big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i)\big) \barwedge \big(\mathcal{O}(l_{11}) \otimes \cdots \otimes \mathcal{O}(l_{i,j_i})\big) \\
&\xrightarrow{\gamma \barwedge 1} \mathcal{O}(j_1 + \cdots + j_i) \barwedge \big(\mathcal{O}(l_{11}) \otimes \cdots \otimes \mathcal{O}(l_{i,j_i})\big) \\
&\xrightarrow{\gamma} \mathcal{O}(l_{11} + \cdots + l_{ij_i}).
\end{aligned}
$$

To formulate the unital property of $\gamma$, we first need to consider the unit object for the operation $\barwedge$.

Definition 7.3. *Let $\overline{S}_k$ be the object of $\Sigma_k ss\mathcal{S}_k$ which has a copy of $S^0$ in each multidegree (with each morphism of $\Sigma_k \ltimes (\Delta^{op}_{\text{inj}})^{\times k}$ acting as the identity of $S^0$), and let $\overline{\mathbf{S}}$ be the object of $\Sigma ss\mathcal{S}$ with $k$-th term $\overline{S}_k$.*

Remark 7.4. (i) $\overline{\mathbf{S}} \barwedge \mathbf{X} \cong \mathbf{X}$ for any $\mathbf{X} \in \Sigma ss\mathcal{S}$.
(ii) $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are both equal to $\overline{\mathbf{S}}$.
(iii) $\overline{\mathbf{S}}$ is a commutative monoid in $\Sigma ss\mathcal{S}$ with multiplication

$$m : \overline{\mathbf{S}} \otimes \overline{\mathbf{S}} \to \overline{\mathbf{S}}$$

given by

$$m([\alpha, s_1 \wedge s_2]) = t,$$

where $s_1$ and $s_2$ are any nontrivial simplices and $t$ is the non-trivial simplex in the relevant multidegree.

Lemma 7.5. *The operation $\gamma$ has the following unital property: the diagrams*

$$
\begin{array}{ccc}
\mathcal{O}(j) \barwedge \overline{\mathbf{S}}^{\otimes j} & \xrightarrow{=} & \mathcal{O}(j) \barwedge \mathcal{O}(1)^{\otimes j} \\
{\scriptstyle 1 \barwedge m} \downarrow & & \downarrow {\scriptstyle \gamma} \\
\mathcal{O}(j) \barwedge \overline{\mathbf{S}} & \xrightarrow{\cong} & \mathcal{O}(j)
\end{array}
$$

*and*

$$\begin{array}{ccc} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathcal{O}(j) & \xrightarrow{\cong} & \mathcal{O}(j) \\ {\scriptstyle =}\downarrow & \nearrow{\scriptstyle \gamma} & \\ \mathcal{O}(1) \,\overline{\wedge}\, \mathcal{O}(j) & & \end{array}$$

*commute.*

To complete the analogy between $\gamma$ and the composition map of an operad, we need an equivariance property.

DEFINITION 7.6. *Define a right action of* $\Sigma_j$ *on* $\mathcal{O}(j)$ *by*

$$(a\alpha)(\sigma) = a(\sigma) \cdot \alpha,$$

*where* $a \in \mathrm{Map}(U(\Delta^{\mathbf{n}}), \Sigma_j)_+$, $\sigma \in U(\Delta^{\mathbf{n}})$, *and* $\cdot$ *is multiplication in* $\Sigma_j$.

LEMMA 7.7. *(i) The following diagram commutes for all* $\alpha \in \Sigma_i$.

$$\begin{array}{ccc} \mathcal{O}(i) \,\overline{\wedge}\, \big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i)\big) & \longrightarrow & \mathcal{O}(i) \,\overline{\wedge}\, \big(\mathcal{O}(j_{\alpha^{-1}(1)}) \otimes \cdots \otimes \mathcal{O}(j_{\alpha^{-1}(i)})\big) \\ & & \downarrow{\scriptstyle \gamma} \\ {\scriptstyle \alpha \overline{\wedge}(\beta_1 \otimes \cdots \otimes \beta_i)}\downarrow & & \mathcal{O}(j_1 + \cdots + j_i) \\ & & \downarrow{\scriptstyle \gamma_M(\alpha, \beta_1, \ldots, \beta_i)} \\ \mathcal{O}(i) \,\overline{\wedge}\, \big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i)\big) & \xrightarrow{\gamma} & \mathcal{O}(j_1 + \cdots + j_i) \end{array}$$

Now we use the data defined so far to construct a monad in the category $\Sigma ss\mathcal{S}$.

DEFINITION 7.8. *(i) For* $\mathbf{X} \in \Sigma ss\mathcal{S}$, *give* $\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\otimes j}$ *the diagonal right* $\Sigma_j$ *action.*

*(ii) Define a functor* $\mathbb{O} : \Sigma ss\mathcal{S} \to \Sigma ss\mathcal{S}$ *by*

$$\mathbb{O}(\mathbf{X}) = \bigvee_{j \geq 0} \big(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\otimes j}\big)/\Sigma_j.$$

*(iii) Define a natural transformation*

$$\iota : \mathbf{X} \to \mathbb{O}\mathbf{X}$$

*to be the composite*

$$\mathbf{X} \xrightarrow{\cong} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X} = \mathcal{O}(1) \,\overline{\wedge}\, \mathbf{X} \hookrightarrow \mathbb{O}(\mathbf{X}).$$

*(iv) Define*

$$\mu : \mathbb{O}\mathbb{O}\mathbf{X} \to \mathbb{O}\mathbf{X}$$

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*to be the natural transformation induced by the maps*

$$\begin{aligned}\mathcal{O}(i) \barwedge \Big( \big(\mathcal{O}(j_1) \barwedge \mathbf{X}^{\otimes j_1}\big) \otimes \cdots \otimes \big(\mathcal{O}(j_i) \barwedge \mathbf{X}^{\otimes j_i}\big)\Big) \\ \xrightarrow{1 \barwedge \chi} \Big(\mathcal{O}(i) \barwedge \big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i)\big)\Big) \barwedge \mathbf{X}^{\otimes(j_1+\cdots+j_i)} \\ \xrightarrow{\gamma \barwedge 1} \mathcal{O}(j_1+\cdots+j_i) \barwedge \mathbf{X}^{\otimes(j_1+\cdots+j_i)}.\end{aligned}$$

Proposition 7.9. *The transformations $\mu$ and $\iota$ define a monad structure on $\mathbb{O}$.*

*Proof.* This is immediate from lemmas 7.2 and 7.5. □

We conclude this section by giving the action of $\mathbb{O}$ on $\mathbf{R}$. Observe that the map

$$\phi_j : \mathcal{O}(j) \barwedge \mathbf{R}^{\otimes j} \to \mathbf{R}$$

of definition 6.9 induces a map

$$\big(\mathcal{O}(j) \barwedge \mathbf{R}^{\otimes j}\big)/\Sigma_j \to \mathbf{R}. \tag{7.1}$$

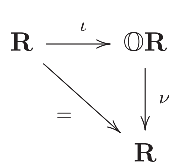
Definition 7.10. *Define*

$$\nu : \mathbb{O}\mathbf{R} \to \mathbf{R}$$

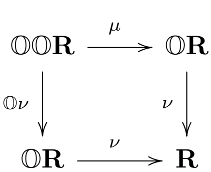
*to be the map whose restriction to $\big(\mathcal{O}(j) \barwedge \mathbf{X}^{\otimes j}\big)/\Sigma_j$ is the map (7.1).*

Proposition 7.11. *$\nu$ is an action of $\mathbb{O}$ on $\mathbf{R}$.*

*Proof.* We need to show that the diagrams




and

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commute. The first is obvious and for the second it suffices to check that the composite

$$\mathcal{O}(i) \mathbin{\overline{\wedge}} \Big( \big(\mathcal{O}(j_1) \mathbin{\overline{\wedge}} \mathbf{R}^{\otimes j_1}\big) \otimes \cdots \otimes \big(\mathcal{O}(j_i) \mathbin{\overline{\wedge}} \mathbf{R}^{\otimes j_i}\big) \Big) \xrightarrow{1 \overline{\wedge} \chi} \mathcal{O}(i) \mathbin{\overline{\wedge}} \big(\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_i)\big) \mathbin{\overline{\wedge}} \mathbf{R}^{\otimes(j_1+\cdots+j_i)} \xrightarrow{\gamma \overline{\wedge} 1} \mathcal{O}(j_1+\cdots+j_i) \mathbin{\overline{\wedge}} \mathbf{R}^{\otimes(j_1+\cdots+j_i)} \xrightarrow{\phi} \mathbf{R}$$

is the same as the composite

$$\mathcal{O}(i) \mathbin{\overline{\wedge}} \Big( \big(\mathcal{O}(j_1) \mathbin{\overline{\wedge}} \mathbf{R}^{\otimes j_1}\big) \otimes \cdots \otimes \big(\mathcal{O}(j_i) \mathbin{\overline{\wedge}} \mathbf{R}^{\otimes j_i}\big) \Big) \xrightarrow{1 \overline{\wedge} (\phi \otimes \cdots \phi)} \mathcal{O}(i) \mathbin{\overline{\wedge}} \mathbf{R}^{\otimes i} \xrightarrow{\phi} \mathbf{R}.$$

□

## 8. A monad in $\mathcal{S}p_{mss}$

We begin by giving $\mathcal{O}(j) \mathbin{\overline{\wedge}} \mathbf{X}^{\wedge j}$ the structure of a multisemisimplicial symmetric spectrum when $\mathbf{X} \in \mathcal{S}p_{mss}$. The definition is motivated by lemma 6.5. Recall definition 6.4.

Definition 8.1. *Let $j, k \geq 0$. Let $s$ be the 1-simplex of $S^1$. Define*

$$\omega : S^1 \wedge (\mathcal{O}(j) \mathbin{\overline{\wedge}} \mathbf{X}^{\wedge j})_k \to (\mathcal{O}(j) \mathbin{\overline{\wedge}} \mathbf{X}^{\wedge j})_{k+1}$$

*as follows: for $a \in (\mathcal{O}(j)_k)_{\mathbf{n}}$ and $x \in ((X^{\wedge j})_k)_{\mathbf{n}}$, let*

$$\omega(s \wedge (a \wedge x)) = (a \circ \Pi) \wedge \omega(s \wedge x).$$

*The identification given in definition 4.14 was omitted in this notation.*

Definition 8.2. *(i) For $\mathbf{X} \in \mathcal{S}p_{mss}$, give $\mathcal{O}(j) \mathbin{\overline{\wedge}} \mathbf{X}^{\wedge j}$ the diagonal right $\Sigma_j$ action.*
*(ii) Define a functor $\mathbb{P} : \mathcal{S}p_{mss} \to \mathcal{S}p_{mss}$ by*

$$\mathbb{P}(\mathbf{X}) = \bigvee_{j \geq 0} (\mathcal{O}(j) \mathbin{\overline{\wedge}} \mathbf{X}^{\wedge j})/\Sigma_j.$$

To give $\mathbb{P}$ a monad structure we need

Lemma 8.3. *The composite in definition 7.8(iv) induces a map*

$$\mathcal{O}(i) \mathbin{\overline{\wedge}} \Big( \big(\mathcal{O}(j_1) \mathbin{\overline{\wedge}} \mathbf{X}^{\wedge j_1}\big) \wedge \cdots \wedge \big(\mathcal{O}(j_i) \mathbin{\overline{\wedge}} \mathbf{X}^{\wedge j_i}\big) \Big) \to \mathcal{O}(j_1+\cdots+j_i) \mathbin{\overline{\wedge}} \mathbf{X}^{\wedge(j_1+\cdots+j_i)}$$

*in $\mathcal{S}p_{mss}$.*

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Definition 8.4. *(i) Define a natural transformation*

$$\iota : \mathbf{X} \to \mathbb{P}\mathbf{X}$$

*to be the composite*

$$\mathbf{X} \xrightarrow{\cong} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X} = \mathcal{O}(1) \,\overline{\wedge}\, \mathbf{X} \hookrightarrow \mathbb{P}(\mathbf{X}).$$

*(ii) Define*

$$\mu : \mathbb{P}\mathbb{P}\mathbf{X} \to \mathbb{P}\mathbf{X}$$

*to be the natural transformation induced by the maps constructed in lemma 8.3.*

Proposition 8.5. *The transformations $\mu$ and $\iota$ define a monad structure on $\mathbb{P}$.*

*Proof.* This follows from proposition 7.9 by passage to quotients. □

Next we give the action of $\mathbb{P}$ on $\mathbf{R}$. By definition 8.1 and lemma 6.5, the map

$$\psi_j : \mathcal{O}(j) \,\overline{\wedge}\, \mathbf{R}^{\wedge j} \to \mathbf{R}$$

of lemma 6.10 is a map in $\mathcal{S}p_{mss}$. It induces a map

$$\big(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{R}^{\wedge j}\big)/\Sigma_j \to \mathbf{R} \tag{8.1}$$

in $\mathcal{S}p_{mss}$.

Definition 8.6. *Define*

$$\nu : \mathbb{P}\mathbf{R} \to \mathbf{R}$$

*to be the map whose restriction to $\big(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j$ is the map (8.1).*

Proposition 8.7. *$\nu$ is an action of $\mathbb{P}$ on $\mathbf{R}$.*

*Proof.* This follows from proposition 7.11 by passage to quotients. □

For use in §10, we record a lemma.

Lemma 8.8. *(i) There is a functor $\Upsilon$ from $\mathbb{P}$ algebras to monoids in $\mathcal{S}p_{mss}$ (with respect to $\wedge$) which is the identity on objects.*

*(ii) The geometric realization of $\Upsilon(\mathbf{R})$ is the symmetric ring spectrum $\mathbf{M}$ of [11, theorem 18.5].*

*Proof.* Part (i). Let $\mathbb{A}$ be the monad

$$\mathbb{A}(\mathbf{X}) = \bigvee_{j \geq 0} \mathbf{X}^{\wedge j}.$$

Then a monoid in $\mathcal{S}p_{mss}$ is the same thing as an $\mathbb{A}$-algebra, so it suffices to give a map of monads from $\mathbb{A}$ to $\mathbb{P}$.

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For each $j, k \geq 0$ and each $k$-fold multi-index $\mathbf{n}$, define an element

$$e_{j,k,\mathbf{n}} \in (\mathcal{O}(j)_k)_\mathbf{n}$$

to be the constant function $U(\Delta^\mathbf{n}) \to \Sigma_j$ whose value is the identity element of $\Sigma_j$. Next define a map

$$\overline{\mathbf{S}} \to \mathcal{O}(j)$$

by taking the non-trivial simplex of $(\overline{\mathbf{S}}_k)_\mathbf{n}$ to $e_{j,k,\mathbf{n}}$.

Now the composite

$$\mathbb{A}(\mathbf{X}) = \bigvee_{j\geq 0} \mathbf{X}^{\wedge j} \cong \bigvee_{j\geq 0} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X}^{\wedge j} \to \bigvee_{j\geq 0} \big(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j = \mathbb{P}(\mathbf{X})$$

is a map of monads.

Part (ii) is an easy consequence of the definitions. □

It is worth mentioning that the map from $\mathbb{A}$ to $\mathbb{P}$ does not factor over the commutative monad because $e_{j,k,n}$ is not a fixed point for the $\Sigma_n$-action.

## 9. Degreewise smash product and geometric realization

For the proof of theorem 1.1, we need to know the relation between $\overline{\wedge}$ and geometric realization.

There is a natural map

$$\kappa : |A \,\overline{\wedge}\, B| \to |A| \wedge |B|$$

defined by

$$\kappa([u, x \wedge y]) = [u, x] \wedge [u, y].$$

The analogous map for multisimplicial sets is a homeomorphism, but the situation for multisemisimplicial sets is more delicate.

Definition 9.1. *A multisemisimplicial set has compatible degeneracies if it is in the image of the forgetful functor from multisimplicial sets to multisemisimplicial sets.*

Example 9.2. (i) One can define compatible degeneracies on $\mathcal{O}(j)_k$ for each $j, k \geq 0$ by using the codegeneracy maps between the $\Delta^\mathbf{n}$.

(ii) If $\mathbf{X} \in \mathcal{S}p_{mss}$ and $X_k$ has compatible degeneracies for all $k$ then each $\big(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)_k$ has compatible degeneracies.

Proposition 9.3. *If the underlying multisemisimplicial sets of $A$ and $B$ have compatible degeneracies then $\kappa$ is a weak equivalence.*

*Proof.* Let $\tilde{A}$ and $\tilde{B}$ be multisimplicial sets whose underlying multisemisimplicial sets are $A$ and $B$. Then the underlying multisemisimplicial set of the degreewise

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smash product $\tilde{A} \barwedge \tilde{B}$ is $A \barwedge B$. Consider the following commutative diagram, where $|\ |$ in the bottom row denotes realization of multisimplicial sets, $\tilde{\kappa}$ is defined analogously to $\kappa$, and the vertical maps collapse the degeneracies:

$$\begin{array}{ccc} |A \barwedge B| & \xrightarrow{\kappa} & |A| \wedge |B| \\ \downarrow & & \downarrow \\ |\tilde{A} \barwedge \tilde{B}| & \xrightarrow{\tilde{\kappa}} & |\tilde{A}| \wedge |\tilde{B}| \end{array}$$

The map $\tilde{\kappa}$ is a homeomorphism, and the vertical arrows are weak equivalences by the multisimplicial analogue of [24, lemma A.5], so $\kappa$ is a weak equivalence. □

Next we give a sufficient condition for a multisemisimplicial set to have compatible degeneracies. Let $D^n$ denote the semisimplicial set consisting of the nondegenerate simplices of the standard simplicial $n$-simplex. For a multi-index $\mathbf{n}$, let $D^{\mathbf{n}}$ denote the $k$-fold multisemisimplicial set

$$D^{n_1} \times \cdots \times D^{n_k}.$$

Definition 9.4. *(i) A horn in $D^{\mathbf{n}}$ is a subcomplex $E$ which contains all elements of $D^{\mathbf{n}}$ except for the top-dimensional element and one of its faces.*

*(ii) A $k$-fold multisemisimplicial set $A$ satisfies the multi-Kan condition if every map from a horn in $D^{\mathbf{n}}$ to $A$ extends to a map $D^{\mathbf{n}} \to A$.*

The following result is proved in [16].

Proposition 9.5. *If $A$ satisfies the multi-Kan condition then it has compatible degeneracies.*

Our next result is proved in the same way as [11, lemma 15.12] and does not require the ad theory to be commutative.

Proposition 9.6. *For each $k$, $R_k$ satisfies the multi-Kan condition.*

## 10. Rectification

In this section, we complete the proof of theorem 1.1.

First we consider a monad in $\mathcal{S}p_{mss}$ which is simpler than $\mathbb{P}$.

Definition 10.1. *(i) Define $\mathbb{P}'(\mathbf{X})$ to be $\bigvee_{j \geq 0} \mathbf{X}^{\wedge j}/\Sigma_j$.*

*(ii) For each $j \geq 0$, let*

$$\xi_j : \mathcal{O}(j) \to \overline{\mathbf{S}}$$

*be the map which takes each non-trivial simplex of $\mathcal{O}(j)_k$ to the non-trivial simplex of $\overline{S}_k$ in the same multidegree. Define a natural transformation*

$$\Xi : \mathbb{P} \to \mathbb{P}'$$

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*to be the wedge of the composites*

$$\big(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j \xrightarrow{\xi_j \overline{\wedge} 1} \big(\overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j \xrightarrow{\cong} \mathbf{X}^{\wedge j}/\Sigma_j.$$

PROPOSITION 10.2. *(i) An algebra over $\mathbb{P}'$ is the same thing as a commutative monoid in $\mathcal{S}p_{mss}$.*

*(ii) $\Xi$ is a map of monads.*

*(iii) Suppose that each $X_k$ has compatible degeneracies (see definition 9.1). Let $\mathbb{P}^q$ denote the q-th iterate of $\mathbb{P}$. Then each map*

$$\Xi : \mathbb{P}^q(\mathbf{X}) \to \mathbb{P}'\mathbb{P}^{q-1}(\mathbf{X})$$

*is a weak equivalence.*

Parts (i) and (ii) are immediate from the definitions. Part (iii) will be proved at the end of this section.

*Proof of theorem 1.1.* We apply the monadic bar construction [17, construction 9.6] to obtain simplicial objects $B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})$ and $B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})$ in $\mathcal{S}p_{mss}$. We write $\mathbf{R}_\bullet$ for the constant simplicial object which is $\mathbf{R}$ in each simplicial degree. There are maps of simplicial $\mathbb{P}$-algebras

$$\mathbf{R}_\bullet \xleftarrow{\varepsilon} B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R}) \xrightarrow{\Xi_\bullet} B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R}), \tag{10.1}$$

where $\varepsilon$ is induced by the action of $\mathbb{P}$ on $\mathbf{R}$ (see [17, lemma 9.2(ii)]). The map $\varepsilon$ is a homotopy equivalence of simplicial objects [17, Proposition 9.8] and the map $\Xi_\bullet$ is a weak equivalence in each simplicial degree by propositions 9.5, 9.6, and 10.2(iii). $B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})$ is a simplicial algebra over $\mathbb{P}'$, which by proposition 10.2(i) is the same thing as a simplicial commutative monoid in $\mathcal{S}p_{mss}$. Moreover, by lemma 8.8(i), $\mathbf{R}_\bullet$ and $B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})$ are simplicial monoids, and $\varepsilon$ and $\Xi_\bullet$ are maps of simplicial monoids.

The objects of the diagram (10.1) are simplicial objects in $\mathcal{S}p_{mss}$. We obtain a diagram

$$|\mathbf{R}_\bullet| \xleftarrow{|\varepsilon|} |B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})| \xrightarrow{|\Xi_\bullet|} |B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})| \tag{10.2}$$

of simplicial objects in $\mathcal{S}p$ (the category of symmetric spectra) by applying the geometric realization functor $\mathcal{S}p_{mss} \to \mathcal{S}p$ to the diagram (10.1) in each simplicial degree. The map $|\varepsilon|$ is a homotopy equivalence of simplicial objects and the map $|\Xi_\bullet|$ is a weak equivalence in each simplicial degree. The object $|B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})|$ is a simplicial commutative symmetric ring spectrum, the objects $|\mathbf{R}_\bullet|$ and $|B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})|$ are simplicial symmetric ring spectra, and the maps $|\varepsilon|$ and $|\Xi_\bullet|$ are maps of simplicial symmetric ring spectra.

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Finally, we apply geometric realization to the diagram (10.2). We define $\mathbf{M}^{\mathrm{comm}}$ to be $||B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})||$. Now we have a diagram

$$\mathbf{M} = |\mathbf{R}| \xleftarrow{||\varepsilon||} ||B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})|| \xrightarrow{||\Xi_\bullet||} ||B_\bullet(\mathbb{P}', \mathbb{P}, \mathbf{R})|| = \mathbf{M}^{\mathrm{comm}} \tag{10.3}$$

in $\mathcal{S}p$. The map $||\varepsilon||$ is a homotopy equivalence (cf. [17, corollary 11.9]) and the map $||\Xi_\bullet||$ is a weak equivalence by [12, theorem E]. $\mathbf{M}^{\mathrm{comm}}$ is a commutative symmetric ring spectrum, $\mathbf{M}$ is the symmetric ring spectrum of [11, theorem 18.5] (by lemma 8.8(ii)), $||B_\bullet(\mathbb{P}, \mathbb{P}, \mathbf{R})||$ is a symmetric ring spectrum, and $||\varepsilon||$ and $||\Xi_\bullet||$ are maps of symmetric ring spectra. □

We conclude this section with the proof of part (iii) of proposition 10.2. First we need a lemma (which for later use we state in more generality than we immediately need). Recall that a preorder is a set with a reflexive and transitive relation $\leq$. Examples are $\Sigma_j$, with every element $\leq$ every other, and $U(K)$ (see definition 6.2(i)), with $\leq$ induced by inclusions of cells.

Lemma 10.3. *Let $P$ be a preorder with an element which is $\geq$ all other elements, and let $k \geq 0$. Define a k-fold multisemisimplicial set $A$ by*

$$A_{\mathbf{n}} = \mathrm{Map}_{\mathrm{preorder}}(U(\Delta^{\mathbf{n}}), P).$$

*Then*

*(i) $A$ has compatible degeneracies, and*

*(ii) $A$ is weakly equivalent to a point.*

*Proof.* For (i), we can give $A$ compatible degeneracies by using the codegeneracy maps between the $\Delta^{\mathbf{n}}$.

Part (ii). Let $\tilde{A}$ be a multisimplicial set whose underlying multisemisimplicial set is $A$. Let $d\tilde{A}$ be its diagonal. The multisimplicial analogue of [24, Lemma A.5] implies that $|A|$ is weakly equivalent to $|\tilde{A}|$, and it is well-known that the latter is homeomorphic to $|d\tilde{A}|$. It therefore suffices to show that the simplicial set $d\tilde{A}$ is weakly equivalent to a point.

Let $\Delta^n_{\mathrm{simp}}$ denote the standard simplicial $n$ simplex and let $\partial\Delta^n_{\mathrm{simp}}$ denote its boundary. Then it suffices by [7, theorem I.11.2] to show that every map from $\partial\Delta^n_{\mathrm{simp}}$ to $d\tilde{A}$ extends to $\Delta^n_{\mathrm{simp}}$.

Let $D^n$ (resp., $\partial D^n$) be the semisimplicial set consisting of the nondegenerate simplices of $\Delta^n_{\mathrm{simp}}$ (resp., $\partial\Delta^n_{\mathrm{simp}}$). Since $\Delta^n_{\mathrm{simp}}$ (resp., $\partial\Delta^n_{\mathrm{simp}}$) is the free simplicial set generated by $D^n$ (resp., $\partial D^n$), it suffices to show that every semisimplicial map from $\partial D^n$ to $d\tilde{A}$ extends to $D^n$, and this is obvious from the definition of $A$. □

Note that if $P$ is $\Sigma_j$ with the preorder described above then $A_+$ is $\tilde{\mathcal{O}}(j)_k$.

*Proof of 10.2(iii).* We begin with the case $q = 1$, so we want to show that the map $\Xi : \mathbb{P}(\mathbf{X}) \to \mathbb{P}'(\mathbf{X})$ is a weak equivalence. It suffices to show that the map

$$(\mathcal{O}(j) \,\overline{\wedge}\, \mathbf{X}^{\wedge j})/\Sigma_j \to \mathbf{X}^{\wedge j}/\Sigma_j$$

is a weak equivalence for each $j$. Proposition A.2 and remark A.3 show that the $\Sigma_j$ actions are free away from the basepoint. We will now employ the following

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fact: if $X$ is a cellular $G$-spectrum with a free action away from the base point then the canonical map from $EG_+ \wedge_G X$ to $X/G$ is a weak equivalence. Moreover, $(EG)_+ \wedge_G X$ sits in a fibration with base $BG$ and fibre $X$. Hence, it suffices to show that each map

$$\mathcal{O}(j) \barwedge \mathbf{X}^{\wedge j} \to \mathbf{X}^{\wedge j}$$

is a weak equivalence. Now the object $\left(\mathcal{O}(j) \barwedge \mathbf{X}^{\wedge j}\right)_k$ comes from

$$\bigvee_{k_1+\cdots+k_j=k} \mathcal{O}(j)_k \barwedge I^{\Sigma_k}_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_j}} \left(\mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j}\right)$$

and we have

$$\begin{aligned} &\mathcal{O}(j)_k \barwedge I^{\Sigma_k}_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_j}} \left(\mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j}\right) \\ &\quad \cong I^{\Sigma_k}_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_j}} \Big(\mathcal{O}(j)_k \barwedge \left(\mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j}\right)\Big), \end{aligned}$$

so it suffices by proposition 5.4 to show that each map

$$\mathcal{O}(j)_k \barwedge (\mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j}) \to \mathbf{X}_{k_1} \wedge \cdots \wedge \mathbf{X}_{k_j}$$

is a weak equivalence, and this follows from example 9.2, proposition 9.3, and lemma 10.3.

The general case follows from the case $q = 1$ and example 9.2(ii). □

## 11. Proof of theorem 1.2

It is well-known that Thom spectra are commutative symmetric ring spectra (see for example [23]; we recall this below). In this section, we show that the Thom spectrum $M\mathrm{STop}$ obtained from the bar construction is weakly equivalent, in the category of commutative symmetric ring spectra, to the commutative symmetric ring spectrum $(\mathbf{M}_{\mathrm{STop}})^{\mathrm{comm}}$ given by theorem 1.1.

Our first task is to construct the following chain of weak equivalences in the category of symmetric spectra

$$\mathbf{M}_{\mathrm{STop}} \xleftarrow{f_1} \mathbf{Y} \xrightarrow{f_2} \mathbf{X} \xrightarrow{f_3} M\mathrm{STop}. \tag{11.1}$$

First recall that $M\mathrm{STop}$ has as $k$th space the Thom space $T(\mathrm{STop}(k))$. The $\Sigma_k$ action on $T(\mathrm{STop}(k))$ is induced by the conjugation action on $\mathrm{STop}(k)$.

For the construction of $\mathbf{X}$, we need some facts about multisimplicial sets. Given a space $Z$ and $k \geq 1$, let $S^{k\text{-multi}}_\bullet(Z)$ be the $k$-fold multisimplicial set whose simplices in multidegree $\mathbf{n}$ are the maps $\Delta^{\mathbf{n}} \to Z$. There is a natural map

$$|S^{k-\mathrm{multi}}_\bullet(Z)| \to Z$$

(where $|\ |$ denotes realization of the underlying multisimplicial set) which is a weak equivalence by [1] and the multisimplicial analogue of [24, lemma A.5]. If $Z$ is a

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based space, there are natural maps

$$\lambda : \Sigma|S_\bullet^{k-\text{multi}}(Z)| \to |S^1 \wedge S_\bullet^{k-\text{multi}}(Z)|$$

and

$$\kappa : S^1 \wedge S_\bullet^{k-\text{multi}}(Z) \to S_\bullet^{(k+1)-\text{multi}}(\Sigma Z)$$

defined as follows. Given $t \in [0,1]$, $u \in \Delta^{\mathbf{n}}$, and $g : \Delta^{\mathbf{n}} \to Z$, let $\bar{t}$ denote the image of $t$ under the oriented affine homeomorphism $[0,1] \to \Delta^1$, and define

$$\lambda(t \wedge [u,g]) = [(\bar{t}, u), s \wedge g],$$

where $s$ is the non-trivial simplex of $S^1$. Define

$$\kappa(s \wedge g)(\bar{t}, u) = t \wedge g(u).$$

Then the diagram

$$\begin{array}{ccc}
\Sigma|S_\bullet^{k-\text{multi}}(Z)| & \xrightarrow{\lambda} & |S^1 \wedge S_\bullet^{k-\text{multi}}(Z)| \\
\downarrow & & \downarrow{\scriptstyle |\kappa|} \\
\Sigma Z & \longleftarrow & |S_\bullet^{(k+1)-\text{multi}}(\Sigma Z)|
\end{array}$$

commutes.

Now let $X_k = |S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))|$. We define the $\Sigma_k$ action on $X_k$ as follows. For $\alpha \in \Sigma_k$ and $g : \Delta^{\mathbf{n}} \to T(\text{STop}(k)))$, let $\alpha(\mathbf{n}) = (n_{\alpha^{-1}(1)}, \ldots, n_{\alpha^{-1}(k)})$ and let $\alpha(g)$ be the composite

$$\Delta^{\alpha(\mathbf{n})} \xrightarrow{\alpha^{-1}} \Delta^{\mathbf{n}} \xrightarrow{g} T(\text{STop}(k)) \xrightarrow{\alpha} T(\text{STop}(k)).$$

This makes $S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))$ an object of $\Sigma_k ss\mathcal{S}_k$, and now proposition 5.3 gives the $\Sigma_k$ action on $X_k$. Next define the structure map

$$\Sigma X_k \to X_{k+1}$$

to be the composite

$$\begin{aligned}
\Sigma|S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))| &\xrightarrow{\lambda} |S^1 \wedge S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))| \\
&\xrightarrow{|\kappa|} |S_\bullet^{(k+1)-\text{multi}}(\Sigma T(\text{STop}(k)))| \to |S_\bullet^{(k+1)-\text{multi}}(T(\text{STop}(k+1)))|,
\end{aligned}$$

where the last map is induced by the structure map of $M\text{STop}$. Let $\mathbf{X}$ be the symmetric spectrum consisting of the spaces $X_k$ with these structure maps. Define $f_3 : \mathbf{X} \to M\text{STop}$ to be the sequence of weak equivalences

$$|S_\bullet^{k-\text{multi}}(T(\text{STop}(k)))| \to T(\text{STop}(k)).$$

The commutativity of diagram (11.2) shows that $f_3$ is a map of symmetric spectra.

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Next let $S_\bullet^{k-\mathrm{multi},\pitchfork}(T(\mathrm{STop}(k)))$ be the sub-multisemisimplicial set of $S_\bullet^{k-\mathrm{multi}}(T(\mathrm{STop}(k)))$ consisting of maps whose restrictions to each face of $\Delta^{\mathbf{n}}$ are transverse to the zero section (see [5] for topological transversality). Let $\mathbf{Y}$ be the subspectrum of $\mathbf{X}$ with $k$th space $|S_\bullet^{k-\mathrm{multi},\pitchfork}(T(\mathrm{STop}(k)))|$, and let $f_2 : \mathbf{Y} \to \mathbf{X}$ be the inclusion.

Lemma 11.1. *$f_2$ is a weak equivalence.*

*Proof.* Since $S_\bullet^{k-\mathrm{multi},\pitchfork}(T(\mathrm{STop}(k)))$ and $S_\bullet^{k-\mathrm{multi}}(T(\mathrm{STop}(k)))$ satisfy the multi-Kan condition, they have compatible degeneracies by proposition 9.5. It therefore suffices to show that the inclusion $S_\bullet^{k-\mathrm{multi},\pitchfork}(T(\mathrm{STop}(k))) \subset S_\bullet^{k-\mathrm{multi}}(T(\mathrm{STop}(k)))$ induces a weak equivalence on the diagonal semisimplicial sets, and this follows from [5, §9.6] and the definition of homotopy groups [18, definition 3.6]. □

It remains to construct $f_1$. Let $S \subset T(\mathrm{STop}(k))$ be the zero section. First we observe that, if $g : \Delta^{\mathbf{n}} \to T(\mathrm{STop}(k))$ is a map whose restriction to each face is transverse to $S$, we obtain an element $F \in \mathrm{ad}_{\mathrm{STop}}(\Delta^{\mathbf{n}})$ by letting $F(\sigma, o)$ be $g^{-1}(S) \cap \sigma$ with the orientation determined by $o$. This construction gives a map

$$S_\bullet^{k-\mathrm{multi},\pitchfork}(T(\mathrm{STop}(k))) \to (\mathbf{R}_{\mathrm{STop}})_k,$$

in $\Sigma_k ss\mathcal{S}_k$, and applying geometric realization gives a $\Sigma_k$ equivariant map $Y_k \to (\mathbf{M}_{\mathrm{STop}})_k$; we let $f_1$ be the sequence of these maps.

Lemma 11.2. *$f_1$ is a weak equivalence.*

*Proof.* For a $k$-fold multisemisimplicial set $A$, let $A'$ be the semisimplicial set whose $n$th set is $A_{0,\ldots,0,n}$. There is an evident map

$$\phi : |A'| \to |A|.$$

If $A$ is $S_\bullet^{k-\mathrm{multi}}(Z)$, then $A'$ is $S_\bullet(Z)$, and if $A$ is $R_k$ then $A'$ is the semisimplicial set $P_k$ of [11, definition 15.4(i)], with realization $(\mathbf{Q}_{\mathrm{STop}})_k$ [11, definitions 15.4(ii) and 15.8]. Now we have a commutative diagram

$$\begin{array}{ccccccc}
(\mathbf{M}_{\mathrm{STop}})_k & \xleftarrow{f_1} & Y_k & \xrightarrow{f_2} & X_k & \xrightarrow{f_3} & T(\mathrm{STop}(k)) \\
\uparrow{\scriptstyle \phi_1} & & \uparrow{\scriptstyle \phi_2} & & & & \uparrow{\scriptstyle =} \\
(\mathbf{Q}_{\mathrm{STop}})_k & \xleftarrow{g_1} & |S_\bullet^{\pitchfork}(T(\mathrm{STop}(k)))| & \xrightarrow{g_2} & |S_\bullet(T(\mathrm{STop}(k)))| & \xrightarrow{g_3} & T(\mathrm{STop}(k))
\end{array}$$

Here $g_3$ is the usual weak equivalence, and $g_2$ is a weak equivalence by [5, §9.6] and the definition of homotopy groups [18, definition 3.6], so $\phi_2$ is a weak equivalence. $g_1$ was shown to be a weak equivalence in [11, Appendix B], and $\phi_1$ was shown to be a weak equivalence in [11, §15], so $f_1$ is a weak equivalence as required. □

This completes the construction of diagram (11.1).

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Next we recall that $M\mathrm{STop}$ is a commutative symmetric ring spectrum with product

$$T(\mathrm{STop}(k)) \wedge T(\mathrm{STop}(l)) \to T(\mathrm{STop}(k+l)).$$

$\mathbf{X}$ is also a commutative symmetric ring spectrum, with the product

$$|S_\bullet^{k-\mathrm{multi}}(T(\mathrm{STop}(k)))| \wedge |S_\bullet^{l-\mathrm{multi}}(T(\mathrm{STop}(l)))|$$
$$\to |S_\bullet^{(k+l)-\mathrm{multi}}(T(\mathrm{STop}(k)) \wedge T(\mathrm{STop}(l)))| \to |S_\bullet^{(k+l)-\mathrm{multi}}(T(\mathrm{STop}(k+l)))|,$$

and $\mathbf{Y}$ is a commutative symmetric ring spectrum with the product it inherits from $\mathbf{X}$. The maps $f_2$ and $f_3$ are maps of symmetric ring spectra, so to complete the proof of theorem 1.2, it suffices to show

Lemma 11.3. $(\mathbf{M}_{\mathrm{STop}})^{\mathrm{comm}}$ *and* $\mathbf{Y}$ *are isomorphic in the homotopy category of commutative symmetric ring spectra.*

The proof of lemma 11.3 is outsourced to Appendix B. The results of Appendix B use material from §17 and 18.

## 12. Relaxed symmetric Poincaré complexes

For a commutative ring $R$ with the trivial involution, we would like to apply theorem 1.1 to obtain a commutative model for the symmetric L-spectrum of $R$. However, the ad theory $\mathrm{ad}^R$ defined in §9 of [11], with the product defined in [11, definition 9.12], is not commutative. The difficulty is that this product is defined using a noncommutative coproduct

$$\Delta : W \to W \otimes W$$

for the standard resolution $W$ of $\mathbb{Z}$ by $\mathbb{Z}[\mathbb{Z}/2]$-modules. In this section and the next, we give an equivalent ad theory which is commutative.

Fix a ring $R$ with involution. For a complex $C$ of left $R$-modules, let $C^t$ be the complex of right $R$-modules obtained from $C$ by applying the involution of $R$. As usual, give $C^t \otimes_R C$ the $\mathbb{Z}/2$-action which switches the factors. Write $\mathcal{C}h_{hf}$ for the category of homotopy finite chain complexes as in [11, definition 9.2(v)].

Definition 12.1. *A relaxed quasi-symmetric complex of dimension $n$ is a quadruple $(C, D, \beta, \varphi)$, where $C$ is an object of $\mathcal{C}h_{hf}$, $D$ is an object of $\mathcal{C}h_{hf}$ with a $\mathbb{Z}/2$ action, $\beta$ is a quasi-isomorphism $C^t \otimes_R C \to D$ which is also a $\mathbb{Z}/2$ equivariant map, and $\varphi$ is an element of $D_n^{\mathbb{Z}/2}$.*

Example 12.2. (i) If $(C, \varphi)$ is a quasi-symmetric complex as defined in [11, definition 9.3], then the quadruple $(C, (C^t \otimes_R C)^W, \beta, \varphi)$ is a relaxed quasi-symmetric complex, where $\beta : C^t \otimes_R C \to (C^t \otimes_R C)^W$ is induced by the augmentation $W \to \mathbb{Z}$.

(ii) Relaxed quasi-symmetric complexes arise naturally from the construction of the symmetric signature of a Witt space given in [6]; see [3].

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Definition 12.3. *We define a category $\mathcal{A}^R_{rel}$ (the rel stands for relaxed) as follows. The objects of $\mathcal{A}^R_{rel}$ are the relaxed quasi-symmetric complexes. A morphism $(C, D, \beta, \varphi) \to (C', D', \beta', \varphi')$ is a pair $(f : C \to C', g : D \to D')$, where $f$ and $g$ are $R$-linear chain maps, $g$ is $\mathbb{Z}/2$ equivariant, $g\beta = \beta'(f \otimes f)$, and (if $\dim \varphi = \dim \varphi'$) $g_*(\varphi) = \varphi'$.*

Definition 12.4. *A morphism $(f, g) : (C, D, \beta, \varphi) \to (C', D', \beta', \varphi')$ between objects of the same dimension is a quasi-isomorphism if $f$ (and hence also $g$) is a quasi-isomorphism.*

$\mathcal{A}^R_{\mathrm{rel}}$ is a $\mathbb{Z}$-graded category, where $d$ was defined above, $i$ takes $(C, D, \beta, \varphi)$ to $(C, D, \beta, -\varphi)$ and $\emptyset_n$ is the $n$-dimensional object for which $C$ and $D$ are zero in all degrees. Since the set of morphisms between objects of different dimensions is independent of the chains $\varphi$ the category $\mathcal{A}^R_{\mathrm{rel}}$ is a balanced in the obvious way [11, definition 5.1].

Remark 12.5. The construction of example 12.2(i) gives a morphism

$$\mathcal{A}^R \to \mathcal{A}^R_{\mathrm{rel}}$$

of $\mathbb{Z}$-graded categories.

Next we must say what the $K$-ads with values in $\mathcal{A}^R_{\mathrm{rel}}$ are. We need some preliminary definitions and a lemma. For a balanced pre $K$-ad $F$ we will use the notation

$$F(\sigma, o) = (C_\sigma, D_\sigma, \beta_\sigma, \varphi_{\sigma,o}).$$

Recall [11, definition 9.7].

Definition 12.6. *A balanced pre $K$-ad $F$ is well-behaved if $C$ and $D$ are well-behaved.*

Next recall [11, definition 12.2].

Lemma 12.7. *Let $F$ be a well-behaved pre $K$-ad. Then*
*(i) $C^t \otimes_R C$ is well-behaved, and*
*(ii) the map*

$$(\beta_\sigma)_* : H_*((C^t \otimes_R C)_\sigma/(C^t \otimes_R C)_{\partial\sigma}) \to H_*(D_\sigma/D_{\partial\sigma})$$

*is an isomorphism for each $\sigma$.*

*Proof.* For part (i), first recall (by [11, definitions 9.7(b) and 9.6(ii)]) that $C$ takes morphism to cofibrations, that is, split monomorphisms in each degree. Moreover, the canonical map from $C_{\partial\sigma}$ to $C_\sigma$ is a cofibration for each $\sigma$. Let us fix a cell $\sigma$ of $K$ for the rest of the proof. For each degree $n$, we will construct a set $S_\tau$ and a basis $(b_s)_{s\in S_\tau}$ of $C_\tau$ by induction for all $\tau \subset \sigma$ in a functorial way. We will omit the degree from the notation for this part. For points $\tau$, we simply choose a basis. These add up to a basis of $C_{\partial\rho}$ for 1-cells $\rho$. Given a cell $\tau$ of dimension $k$, we may

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suppose that we have already constructed a basis of $C_{\partial\tau}$. Since $C_{\partial\tau} \to C_\tau$ is split injective and all modules, including the quotient module, are free we may extend this basis to a basis of $C_\tau$. This way, we constructed a basis of $C_{\partial\rho}$ for all $\rho$ of dimension $k+1$ as well: the set of all $b_s$ with $s \in \bigcup_{\tau \subsetneq \rho} S_\tau$ gives a basis because the free functor commutes with colimits. We have to show that the map

$$(C^t \otimes_R C)_{\partial\sigma} = \mathrm{colim}_{\tau \subsetneq \sigma} C^t_\tau \otimes_R C_\tau \longrightarrow C^t_\sigma \otimes C_\sigma$$

is a cofibration. A basis of the target in degree $n$ is indexed by the union $\bigcup_{p+q=n} S_{\sigma,p} \times S_{\sigma,q}$ where we now have to take care of the different degrees in the notation. A basis for the source is the subset given by the union of all pairs coming from $S_{\tau,p} \times S_{\tau,q}$ for $\tau \subsetneq \sigma$. Since the map is induced by the free functor it is split injective.

For part (ii), first observe that the fact that $C^t \otimes_R C$ and $D$ are well-behaved implies that they are Reedy cofibrant [8, definition 15.3.3(2)]. The colim that defines $(C^t \otimes_R C)_{\partial\sigma}$ is a hocolim by [8, theorem 19.9.1(1) and proposition 15.10.2(2)], and similarly for $D_{\partial\sigma}$, so the map

$$(\beta_\sigma)_* : H_*((C^t \otimes_R C)_{\partial\sigma}) \to H_*(D_{\partial\sigma})$$

is an isomorphism by [8, theorem 19.4.2(1)], and this implies the lemma. □

Recall [11, example 3.12 ].

Definition 12.8. *A balanced pre K-ad F is closed if, for each $\sigma$, the map from the cellular chains* $\mathrm{cl}(\sigma)$ *to $D_\sigma$ which takes $\langle\tau, o\rangle$ to $\varphi_{\tau,o}$ is a chain map.*

Note that if $F$ is closed then $\varphi_{\sigma,o}$ represents an element $[\varphi_{\sigma,o}] \in H_*(D_\sigma/D_{\partial\sigma})$.

Notation 12.9. For a balanced pre $K$-ad $F$ and a cell $\sigma$ of $K$, let

$$j_\sigma : (C^t \otimes_R C)_\sigma/(C^t \otimes_R C)_{\partial\sigma} \to (C_\sigma/C_{\partial\sigma})^t \otimes_R C_\sigma$$

be the map that takes $[c \otimes c']$ to $[c] \otimes c'$.

The next definition is a little more complicated than the corresponding definition in [11, §9] in order to satisfy the extra condition in definition 3.3.

Definition 12.10. *(i) A balanced K-ad is a pre K-ad F with the following properties:*

*(a) it is balanced, well-behaved, and closed, and*

*(b) for each $\sigma$ the slant product with $(j_\sigma)_*(\beta_\sigma)_*^{-1}([\varphi_{\sigma,o}])$ is an isomorphism*

$$H^*(\mathrm{Hom}_R(C_\sigma, R)) \to H_{\dim\sigma - \deg F - *}(C_\sigma/C_{\partial\sigma}).$$

*(ii)A K-ad is a pre K-ad which is naturally quasi-isomorphic to a balanced K-ad.*

*We mention that, by the definition of an ad theory, a (K, L)-ad is just a (K, L)-pread which is a K-ad.*

Write $\mathrm{ad}^R_{\mathrm{rel}}$ for the set of $K$ ads with values in $\mathcal{A}^R_{\mathrm{rel}}$.

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REMARK 12.11. The morphism of remark 12.5 takes ads to ads.

THEOREM 12.12 $\mathrm{ad}^R_{\mathrm{rel}}$ *is an ad theory. Moreover, when $R$ is commutative with the trivial involution,* $\mathrm{ad}^R_{rel}$ *is a commutative ad theory.*

For the proof of theorem 12.12, we need a product operation.

DEFINITION 12.13. *(i) For $i = 1, 2$, let $R_i$ be a ring with involution and let $(C^i, D^i, \beta^i, \varphi^i)$ be an object of $\mathcal{A}^R_{rel}$. Define*

$$(C^1, D^1, \beta^1, \varphi^1) \otimes (C^2, D^2, \beta^2, \varphi^2)$$

*to be the following object of $\mathcal{A}^{R_1 \otimes R_2}_{rel}$:*

$$(C^1 \otimes C^2, D^1 \otimes D^2, \gamma, \varphi^1 \otimes \varphi^2),$$

*where $\gamma$ is the composite*

$$(C^1 \otimes C^2)^t \otimes_{R_1 \otimes R_2} (C^1 \otimes C^2) \cong ((C^1)^t \otimes_{R_1} C^1) \otimes ((C^2)^t \otimes_{R_2} C^2) \xrightarrow{\beta^1 \otimes \beta^2} D^1 \otimes D^2.$$

*(ii) For $i = 1, 2$, suppose given a ball complex $K_i$ and a pre $K_i$-ad $F_i$ of degree $k_i$ with values in $\mathcal{A}^{R_i}_{rel}$. Define a pre $(K_1 \times K_2)$-ad $F_1 \otimes F_2$ with values in $\mathcal{A}^{R_1 \otimes R_2}_{rel}$ by*

$$(F_1 \otimes F_2)(\sigma \times \tau, o_1 \times o_2) = i^{k_2 \dim \sigma} F_1(\sigma, o_1) \otimes F_2(\tau, o_2).$$

LEMMA 12.14. *For $i = 1, 2$, suppose given a ball complex $K_i$ and a $K_i$-ad $F_i$ with values in $\mathcal{A}^{R_i}_{\mathrm{rel}}$. Then $F_1 \otimes F_2$ is a $(K_1 \times K_2)$-ad.*

*Proof of 12.12.* We only need to verify parts (d), (f), and (g) of [11, definition 3.10].

For part (d), we have to show that a pre $K$-ad is a $K$-ad if it restricts to a $\sigma$-ad for each closed cell $\sigma$ of $K$. It suffices to consider the case of a $K$-pread $F$ with $K = L \sqcup_{\partial\sigma} \sigma$ whose restriction to $L$ is naturally quasi-isomorphic to a balanced $L$-ad $G$ and to a balanced $\sigma$-ad $H$ on $\sigma$. We have to give a quasi-isomorphism of $F$ to a balanced $K$-ad $I$. In the following, we will concentrate on the first complex in the datum of a pread and we will use the same letter for this complex. The second complex and all other entries will be clear then. Define the restriction of $I$ to $L$ be $G$. It remains to define $I_\sigma$, a map to $F_\sigma$ and the maps from its lower dimensional cells. For each $\tau \subset \partial\sigma$, we are given a quasi-isomorphism $g_\tau$ from $G_\tau$ to $F_\tau$ and an $h_\tau$ from $H_\tau$ to $F_\tau$. Since $G_\tau$ and $H_\tau$ are cofibrant we find a map $f_\tau : G_\tau \to H_\tau$ such that $h_\tau f_\tau$ is homotopic to $g_\tau$. This only uses the fact that isomorphisms in the homotopy category between cofibrant objects can be represented by chain maps up to homotopy. Similarly, using the fact that the restriction of $G$ to $\partial\sigma$ is balanced we find a map $f_{\partial\sigma} : G_{\partial\sigma} \to H_{\partial\sigma}$ whose restriction to $\tau \subset \partial\sigma$ is homotopic to $H_{\tau\subset\partial\sigma} f_\tau$. We also find a system of compatible homotopies, that is, a homotopy

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between $h_{\partial\sigma} f_{\partial\sigma}$ and $g_{\partial\sigma}$. Let $I_\sigma$ be the mapping cylinder of

$$H_{\partial\sigma\subset\sigma} f_{\partial\sigma} : G_{\partial\sigma} \longrightarrow H_\sigma.$$

Then the constructed homotopy and the map $h_\sigma$ complete the required quasi-isomorphism $I_\sigma \to F_\sigma$. The other maps are obvious.

For part (f), let $F$ be a $K^{'}$-ad. We may assume it is balanced. Let

$$F(\sigma, o) = (C_\sigma, D_\sigma, \beta_\sigma, \varphi_{\sigma,o}).$$

We need to define a $K$-ad $E$ which agrees with $F$ on each residual subcomplex of $K$. As in the proof of [11, theorem 6.5], we may assume by induction that $K$ is a ball complex structure for the $n$ disk with one $n$ cell $\tau$, and that $K^{'}$ is a subdivision of $K$ which agrees with $K$ on the boundary. We only need to define $E$ on the top cell $\tau$ of $K$. We define $E(\tau, o)$ to be $(C_\tau, D_\tau, \beta_\tau, \varphi_{\tau,o})$, where

- $C_\tau = \mathrm{colim}_{\sigma\in K'} C_\sigma$,
- $D_\tau = \mathrm{colim}_{\sigma\in K'} D_\sigma$,
- $\beta = \mathrm{colim}_{\sigma\in K'} \beta_\sigma$, and
- $\varphi_{\tau,o} = \sum \varphi_{\sigma,o'}$, where $(\sigma, o')$ runs through the $n$-dimensional cells of $K^{'}$ with orientations induced by $o$.

The fact that $E$ satisfies part (a) of definition 12.10 is a consequence of [11, proposition A.1(ii)]. We will deduce the isomorphism in part (b) of definition 12.10 from [11, proposition 12.4], and for this we need some facts from [11, §12].

First recall that for a well-behaved functor

$$B : \mathcal{C}ell^\flat(K') \to \mathcal{C}h_{hf},$$

we write

$$\mathrm{Nat}(\mathrm{cl}, B)$$

for the chain complex of natural transformations of graded abelian groups; the differential is given by

$$\partial(\nu) = \partial \circ \nu - (-1)^{|\nu|} \nu \circ \partial.$$

Recall [11, definition 12.3] and also the map $\Phi$ defined just before the statement of [11, lemma 12.6]. Consider the diagram

$$\begin{array}{ccc} H_*(\mathrm{Nat}(\mathrm{cl}, D)) & \xrightarrow{\Phi} & H_{*+n}(D_\tau, D_{\partial\tau}) \\ \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\beta} \\ H_*(\mathrm{Nat}(\mathrm{cl}, C^t \otimes_R C)) & \xrightarrow{\Phi} & H_{*+n}((C^t \otimes_R C)_\tau, (C^t \otimes_R C)_{\partial\tau}). \end{array}$$

The horizontal maps are isomorphisms by [11, lemma 12.6], and the right-hand vertical map is an isomorphism by the proof of lemma 12.7(ii). Hence the left-hand vertical map is an isomorphism.

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The collection $\{\varphi_{\sigma,o}\}$ gives a cycle $\nu$ in $\mathrm{Nat}(\mathrm{cl}, D)$. Let $\mu \in \mathrm{Nat}(\mathrm{cl}, C^t \otimes_R C)$ be a representative for $\beta^{-1}([\nu])$. Now fix an orientation $o$ for $\tau$. Let $\psi \in C^t_\tau \otimes_R C_\tau$ be $\sum \mu(\langle \sigma, o' \rangle)$, where $(\sigma, o')$ runs through the $n$-dimensional cells of $K'$ with orientations induced by $o$. Then $\psi$ is a representative of $(j_\sigma)_* \beta^{-1}([\varphi_{\tau,o}])$, so it suffices to show that the cap product with $\psi$ is an isomorphism $H^*(\mathrm{Hom}_R(C_\tau, R)) \to H_{n-\deg F-*}(C_\tau / C_{\partial\tau})$, and this follows from [11, proposition 12.4].

It remains to verify part (g) of [11, definition 3.10]. Let $0, 1, \iota$ denote the three cells of the unit interval $I$, with their standard orientations. As in the proof of [11, theorem 9.11], it suffices to construct a relaxed symmetric Poincaré $I$-ad $H$ over $\mathbb{Z}$ which takes both 0 and 1 to the object $(\mathbb{Z}, \mathbb{Z}, \gamma, 1)$, where $\gamma$ is the isomorphism $\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$. The proof of [11, theorem 9.11] gives a symmetric Poincaré $I$-ad $G$ with $G(0) = G(1) = (\mathbb{Z}, \epsilon)$, where $\epsilon : W \to \mathbb{Z} \otimes \mathbb{Z}$ is the composite of the augmentation with $\gamma^{-1}$. Let us denote the object $G(\iota)$ by $(C, \varphi)$. Applying remark 12.11 to $G$ gives a relaxed symmetric Poincaré $I$-ad $G'$ with $G'(\iota) = (C, (C \otimes C)^W, \beta, \varphi)$, where $\beta$ is induced by the augmentation. Let $e_0$ (resp., $e_1$) be the inclusion $0 \hookrightarrow \iota$ (resp., $1 \hookrightarrow \iota$) and for $i = 0, 1$ let $g_i = G(e_i) : \mathbb{Z} \to C$. Then

$$\partial\varphi = (g_1 \otimes g_1) \circ \epsilon - (g_0 \otimes g_0) \circ \epsilon$$

because $G$ is closed. We can therefore construct the required $I$-ad $H$ from $G'$ by replacing $G'(0)$ and $G'(1)$ by $(\mathbb{Z}, \mathbb{Z}, \gamma, 1)$. □

## 13. Equivalence of the spectra associated with $\mathrm{ad}^R$ and $\mathrm{ad}^R_{\mathbf{rel}}$

By remark 12.11, the morphism

$$\mathcal{A}^R \to \mathcal{A}^R_{\mathrm{rel}}$$

of remark 12.5 induces a map of spectra

$$\mathbf{Q}^R \to \mathbf{Q}^R_{\mathrm{rel}}$$

(see [11, §15]) and a map of symmetric spectra

$$\mathbf{M}^R \to \mathbf{M}^R_{\mathrm{rel}}$$

(see [11, §17]).

THEOREM 13.1 *The maps*

$$\mathbf{Q}^R \to \mathbf{Q}^R_{\mathrm{rel}}$$

*and*

$$\mathbf{M}^R \to \mathbf{M}^R_{\mathrm{rel}}$$

*are weak equivalences.*

REMARK 13.2. The method that will be used to prove theorem 1.3 can be used to show that $\mathbf{M}^R \to \mathbf{M}^R_{\mathrm{rel}}$ is weakly equivalent to a map of symmetric ring spectra.

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Recall [11, definitions 4.1 and 4.2]. By [11, theorem 16.1, remark 14.2(i), and corollary 17.9(iii)], theorem 13.1 follows from

Proposition 13.3. *The map of bordism groups*

$$\Omega_*^R \to (\Omega_{\mathrm{rel}}^R)_*$$

*is an isomorphism.*

The proof of proposition 13.3 will occupy the rest of this section. The following lemma proves surjectivity. As usual, for a chain complex $A$ with a $\mathbb{Z}/2$ action, we write $A^{h\mathbb{Z}/2}$ for $(A^W)^{\mathbb{Z}/2}$. The augmentation induces a map $A^{\mathbb{Z}/2} \to A^{h\mathbb{Z}/2}$.

Lemma 13.4. *Let*

$$(C, D, \beta, \varphi)$$

*be a relaxed symmetric Poincaré $*$-ad and let*

$$\psi \in (C^t \otimes_R C)^{h\mathbb{Z}/2}$$

*represent the image of $\varphi$ under the map*

$$H_*(D^{\mathbb{Z}/2}) \to H_*(D^{h\mathbb{Z}/2}) \xleftarrow{\cong} H_*((C^t \otimes_R C)^{h\mathbb{Z}/2})$$

*(where the isomorphism is induced by $\beta$). Then $(C, \psi)$ is a symmetric Poincaré $*$-ad, and $(C, D, \beta, \varphi)$ is bordant to*

$$(C, (C^t \otimes_R C)^W, \gamma, \psi),$$

*where $\gamma$ is induced by the augmentation.*

For the proof, we need another lemma.

Lemma 13.5. *Let $(C, D, \beta, \varphi)$ be a relaxed symmetric Poincaré $*$-ad.*

*(i) If $\psi \in D^{\mathbb{Z}/2}$ is any representative for the homology class $[\varphi] \in H_*(D^{\mathbb{Z}/2})$ then $(C, D, \beta, \psi)$ is bordant to $(C, D, \beta, \varphi)$.*

*(ii) If*

$$(f, g) : (C, D, \beta, \varphi) \to (C', D', \beta', \varphi')$$

*is a map of $*$-ads of the same dimension for which $f$ (and hence also $g$) is a quasi-isomorphism then $(C, D, \beta, \varphi)$ and $(C', D', \beta', \varphi')$ are bordant.*

The proof of lemma 13.5 is deferred to the end of the section.

*Proof of lemma 13.4.* The fact that $(C, \psi)$ satisfies [11, definition 9.9] (only part (b) is relevant) is immediate from definition 12.10(b).

To see that $(C, D, \beta, \varphi)$ and $(C, (C^t \otimes_R C)^W, \gamma, \psi)$ are bordant, let $\delta$ denote the composite

$$C^t \otimes_R C \xrightarrow{\beta} D \to D^W,$$

let $\omega \in D^{h\mathbb{Z}/2}$ be the image of $\varphi$, and let $\omega' \in D^{h\mathbb{Z}/2}$ be the image of $\psi$ under the map $(C \otimes C)^{h\mathbb{Z}/2} \to D^{h\mathbb{Z}/2}$ induced by $\beta$. Part (ii) of lemma 13.5 shows that $(C, D, \beta, \varphi)$ and $(C, D^W, \delta, \omega)$ are bordant, and also that $(C, (C^t \otimes_R C)^W, \gamma, \psi)$ and $(C, D^W, \delta, \omega')$ are bordant. But $[\omega] = [\omega']$ in $H_*(D^{h\mathbb{Z}/2})$, so the result follows from part (i) of lemma 13.5. □

Next we show that the map in proposition 13.3 is 1-1. So let $(C_0, \varphi_0)$ and $(C_1, \varphi_1)$ be symmetric Poincaré $*$-ads and let $F$ be a relaxed symmetric Poincaré bordism between them. Let $0, 1, \iota$ denote the three cells of the unit interval $I$, with their standard orientations. Denote the object $F(\iota)$ by $(C, D, \beta, \varphi)$. It suffices to show that there is a symmetric Poincaré $I$-ad $G$ with

$$G(0) = (C_0, \varphi_0), \quad G(1) = (C_1, \varphi_1), \quad G(\iota) = (C, \chi) \tag{13.1}$$

for an element $\chi$ which we will now construct.

$\varphi$ represents an element

$$[\varphi] \in H_*(D^{\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}).$$

The map $D \to D^W$ induced by the augmentation gives a map

$$\begin{aligned} &H_*(D^{\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}) \\ &\qquad \to H_*(D^{h\mathbb{Z}/2}, ((C_0^t \otimes_R C_0)^W)^{h\mathbb{Z}/2} \oplus ((C_1^t \otimes_R C_1)^W)^{h\mathbb{Z}/2}); \end{aligned}$$

let $x$ be the image of $\varphi$ under this map. The map $\beta : C^t \otimes_R C \to D$ gives an isomorphism

$$\begin{aligned} (\beta^{h\mathbb{Z}/2})_* : &H_*((C^t \otimes_R C)^{h\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}) \\ &\qquad \to H_*(D^{h\mathbb{Z}/2}, ((C_0^t \otimes_R C_0)^W)^{h\mathbb{Z}/2} \oplus ((C_1^t \otimes_R C_1)^W)^{h\mathbb{Z}/2}); \end{aligned}$$

let $y = (\beta^{h\mathbb{Z}/2})_*^{-1}(x)$.

LEMMA 13.6. *The image of $y$ under the boundary map*

$$\begin{aligned} &H_*((C^t \otimes_R C)^{h\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}) \\ &\qquad \xrightarrow{\partial} H_{*-1}((C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}) \end{aligned}$$

*is* $-[\varphi_0] + [\varphi_1]$.

Before proving this we conclude the proof of proposition 13.3. The lemma implies that there is a representative $\chi$ of $y$ with

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$$\partial\chi = -\varphi_0 + \varphi_1. \tag{13.2}$$

It suffices to show that, with this choice of $\chi$, the symmetric Poincaré pre $I$-ad $G$ given by Eq. (13.1) is an ad. Equation (13.2) says that $G$ is closed, and part (b) of [11, definition 9.9] follows from definition 12.10(b) and the fact that the image of $[\chi]$ under the map

$$\begin{aligned} &H_*((C^t \otimes_R C)^{h\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}) \\ &\quad \to H_*((C^t \otimes_R C)^W, (C_0^t \otimes_R C_0)^W \oplus (C_1^t \otimes_R C_1)^W) \\ &\quad \xleftarrow{\cong} H_*(C^t \otimes_R C, (C_0^t \otimes_R C_0) \oplus (C_1^t \otimes_R C_1)) \end{aligned}$$

is the same as the image of $[\varphi]$ under the map

$$\begin{aligned} &H_*(D^{\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}) \\ &\quad \to H_*(D, (C_0^t \otimes_R C_0)^W \oplus (C_1^t \otimes_R C_1)^W) \\ &\quad \xrightarrow{\beta_*^{-1}} H_*(C^t \otimes_R C, (C_0^t \otimes_R C_0) \oplus (C_1^t \otimes_R C_1)). \end{aligned}$$

*Proof of lemma 13.6.* We know that the image of $[\varphi]$ under the boundary map

$$\begin{aligned} &H_*(D^{\mathbb{Z}/2}, (C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}) \\ &\quad \xrightarrow{\partial} H_{*-1}((C_0^t \otimes_R C_0)^{h\mathbb{Z}/2} \oplus (C_1^t \otimes_R C_1)^{h\mathbb{Z}/2}) \end{aligned}$$

is $-[\varphi_0] + [\varphi_1]$, so it suffices to show that for $i = 0, 1$ the maps

$$(C_i^t \otimes_R C_i)^{h\mathbb{Z}/2} \to ((C_i^t \otimes_R C_i)^W)^{h\mathbb{Z}/2}$$

induced by $D \to D^W$ and by $\beta$ give the same map in homology. If we think of these as maps

$$a_i, b_i : ((C_i^t \otimes_R C_i)^W)^{\mathbb{Z}/2} \to ((C_i^t \otimes_R C_i)^{W\otimes W})^{\mathbb{Z}/2}$$

(with diagonal $\mathbb{Z}/2$ action on $W \otimes W$) then $a_i$ and $b_i$ are induced by the maps

$$e_1, e_2 : W \otimes W \to W$$

given by the augmentations on the two factors. Now the $\mathbb{Z}/2$ equivariant map

$$\Delta : W \to W \otimes W$$

of [20, p. 175] has the property that $e_1 \circ \Delta$ and $e_2 \circ \Delta$ are both the identity map, so if

$$d : ((C_i^t \otimes_R C_i)^{W\otimes W})^{\mathbb{Z}/2} \to ((C_i^t \otimes_R C_i)^W)^{\mathbb{Z}/2}$$

is the map induced by $\Delta$ then $d \circ a_i$ and $d \circ b_i$ are both the identity map. But $\Delta$ is a $\mathbb{Z}/2$ chain homotopy equivalence, so $d$ is a homology isomorphism and it follows that $a_i$ and $b_i$ induce the same map in homology as required. □

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It remains to prove lemma 13.5. Let $F$ be the cylinder of $(C, D, \beta, \varphi)$ (which was constructed in the last paragraph of the proof of theorem 12.12). Then $F(0)$ and $F(1)$ are both $(C, D, \beta, \varphi)$. Write

$$F(\iota) = (C_\iota, D_\iota, \beta_\iota, \varphi_\iota)$$

and let

$$(h, k) : (C, D, \beta, \varphi) \to (C_\iota, D_\iota, \beta_\iota, \varphi_\iota)$$

be the map $F(1) \to F(\iota)$.

For part (i), the hypothesis gives an element $\rho \in D^{\mathbb{Z}/2}$ with $\partial\rho = \psi - \varphi$. Let $\rho' \in D_\iota^{\mathbb{Z}/2}$ be the image of $\rho$ under $k : D \to D_\iota$. Define an $I$-ad $G$ by

$$G(0) = (C, D, \beta, \varphi), \quad G(1) = (C, D, \beta, \psi), \quad G(\iota) = (C_\iota, D_\iota, \beta_\iota, \varphi_\iota + \rho').$$

Then $G$ is the desired bordism.

For part (ii), we first show that $(C, D, \beta, \varphi)$ is bordant to $(C, D', \beta_1, \varphi')$, where $\beta_1$ is the composite

$$C^t \otimes_R C \xrightarrow{\beta} D \xrightarrow{g} D'$$

The idea is to construct a suitable mapping cylinder. Let $D_1$ be the pushout of the diagram

$$\begin{array}{ccc} D & \xrightarrow{k} & D_\iota \\ {\scriptstyle g}\downarrow & & \\ D' & & \end{array}$$

Let $\varphi_1$ be the image of $\varphi_\iota$ in $D_1$ and let $\beta_2$ be the composite

$$C_\iota^t \otimes_R C_\iota \to D_\iota \to D_1.$$

Define an $I$-ad $H$ by

$$H(0) = (C, D, \beta, \varphi), \quad H(1) = (C, D', \beta_1, \varphi'), \quad H(\iota) = (C_\iota, D_1, \beta_2, \varphi_1).$$

Then $H$ is the desired bordism.

To conclude the proof we show that $(C, D', \beta_1, \varphi')$ is bordant to $(C', D', \beta', \varphi')$. Let $C_1$ be the pushout of the diagram

$$\begin{array}{ccc} C & \xrightarrow{h} & C_\iota \\ {\scriptstyle f}\downarrow & & \\ C' & & \end{array}$$

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Let $(C'_\iota, D'_\iota, \beta'_\iota, \varphi'_\iota)$ be the cylinder of $(C', D', \beta', \varphi')$. Let $\beta_3$ be the map

$$C_1^t \otimes_R C_1 \to D'_\iota.$$

Define an I-ad $H_1$ by

$$H_1(0) = (C, D', \beta_1, \varphi'), \quad H_1(1) = (C', D', \beta', \varphi'), \quad H_1(\iota) = (C_1, D'_\iota, \beta_3, \varphi'_\iota).$$

Then $H_1$ is the desired bordism.

## 14. The symmetric signature revisited

Fix a group $\pi$, a simply connected free $\pi$-space $Z$, and a homomorphism $w : \pi \to \{\pm 1\}$, and recall the symmetric spectrum $\mathbf{M}_{\pi,Z,w}$ [11, §7 and 17] which represents $w$-twisted Poincaré bordism over $Z/\pi$.

Let $R$ denote the group ring $\mathbb{Z}[\pi]$ with the $w$-twisted involution [21, p. 196].

We now restrict our attention to strict ball complexes. Recall from §2 that a ball complex is *strict* if each component of the intersection of two cells is a single cell. In [11, §10], we gave a functor

$$\mathrm{sig} : \mathcal{A}_{\pi,Z,w} \to \mathcal{A}^R$$

which induces a natural transformation

$$\mathrm{sig} : \mathrm{ad}_{\pi,Z,w}(K) \to \mathrm{ad}^R(K)$$

for strict ball complexes $K$.

Remark 14.1. (i) The restriction to strict ball complexes was not mentioned in [11] but is necessary, because if $K$ has a cell $\tau$ whose boundary is not strict and if $F$ is a $K$-ad $(X_\sigma, f_\sigma, \xi_{\sigma,o})$ then the map

$$\mathrm{colim}_{\sigma\subset\partial\tau} S_*(X_\sigma) \to S_*(X_{\partial\tau}) \tag{14.1}$$

is not a monomorphism (because simplices with support in $\sigma \cap \sigma'$ but not in a cell of $\sigma \cap \sigma'$ will have two representatives in the colimit), and hence $\mathrm{sig} \circ F$ is not well-behaved.

(ii) If $K$ is strict then the map (14.1) has a left inverse for all $\tau$ (because its image is the subcomplex of $S_*(X_\tau)$ generated by the simplices that land in some $X_\sigma$ with $\sigma \subset \partial\tau$, and the left inverse takes each such simplex to a representative for it in the colimit system; all such representatives are identified because $K$ is strict). Hence the map (14.1) is the inclusion of a direct summand, as required for $\mathrm{sig} \circ F$ to be well-behaved.

(iii) The restriction to strict ball complexes does not affect the results about the symmetric signature in [11] because the only ball complexes that occur in [11, §15, 17–19] are products of simplices, and these are strict.

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In this section, we give a functor

$$\mathrm{sig}_{\mathrm{rel}} : \mathcal{A}_{\pi,Z,w} \to \mathcal{A}^R_{\mathrm{rel}}$$

which induces a natural transformation

$$\mathrm{sig}_{\mathrm{rel}} : \mathrm{ad}_{\pi,Z,w}(K) \to \mathrm{ad}^R_{\mathrm{rel}}(K)$$

for strict ball complexes $K$.

Let $(X, f, \xi)$ be an object of $\mathcal{A}_{\pi,Z,w}$ [11, definition 7.3].

In the special case where $\pi$ is the trivial group and $Z$ is a point, the definition is easy:

$$\mathrm{sig}_{\mathrm{rel}}(X, f, \xi) = (S_*X, S_*(X \times X), \beta, \varphi),$$

where $\beta$ is the cross product $S_*X \otimes S_*X \xrightarrow{\times} S_*(X \times X)$ and $\varphi$ is the image of $\xi$ under the diagonal map.

The definition in the general case is similar. Recall that we write $\widetilde{X}$ for the pullback of $Z$ along $f$ and $\mathbb{Z}^w$ for $\mathbb{Z}$ with the right $R$ action determined by $w$. Also recall [11, definition 7.1].

DEFINITION 14.2. *(i) Give $S_*(\widetilde{X})$ the left $R$ module structure determined by the action of $\pi$ on $\widetilde{X}$ and give $S_*(\widetilde{X} \times \widetilde{X})$ and $S_*(\widetilde{X}) \otimes S_*(\widetilde{X})$ the left $R$ module structures determined by the diagonal actions of $\pi$.*

*(ii) Define*

$$\mathrm{sig}_{\mathrm{rel}}(X, f, \xi) = (S_*(\widetilde{X}), \mathbb{Z}^w \otimes_R S_*(\widetilde{X} \times \widetilde{X}), \beta, \varphi),$$

*where $\beta$ is the composite*

$$S_*(\widetilde{X})^t \otimes_R S_*(\widetilde{X}) \cong \mathbb{Z}^w \otimes_R (S_*(\widetilde{X}) \otimes S_*(\widetilde{X})) \xrightarrow{1\otimes\times} \mathbb{Z}^w \otimes_R S_*(\widetilde{X} \times \widetilde{X})$$

*and $\varphi$ is the image of $\xi$ under the map*

$$S_*(X, \mathbb{Z}^f) = \mathbb{Z}^w \otimes_R S_*(\widetilde{X}) \to \mathbb{Z}^w \otimes_R S_*(\widetilde{X} \times \widetilde{X})$$

*(where the unmarked arrow is induced by the diagonal map).*

REMARK 14.3. (i) For set-theoretic reasons one should modify this definition as in [11, §10]; we leave this to the reader.

(ii) For strict ball complexes, $\mathrm{sig}_{\mathrm{rel}}$ takes ads to ads, because the composite of the cross product with the Alexander–Whitney map is naturally chain homotopic to the map induced by the diagonal.

Next we compare sig to $\mathrm{sig}_{\mathrm{rel}}$.

Let us denote by $\delta$ both the map $\mathcal{A}^R \to \mathcal{A}^R_{\mathrm{rel}}$ of remark 12.5 and the map $\mathbf{M}^R \to \mathbf{M}^R_{\mathrm{rel}}$ which it induces.

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Proposition 14.4. *The diagram*

$$\begin{array}{ccccc} & & \mathbf{M}_{\pi,Z,w} & & \\ & \swarrow^{\text{sig}} & & \searrow^{\text{sig}_{\text{rel}}} & \\ \mathbf{M}^R & & \xrightarrow{\quad\delta\quad} & & \mathbf{M}^R_{\text{rel}} \end{array}$$

*commutes in the homotopy category of symmetric spectra.*

We will derive this from a more general result. Recall definition 12.4.

Proposition 14.5. *Let $\Theta$ be an ad theory, let $R$ be a ring with involution, and let $S_1$ and $S_2$ be morphisms of ad theories $\Theta \to \text{ad}^R_{\text{rel}}$. Suppose that there is a natural quasi-isomorphism $\nu : S_1 \to S_2$. Then the maps $\mathbf{M}_\Theta \to \mathbf{M}^R_{\text{rel}}$ induced by $S_1$ and $S_2$ are homotopic.*

*Proof of proposition 14.4.* The extended Eilenberg–Zilber map

$$W \otimes S_*(Y \times Z) \to S_*(Y) \otimes S_*(Z)$$

[6, proof of proposition 5.8] gives a map

$$S_*(Y \times Z) \to ((S_*(Y) \otimes S_*(Z))^W,$$

and this gives a natural quasi-isomorphism

$$\nu : \text{sig}_{\text{rel}} \to \delta \circ \text{sig}.$$

□

The rest of this section is devoted to the proof of proposition 14.5. The basic idea is similar to the proof of theorem 1.1.

Definition 14.6. *Let $P$ be the poset whose two elements are the functors $S_1$ and $S_2$, with $S_1 \le S_2$.*

Recall definition 6.2(i) and note that $U(K)$ has a poset structure given by inclusions of cells. Our next definition is analogous to definition 6.2(ii).

Definition 14.7. *Let $k \ge 0$, let $\mathbf{n}$ be a $k$-fold multi-index, and let $F \in \text{pre}^k_\Theta(\Delta^{\mathbf{n}})$. Let*

$$b : U(\Delta^{\mathbf{n}}) \to P$$

*be a map of posets.*

*(i) For an object $(\sigma, o)$ of $\mathcal{C}ell(\Delta^{\mathbf{n}})$ define the object $b_*(F)(\sigma, o)$ of $\mathcal{A}^R_{rel}$ to be $b(\sigma)(F(\sigma, o))$.*

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*(ii) For a morphism $f : (\sigma, o) \to (\sigma', o')$ of $\mathcal{C}ell(\Delta^{\mathbf{n}})$ define the morphism*

$$b_*(F)(f) : b_*(F)(\sigma, o) \to b_*(F)(\sigma', o')$$

*to be*

$$\begin{cases} S_1(f) & \textit{if } b(\sigma', o') = S_1, \\ S_2(f) & \textit{if } b(\sigma, o) = S_2, \\ \nu \circ S_1(f) & \textit{otherwise.} \end{cases}$$

LEMMA 14.8. *$b_*$ takes ads to ads.*

*Proof.* Suppose $F$ is a $K$-ad in $\Theta$. Then $\nu$ provides a quasi-isomorphism $\kappa$ from $S_1(F)$ to $b_*(F)$: let $\kappa(F)_{(\sigma,o)}$ be the identity if $b(\sigma, o) = S_1$ and let it coincide with $\nu(F)_{(\sigma,o)}$ else. Since $S_1(F)$ is quasi-isomorphic to a balanced $K$-ad the same is true for $b_*(F)$. This implies that $b_*(F)$ is a $K$-ad. □

Recall that we write $\mathbf{R}$ for the object of $\mathcal{S}p_{mss}$ associated with an ad theory (example 4.13). Then $b_*$ gives a map

$$((\mathbf{R}_\Theta)_k)_{\mathbf{n}} \to ((\mathbf{R}^R_{\mathrm{rel}})_k)_{\mathbf{n}}.$$

DEFINITION 14.9. *(i) For $k \geq 0$ define an object $\mathcal{P}_k$ of $\Sigma_k ss\mathcal{S}_k$ by*

$$(\mathcal{P}_k)_{\mathbf{n}} = \mathrm{Map}_{posets}(U(\Delta^{\mathbf{n}}), P)_+$$

*(where the $+$ denotes a disjoint basepoint); the morphisms in $(\Delta^{op}_{\mathrm{inj}})^{\times k}$ act in the evident way, and the morphisms of the form $(\alpha, \mathrm{id})$ with $\alpha \in \Sigma_k$ act by permuting the factors in $\Delta^{\mathbf{n}}$.*

*(ii) Define $\mathcal{P}$ to be the object of $\Sigma ss\mathcal{S}$ with k-th term $\mathcal{P}_k$.*

Next we give $\mathcal{P} \,\overline{\wedge}\, \mathbf{R}_\Theta$ the structure of a multisemisimplicial symmetric spectrum (cf. definition 8.1). Recall definition 6.4 and let $s$ be the 1-simplex of $S^1$. Define

$$\omega : S^1 \wedge (\mathcal{P} \,\overline{\wedge}\, \mathbf{R}_\Theta)_k \to (\mathcal{P} \,\overline{\wedge}\, \mathbf{R}_\Theta)$$

as follows: for $b \in (\mathcal{P}_k)_{\mathbf{n}}$ and $x \in ((\mathbf{R}_\Theta)_k)_{\mathbf{n}}$, let

$$\omega(s \wedge (b \wedge x)) = (b \circ \Pi) \wedge \omega(s \wedge x).$$

It follows from the definitions that we obtain a map

$$\beta : \mathcal{P} \,\overline{\wedge}\, \mathbf{R}_\Theta \to \mathbf{R}^R_{\mathrm{rel}}$$

in $\mathcal{S}p_{mss}$ by

$$\beta(b \wedge F) = b_*(F).$$

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For each $k$ and $\mathbf{n}$, define elements $c_{k,\mathbf{n}}, d_{k,\mathbf{n}} \in (\mathcal{P}_k)_{\mathbf{n}}$ to be the constant functions $U(\Delta^{\mathbf{n}}) \to P$ whose values are respectively $S_1$ and $S_2$. Then define maps

$$c, d : \overline{\mathbf{S}} \to \mathcal{P}$$

in $\Sigma ss\mathcal{S}$ by taking the non-trivial simplex of $(\overline{\mathbf{S}}_k)_{\mathbf{n}}$ to $c_{k,\mathbf{n}}$, resp., $d_{k,\mathbf{n}}$. Finally, define maps

$$\mathbf{c}, \mathbf{d} : \mathbf{R}_\Theta \to \mathcal{P} \,\overline{\wedge}\, \mathbf{R}_\Theta$$

in $\mathcal{S}p_{mss}$ by letting $\mathbf{c}$ be the composite

$$\mathbf{R}_\Theta \cong \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{R}_\Theta \xrightarrow{c\wedge 1} \mathcal{P} \,\overline{\wedge}\, \mathbf{R}_\Theta$$

and similarly for $\mathbf{d}$.

Now $\beta \circ \mathbf{c}$ is the map $S_1$ and $\beta \circ \mathbf{d}$ is the map $S_2$, so to complete the proof of proposition 14.5 it suffices to show:

Lemma 14.10. *$\mathbf{c}$ and $\mathbf{d}$ are homotopic in $\mathcal{S}p_{mss}$.*

*Proof of lemma 14.10.* For each $k \geq 0$ and each $\mathbf{n}$ let $e_{k,\mathbf{n}} : (\mathcal{P}_k)_{\mathbf{n}} \to S^0$ be the map which takes every simplex except the basepoint to the non-trivial element of $S^0$, and let

$$e : \mathcal{P} \to \overline{\mathbf{S}}$$

be the map given by the $e_{k,\mathbf{n}}$. Let

$$\mathbf{e} : \mathcal{P} \,\overline{\wedge}\, \mathbf{R}_\Theta \to \mathbf{R}_\Theta$$

be the composite

$$\mathcal{P} \,\overline{\wedge}\, \mathbf{R}_\Theta \xrightarrow{e\wedge 1} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{R}_\Theta \cong \mathbf{R}_\Theta.$$

Then $\mathbf{e} \circ \mathbf{c}$ and $\mathbf{e} \circ \mathbf{d}$ are both equal to the identity. But $\mathbf{e}$ is a weak equivalence by proposition 9.3 and lemma 10.3, and the result follows. □

## 15. Background for the proof of theorem 1.3

Notation 15.1. In order to distinguish the product in $\mathcal{A}_{e,*,1}$ from the Cartesian product of categories, we will denote the former by $\boxtimes$ from now on.

We now turn to the proof of theorem 1.3, which will follow the general outline of the proof of theorem 1.1. The key ingredient in that proof was the action of the monad $\mathbb{P}$ on $\mathbf{R}$. That action was constructed from the family of operations given in definition 6.2(ii), and this family in turn was constructed from the family of functors

$\eta_\bigstar$ given in definition 3.5(ii). For our present purpose, we need the functors $\eta_\bigstar$ and also a family of functors

$$\mathbf{d}_\blacksquare : \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \to \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}},$$

where each $\mathcal{A}_i$ is equal to $\mathcal{A}_{e,*,1}$ or $\mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$; these will be built from the symmetric monoidal structures of $\mathcal{A}_{e,*,1}$ and $\mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$ and the functor

$$\mathrm{sig}_{\mathrm{rel}} : \mathcal{A}_{e,*,1} \to \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}.$$

It is convenient to represent this situation by a function $r$ from $\{1, \dots, j\}$ to a two element set $\{u, v\}$, with $\mathcal{A}_i = \mathcal{A}_{e,*,1}$ if $r(i) = u$ and $\mathcal{A}_i = \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$ if $r(i) = v$.

Example 15.2. A typical example is the functor

$$\mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}} \times (\mathcal{A}_{e,*,1})^{\times 5} \times \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}} \to \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$$

which takes $(x_1, \dots, x_7)$ to

$$i^\epsilon \mathrm{sig}_{\mathrm{rel}}(x_4 \boxtimes x_3) \otimes x_7 \otimes \mathrm{sig}_{\mathrm{rel}}(x_6 \boxtimes x_2 \boxtimes x_5) \otimes x_1,$$

where $i^\epsilon$ is the sign that arises from permuting $(x_1, \dots, x_7)$ into the order $(x_4, x_3, x_7, x_6, x_2, x_5, x_1)$. In definition 15.4(iv), we will represent such a functor by a surjection $h$ which keeps track of which inputs go to which output factors and a permutation $\eta$ which keeps track of the order in which the inputs to each $\mathrm{sig}_{\mathrm{rel}}$ factor are multiplied. In the present example, $h$ is the surjection $\{1, \dots, 7\} \to \{1, 2, 3, 4\}$ with

$$h^{-1}(1) = \{3, 4\}, h^{-1}(2) = 7, h^{-1}(3) = \{2, 5, 6\}, h^{-1}(4) = 1$$

and $\eta$ is the permutation $(256)(34)$.

In order to get the signs right we need a preliminary definition.

Definition 15.3. *(i) For totally ordered sets $S_1, \dots, S_m$, define*

$$\coprod_{i=1}^{n} S_i$$

*to be the disjoint union with the order relation given as follows: $s < t$ if either $s \in S_i$ and $t \in S_j$ with $i < j$, or $s, t \in S_i$ with $s < t$ in the order of $S_i$.*

*(ii) For a surjection*

$$h : \{1, \dots, j\} \to \{1, \dots, m\}$$

*define $\theta(h)$ to be the permutation*

$$\{1, \dots, j\} \cong h^{-1}(1) \coprod \cdots \coprod h^{-1}(m) \cong \{1, \dots, j\};$$

*here the first map restricts to the identity on each $h^{-1}(i)$ and the second is the unique ordered bijection.*

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In example 15.2, $\theta(h)$ takes $1, \ldots, 7$ respectively to $7, 4, 1, 2, 5, 6, 3$.

Definition 15.4. *Let $j \geq 0$ and let $r : \{1, \ldots, j\} \to \{u, v\}$. Let $\mathcal{A}_i$ denote $\mathcal{A}_{e,*,1}$ if $r(i) = u$ and $\mathcal{A}^{\mathbb{Z}}_{rel}$ if $r(i) = v$.*

*(i) Let $1 \leq m \leq j$. A surjection*

$$h : \{1, \ldots, j\} \to \{1, \ldots, m\}$$

*is adapted to $r$ if $r$ is constant on each set $h^{-1}(i)$ and $h$ is monic on $r^{-1}(v)$.*

*(ii) Given a surjection*

$$h : \{1, \ldots, j\} \to \{1, \ldots, m\}$$

*which is adapted to $r$, define*

$$h_{\blacklozenge} : \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \to (\mathcal{A}^{\mathbb{Z}}_{rel})^{\times m}$$

*by*

$$h_{\blacklozenge}(x_1, \ldots, x_j) = (i^{\epsilon} y_1, \ldots, y_m),$$

*where $i^{\epsilon}$ is the sign that arises from putting the objects $x_1, \ldots, x_j$ into the order $x_{\theta(h)^{-1}(1)}, \ldots, x_{\theta(h)^{-1}(j)}$ and*

$$y_i = \begin{cases} \mathrm{sig}_{\mathrm{rel}}(\boxtimes_{l \in h^{-1}(i)} x_l) & \text{if } h^{-1}(i) \subset r^{-1}(u), \\ x_{h^{-1}(i)} & \text{if } h^{-1}(i) \in r^{-1}(v). \end{cases}$$

*(iii) A datum of type $r$ is a pair*

$$(h, \eta),$$

*where $h$ is a surjection which is adapted to $r$ and $\eta$ is an element of $\Sigma_j$ with the property that $h \circ \eta = h$.*

*(iv) Given a datum*

$$\mathbf{d} = (h, \eta),$$

*of type $r$, define*

$$\mathbf{d}_{\blacksquare} : \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \to \mathcal{A}^{\mathbb{Z}}_{rel}$$

*to be the composite*

$$\mathcal{A}_1 \times \cdots \times \mathcal{A}_j \xrightarrow{\eta} \mathcal{A}_{\eta^{-1}(1)} \times \cdots \times \mathcal{A}_{\eta^{-1}(j)} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \xrightarrow{h_{\blacklozenge}} (\mathcal{A}^{\mathbb{Z}}_{rel})^{\times m} \xrightarrow{\otimes} \mathcal{A}^{\mathbb{Z}}_{rel},$$

*where $\eta$ permutes the factors with the usual sign.*

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We also need natural transformations between the functors $\mathbf{d}_{\blacksquare}$. First observe that $\mathrm{sig}_{rel}$ is lax monoidal: there is a natural transformation from the functor

$$(\mathcal{A}_{e,*,1})^{\times l} \xrightarrow{\mathrm{sig_{rel}}^{\times l}} (\mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}})^{\times l} \xrightarrow{\otimes} \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$$

to the functor

$$(\mathcal{A}_{e,*,1})^{\times l} \xrightarrow{\boxtimes} \mathcal{A}_{e,*,1} \xrightarrow{\mathrm{sig_{rel}}} \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$$

given by the maps

$$S_*(X_1) \otimes \cdots \otimes S_*(X_l) \xrightarrow{\times} S_*(X_1 \times \cdots \times X_l)$$

and

$$\begin{aligned} S_*(X_1 \times X_1) \otimes \cdots \otimes S_*(X_l \times X_l) &\xrightarrow{\times} S_*(X_1 \times X_1 \times \cdots \times X_l \times X_l) \\ &\cong S_*((X_1 \times \cdots \times X_l) \times (X_1 \times \cdots \times X_l)). \end{aligned}$$

Combining this with the symmetric monoidal structures of $\mathcal{A}_{e,*,1}$ and $\mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$, we obtain a natural transformation $\mathbf{d}_{\blacksquare} \to \mathbf{d}'_{\blacksquare}$ whenever $\mathbf{d} \leq \mathbf{d}'$, as defined in:

Definition 15.5. *For data of type $r$, define*

$$(h, \eta) \leq (h', \eta')$$

*if each set $h^{-1}(i)$ is contained in some set $h'^{-1}(l)$.*

Our next definition is analogous to definition 6.6 (the presence of the letter $v$ in the symbols $P_{r;v}$ and $\mathcal{O}(r; v)$ will be explained in a moment).

Definition 15.6. *Let $r : \{1 \dots, j\} \to \{u, v\}$.*

*(i) Let $P_{r;v}$ be the preorder whose elements are the data of type $r$, with the order relation given by definition 15.5.*

*(ii) Define an object $\mathcal{O}(r; v)_k$ of $\Sigma_k ss\mathcal{S}_k$ by*

$$(\mathcal{O}(r; v)_k)_{\mathbf{n}} = \mathrm{Map}_{\mathrm{preorder}}(U(\Delta^{\mathbf{n}}), P_{r;v})_+$$

*(where the $+$ denotes a disjoint basepoint); the morphisms in $(\Delta^{op}_{\mathrm{inj}})^{\times k}$ act in the evident way, and the morphisms of the form $(\alpha, \mathrm{id})$ with $\alpha \in \Sigma_k$ act by permuting the factors in $\Delta^{\mathbf{n}}$.*

*(iii) Define $\mathcal{O}(r; v)$ to be the object of $\Sigma ss\mathcal{S}$ with k-th term $\mathcal{O}(r; v)_k$.*

Remark 15.7. (i) Given $r : \{1 \dots, j\} \to \{u, v\}$, let $m = |r^{-1}(v)| + 1$, let

$$h : \{1, \dots, j\} \to \{1, \dots, m\}$$

be any surjection which is adapted to $r$, and let $e$ be the identity element of $\Sigma_j$. Then the datum $(h, e)$ is $\geq$ every element in $P_{r;v}$.

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(ii) Lemma 10.3 shows that each of the objects $\mathcal{O}(r;v)_k$ has compatible degeneracies and is weakly equivalent to a point.

We also need a preorder corresponding to the family of functors

$$\eta_\star : (\mathcal{A}_{e,*,1})^{\times j} \to \mathcal{A}_{e,*,1} :$$

NOTATION 15.8. Let $r_u(j)$ (resp., $r_v(j)$) denote the constant function $\{1,\ldots,j\} \to \{u,v\}$ with value $u$ (resp., $v$).

DEFINITION 15.9. *Let $r : \{1\ldots,j\} \to \{u,v\}$.*

*(i) If $r = r_u(j)$, let $P_{r;u}$ be the set $\Sigma_j$ with the preorder in which every element is $\leq$ every other, and let $\mathcal{O}(r;u)_k$ be the object $\mathcal{O}(j)_k$ of definition 6.6.*

*(ii) Otherwise let $P_{r;u}$ be the empty set and let $\mathcal{O}(r;u)_k$ be the multisemisimplicial set with a point in every multidegree.*

*(iii) In either case, let $\mathcal{O}(r;u)$ be the object of $\Sigma ss\mathcal{S}$ with k-th term $\mathcal{O}(r;u)_k$.*

In the next section, we will use the objects $\mathcal{O}(r;v)$ and $\mathcal{O}(r;u)$ to construct a monad. In preparation for that, we show that the collection of preorders $P_{r;v}$ and $P_{r;u}$ has suitable composition maps. Specifically, we show that it is a coloured operad (also called a multicategory) in the category of preorders.

We refer the reader to [4, §2] for the definition of multicategory; we will mostly follow the notation and terminology given there. In our case, there are two objects $u$ and $v$, and we think of a function $r : \{1,\ldots,j\} \to \{u,v\}$ as a sequence of $u$'s and $v$'s.

REMARK 15.10. Let $r : \{1,\ldots,j\} \to \{u,v\}$ and let $\mathcal{A}_i$ denote $\mathcal{A}_{e,*,1}$ if $r(i) = u$ and $\mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$ if $r(i) = v$. Let us write $\mathcal{A}_r$ for the category $\mathcal{A}_1 \times \cdots \times \mathcal{A}_j$ and $\mathcal{A}_{r;u}$ (resp., $\mathcal{A}_{r;v}$) for the category of functors $\mathcal{A}_r \to \mathcal{A}_{e,*,1}$ (resp., $\mathcal{A}_r \to \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$). Define a multicategory with objects $u$ and $v$ as follows. The multimorphisms from $r$ to $u$ are given by $\mathcal{A}_{r;u}$ and from $r$ to $v$ are given by $\mathcal{A}_{r;v}$. The actions of the symmetric groups are the obvious ones and the composition is provided by the composition of functors. Moreover, definitions 3.5(ii) and 15.4(iv) give inclusion functors

$$\Phi_{r;u} : P_{r;u} \to \mathcal{A}_{r;u}$$

and

$$\Phi_{r;v} : P_{r;v} \to \mathcal{A}_{r;v}.$$

We like to promote these to a functor between multicategories. The source also has two objects, $u$, $v$ say and the functor is the identity on objects. The set of multimorphisms with target $u$ are empty unless the source only involves $u$'s in which case it is $P_{r;u}$ with $r = r_u(j)$. The set of multimorphisms with target $v$ and source $r$ is the $P_{r;v}$. The following definitions are chosen so that the inclusion functors preserve the $\Sigma_j$ actions and the composition operations.

We define the right $\Sigma_j$ action on a multimorphism with $j$ sources as follows. Let $\alpha \in \Sigma_j$ and $r : \{1, \ldots, j\} \to \{u, v\}$. Define $r^\alpha$ to be the composite

$$\{1, \ldots, j\} \xrightarrow{\alpha} \{1, \ldots, j\} \xrightarrow{r} \{u, v\}.$$

If $r = r_u(j)$ then $r^\alpha = r$ and the map

$$\alpha : P_{r;u} \to P_{r^\alpha;u}$$

is the right action of $\Sigma_j$ on itself. If $(h, \eta) \in P_{r,v}$ define $(h, \eta)\alpha \in P_{r^\alpha;v}$ to be $(h \circ \bar{\alpha}, \bar{\alpha}^{-1} \circ \eta \circ \alpha)$, where $\bar{\alpha} \in \Sigma_j$ is the permutation whose restriction to each $\alpha^{-1}h^{-1}(i)$ is the order-preserving bijection to $h^{-1}(i)$.

We define the composition operation as follows. If the composition involves only $u$'s then it is the composition in the operad $\mathcal{M}$ of [17, definition 3.1(i)]. Otherwise let $i, j_1, \ldots, j_i \geq 0$, let $r : \{1, \ldots, i\} \to \{u, v\}$, and for $1 \leq l \leq i$ let $r_l : \{1, \ldots, j_l\} \to \{u, v\}$; assume that if $r(l) = u$ then $r_l$ is $r_u(j_l)$. Let $(h, \eta) \in P_{r;v}$ and for $1 \leq l \leq i$ let $x_l \in P_{r_l;r(l)}$. If $r(l) = v$ then $x_l$ has the form $(h_l, \eta_l)$, otherwise $x_l$ is an element $\eta_l \in \Sigma_{j_l}$ and we write $h_l$ for the map $\{1, \ldots, j_l\} \to \{1\}$. Define the composition operation $\Gamma$ by

$$\Gamma((h, \eta), x_1, \ldots, x_l) = (H, \theta), \tag{15.1}$$

where $\theta$ is the composite $\gamma_{\mathcal{M}}(\eta, \eta_1, \ldots, \eta_i)$ in the operad $\mathcal{M}$ of [17, definition 3.1(i)] and $H$ is the following multivariable composite $h \circ (h_1, \ldots, h_i)$: we are going to make the composite $H$ explicit with $h$ as in example 15.2, the general formula should be clear then. The source of the morphism $(h, \eta)$ is $r = (v, u, u, u, u, u, v)$ and the target is $v$. It is convenient to write $h$ in the form

$$h = (h^{-1}\{1\}, h^{-1}\{2\}, h^{-1}\{3\}, h^{-1}\{4\}) = (\{3, 4\}, \{7\}, \{2, 5, 6\}, \{1\}).$$

Then the multivariable composite $H$ for arbitrary $h_1, h_2, \ldots, h_7$ takes the form

$$(h_3^{-1}\{1\} \cup h_4^{-1}\{1\}, h_7, h_2^{-1}\{1\} \cup h_5^{-1}\{1\} \cup h_6^{-1}\{1\}, h_1).$$

In this notation, the $j_2$ elements in the source of $h_2$ have to be shifted by $j_1$, the ones of $h_3$ should be shifted by $j_1 + j_2$ and so on. Moreover, the round parentheses of $h_7$ and $h_1$ should be ignored. The formula may look strange but notice that $h_k$ is the constant map to $\{1\}$ for all $2 \leq k \leq 6$ because for these we have $r(k) = u$.

Proposition 15.11. *With these definitions, the collection of preorders $P_{r;u}$ and $P_{r;v}$ are the multimorphisms in a multicategory from r to u and from r to v respectively. The functors $\Phi_{r;u}$ and $\Phi_{r;v}$ define a multifunctor between multicategories.*

*Proof.* This is immediate from remark 15.10. □

## 16. A monad in $\Sigma ss\mathcal{S} \times \Sigma ss\mathcal{S}$

In this section, we construct a monad in $\Sigma ss\mathcal{S} \times \Sigma ss\mathcal{S}$ which acts on the pair $(\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}})$. The arguments in this and in the next sections are along the same

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lines with $\mathcal{A}ssoc$ and $\mathcal{C}omm$ replaced by monads encoding (commutative) monoid maps or (symmetric) monoidal transformations.

Definition 16.1. *Let $j \geq 0$ and let $\mathbf{X}, \mathbf{Y} \in \Sigma ss\mathcal{S}$.*
*(i) For $\alpha \in \Sigma_j$ and $r : \{1, \dots, j\} \to \{u, v\}$, define*

$$\tilde{\alpha} : \mathcal{O}(r; v) \to \mathcal{O}(r^\alpha; v)$$

*by*

$$(\tilde{\alpha}(a))(\sigma) = (a(\sigma))\alpha$$

*where $a \in \mathrm{Map}_{\mathrm{preorder}}(U(\Delta^{\mathbf{n}}), P_{r;v})_+$ and $\sigma \in U(\Delta^{\mathbf{n}})$.*
*(ii) Define*

$$(\mathbf{X}, \mathbf{Y})^{\otimes r} = \mathbf{Z}_1 \otimes \cdots \otimes \mathbf{Z}_j,$$

*where $\mathbf{Z}_i$ denotes $\mathbf{X}$ if $r(i) = u$ and $\mathbf{Y}$ if $r(i) = v$.*
*(iii) For $\alpha \in \Sigma_j$ define*

$$\bar{\alpha} : \bigvee_r \mathcal{O}(r; v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\otimes r} \to \bigvee_r \mathcal{O}(r; v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\otimes r}$$

*to be the map which takes the $r$-summand to the $r^\alpha$-summand by means of the map*

$$\mathcal{O}(r; v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\otimes r} \xrightarrow{\tilde{\alpha} \overline{\wedge} \alpha} \mathcal{O}(r^\alpha; v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\otimes r^\alpha}.$$

Note that the maps $\bar{\alpha}$ give $\bigvee_r \mathcal{O}(r; v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\otimes r}$ a right $\Sigma_j$ action.
Recall notation 15.8.

Definition 16.2. *(i) Define a functor $\mathbb{O} : \Sigma ss\mathcal{S} \times \Sigma ss\mathcal{S} \to \Sigma ss\mathcal{S} \times \Sigma ss\mathcal{S}$ by*

$$\mathbb{O}(\mathbf{X}, \mathbf{Y}) = (\mathbb{O}_1(\mathbf{X}), \mathbb{O}_2(\mathbf{X}, \mathbf{Y})),$$

*where*

$$\mathbb{O}_1(\mathbf{X}) = \bigvee_{j \geq 0} \left(\mathcal{O}(r_u(j); u) \,\overline{\wedge}\, \mathbf{X}^{\otimes j}\right)/\Sigma_j$$

*and*

$$\mathbb{O}_2(\mathbf{X}, \mathbf{Y}) = \bigvee_{j \geq 0} \bigvee_{r : \{1, \dots, j\} \to \{u, v\}} \Big(\mathcal{O}(r; v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\otimes r}\Big)/\Sigma_j.$$

*(ii) Define a natural transformation*

$$\iota : (\mathbf{X}, \mathbf{Y}) \to \mathbb{O}(\mathbf{X}, \mathbf{Y})$$

*to be $(\iota_1, \iota_2)$, where $\iota_1$ is the composite*

$$\mathbf{X} \xrightarrow{\cong} \overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X} = \mathcal{O}(r_u(1); u) \,\overline{\wedge}\, \mathbf{X} \hookrightarrow \mathbb{O}(\mathbf{X})$$

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*and $\iota_2$ is the composite*

$$\mathbf{Y} \xrightarrow{\cong} \overline{\mathbf{S}} \barwedge \mathbf{Y} = \mathcal{O}(r_v(1); v) \barwedge \mathbf{Y} \hookrightarrow \mathbb{O}(\mathbf{Y}).$$

For the structure map $\mu : \mathbb{O}\mathbb{O} \to \mathbb{O}$, we need a composition operation for the collection of objects $\mathcal{O}(r; u)$ and $\mathcal{O}(r; v)$. Recall definition 15.3(i) and the map $\Gamma$ defined in Eq. (15.1).

Definition 16.3. *Let $i, j_1, \ldots, j_i \geq 0$, let $r : \{1, \ldots, i\} \to \{u, v\}$, and for $1 \leq l \leq i$ let $r_l : \{1, \ldots, j_l\} \to \{u, v\}$; assume that if $r(l) = u$ then $r_l$ is $r_u(j_l)$. Let*

$$R : \{1, \ldots, \sum j_l\} \to \{u, v\}$$

*be the composite*

$$\{1, \ldots, \sum j_l\} \cong \coprod_{l=1}^{i} \{1, \ldots, j_l\} \to \{u, v\},$$

*where the first map is the unique order-preserving bijection and the second restricts on each $\{1, \ldots, j_l\}$ to $r_l$. Define a map*

$$\gamma : \mathcal{O}(r; v) \barwedge \big(\mathcal{O}(r_1; r(1)) \otimes \cdots \otimes O(r_i; r(i))\big) \to \mathcal{O}(R; v)$$

*in $\Sigma ss\mathcal{S}$ by the formulas*

$$\gamma(a \wedge [e, b_1 \wedge \ldots \wedge b_i])(\sigma_1 \times \cdots \times \sigma_i) = \Gamma(a(\sigma_1 \times \cdots \times \sigma_i), b_1(\sigma_1), \ldots, b_i(\sigma_i))$$

*(where $e$ is the identity element of the relevant symmetric group) and*

$$\gamma(a \wedge [\alpha, b_1 \wedge \ldots \wedge b_i]) = (\alpha, \mathrm{id})_* \gamma((\alpha^{-1}, \mathrm{id})_* a \wedge [e, b_1 \wedge \ldots \wedge b_i]).$$

This operation satisfies the analogues of lemmas 7.2, 7.5, and 7.7.
Now we can define

$$\mu : \mathbb{O}\mathbb{O} \to \mathbb{O}$$

to be $(\mu_1, \mu_2)$, where $\mu_1$ is given by definition 7.8(iv) and $\mu_2$ is defined in a similar way using definition 16.3.

Proposition 16.4. *The transformations $\mu$ and $\iota$ define a monad structure on $\mathbb{O}$.*

We conclude this section by giving the action of $\mathbb{O}$ on the pair $(\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}})$. Recall remark 15.10.

Definition 16.5. *Let $k_1, \ldots, k_j$ be non-negative integers and let $\mathbf{n}_i$ be a $k_i$-fold multi-index for $1 \leq i \leq j$. Let $r : \{1, \ldots, j\} \to \{u, v\}$, and for $1 \leq i \leq j$ let $\mathrm{pre}_i$*

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*denote* $\mathrm{pre}_{e,*,1}$ *if* $r(i) = u$ *and* $\mathrm{pre}^{\mathbb{Z}}_{rel}$ *if* $r(i) = v$*. For any map of preorders*

$$a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to P_{r;v}$$

*define*

$$a_* : \mathrm{pre}_1^{k_1}(\Delta^{\mathbf{n}_1}) \times \cdots \times \mathrm{pre}_j^{k_j}(\Delta^{\mathbf{n}_j}) \to (\mathrm{pre}^{\mathbb{Z}}_{rel})^{k_1+\cdots+k_j}(\Delta^{(\mathbf{n}_1,\ldots,\mathbf{n}_j)})$$

*by*

$$\begin{aligned} &a_*(F_1, \ldots, F_j)(\sigma_1 \times \cdots \times \sigma_j, o_1 \times \cdots \times o_j) \\ &\quad = i^{\epsilon(\zeta)}\Phi_{r;v}(a(\sigma_1 \times \cdots \times \sigma_j))(F_1(\sigma_1, o_1), \ldots, F_j(\sigma_j, o_j)), \end{aligned}$$

*where* $\Phi_{r;v}$ *was defined in remark 15.10 and* $\zeta$ *is the block permutation that takes blocks* $\mathbf{b}_1, \ldots, \mathbf{b}_j, \mathbf{c}_1, \ldots, \mathbf{c}_j$ *of size* $k_1, \ldots, k_j, \dim\sigma_1, \ldots, \dim\sigma_j$ *into the order* $\mathbf{b}_1, \mathbf{c}_1, \ldots, \mathbf{b}_j, \mathbf{c}_j$*.*

Lemma 16.6. *If* $F_i \in \mathrm{ad}_i^{k_i}(\Delta^{\mathbf{n}_i})$ *for* $1 \le i \le j$ *then*

$$a_*(F_1, \ldots, F_j) \in (\mathrm{ad}^{\mathbb{Z}}_{\mathrm{rel}})^{k_1+\cdots+k_j}(\Delta^{(\mathbf{n}_1,\ldots,\mathbf{n}_j)}).$$

*Proof.* This is a straightforward consequence of the fact that the natural transformation from the functor

$$(\mathcal{A}_{e,*,1})^{\times l} \xrightarrow{{\mathrm{sig}_{\mathrm{rel}}}^{\times l}} (\mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}})^{\times l} \xrightarrow{\otimes} \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$$

to the functor

$$(\mathcal{A}_{e,*,1})^{\times l} \xrightarrow{\boxtimes} \mathcal{A}_{e,*,1} \xrightarrow{\mathrm{sig}_{\mathrm{rel}}} \mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$$

given by the cross product is a quasi-isomorphism. □

Definition 16.7. *Let* $j \ge 0$ *and let* $r : \{1, \ldots, j\} \to \{u, v\}$*. Define a map*

$$\phi_r : \mathcal{O}(r; v) \,\overline{\wedge}\, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{rel})^{\otimes r} \to \mathbf{R}^{\mathbb{Z}}_{rel}$$

*in* $\Sigma ss\mathcal{S}$ *by the formulas*

$$\phi_r(a \wedge [e, F_1 \wedge \cdots \wedge F_j]) = a_*(F_1 \wedge \cdots \wedge F_j)$$

*(where* $e$ *denotes the identity element of the relevant symmetric group) and*

$$\phi_r(a \wedge [\alpha, F_1 \wedge \cdots \wedge F_j]) = (\alpha, \mathrm{id})_* \phi_r((\alpha^{-1}, \mathrm{id})_* a \wedge [e, F_1 \wedge \cdots \wedge F_j]).$$

Next observe that the maps $\phi_r$ induce a map

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$$\Big(\bigvee_r \big(\mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}})^{\otimes r}\Big)/\Sigma_j \to \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}} \tag{16.1}$$

for each $j \geq 0$. We define

$$\nu : \mathbb{O}(\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}}) \to (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}})$$

to be the pair $(\nu_1, \nu_2)$, where $\nu_1$ is given by definition 7.10 and $\nu_2$ is given by the maps (16.1).

PROPOSITION 16.8. *$\nu$ is an action of $\mathbb{O}$ on $(\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}})$.*

*Proof.* This is a straightforward consequence of remark 15.10. □

## 17. A monad in $\mathcal{S}p_{mss} \times \mathcal{S}p_{mss}$

First we give $\mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r}$ the structure of a multisemisimplicial symmetric spectrum when $\mathbf{X}, \mathbf{Y} \in \mathcal{S}p_{mss}$. The definition is analogous to definition 8.1. Recall definition 6.4.

DEFINITION 17.1. *Let $j, k \geq 0$. Let $s$ be the 1-simplex of $S^1$. Define*

$$\omega : S^1 \wedge (\mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r})_k \to (\mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r})_{k+1}$$

*as follows: for $a \in (\mathcal{O}(r;v)_k)_{\mathbf{n}}$ and $x \in (((\mathbf{X}, \mathbf{Y})^{\wedge r})_k)_{\mathbf{n}}$, let*

$$\omega(s \wedge (a \wedge x)) = (a \circ \Pi) \wedge \omega(s \wedge x).$$

DEFINITION 17.2. *Define a functor $\mathbb{P} : \Sigma ss\mathcal{S} \times \Sigma ss\mathcal{S} \to \Sigma ss\mathcal{S} \times \Sigma ss\mathcal{S}$ by*

$$\mathbb{P}(\mathbf{X}, \mathbf{Y}) = (\mathbb{P}_1(\mathbf{X}), \mathbb{P}_2(\mathbf{X}, \mathbf{Y})),$$

*where*

$$\mathbb{P}_1(\mathbf{X}) = \bigvee_{j \geq 0} \big(\mathcal{O}(r_u(j); u) \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j$$

*and*

$$\mathbb{P}_2(\mathbf{X}, \mathbf{Y}) = \bigvee_{j \geq 0} \Big(\bigvee_r \big(\mathcal{O}(r;v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r}\Big)/\Sigma_j.$$

The proof that $\mathbb{P}$ inherits a monad structure and an action on $(\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}})$ is the same as the corresponding proof in §8.

For use in the next section we record a lemma. Let $\mathcal{C}$ be the category whose objects are triples $(\mathbf{X}, \mathbf{Y}, f)$, where $\mathbf{X}$ and $\mathbf{Y}$ are monoids in $\mathcal{S}p_{mss}$ and $f$ is a map $\mathbf{X} \to \mathbf{Y}$ in $\mathcal{S}p_{mss}$ which is not required to be a monoid map; the morphisms are commutative diagrams

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$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & \mathbf{X}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathbf{Y} & \longrightarrow & \mathbf{Y}', \end{array}$$

where the horizontal arrows are monoid maps.

LEMMA 17.3. *(i) There is a functor $\Upsilon$ from $\mathbb{P}$ algebras to $\mathcal{C}$ which takes $(\mathbf{X},\mathbf{Y})$ to a map $\mathbf{X}\to\mathbf{Y}$; in particular, $\mathbf{X}$ and $\mathbf{Y}$ have natural monoid structures.*

*(ii) $\Upsilon(\mathbf{R}_{e,*,1},\mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}})$ is the map*

$$\mathrm{sig}_{\mathrm{rel}}:\mathbf{R}_{e,*,1}\to\mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}},$$

*where $\mathbf{R}_{e,*,1}$ and $\mathbf{R}^{\mathbb{Z}}_{\mathrm{rel}}$ have the monoid structures given by lemma 8.8(i).*

*Proof.* Recall notation 15.8, and let $e$ denote the identity element of $\Sigma_j$.

Part (i). The map $f:\mathbf{X}\to\mathbf{Y}$ is the composite

$$\mathbf{X}\cong\overline{\mathbf{S}}\,\overline{\wedge}\,\mathbf{X}=\mathcal{O}(r_u(1);v)\,\overline{\wedge}\,\mathbf{X}\hookrightarrow\mathbb{P}_2(\mathbf{X},\mathbf{Y})\to\mathbf{Y}.$$

The monoid structure on $\mathbf{X}$ is given by lemma 8.8; it remains to give the monoid structure on $\mathbf{Y}$. It suffices to give an action on $\mathbf{Y}$ of the monad $\mathbb{A}$ defined in the proof of lemma 8.8, and for this in turn it suffices to give a suitable natural transformation $\mathbb{A}\to\mathbb{P}_2$.

For each $j\ge 0$ let $h_0$ be the identity map of $\{1,\ldots,j\}$. Then $h_0$ is adapted to $r_v(j)$, so we obtain an element $(h_0,e)\in P_{r_v(j);v}$.

For each $j,k\ge 0$ and each $k$-fold multi-index $\mathbf{n}$, define an element

$$b_{j,k,\mathbf{n}}\in(\mathcal{O}(r_v(j);v)_k)_{\mathbf{n}}$$

to be the constant function $U(\Delta^{\mathbf{n}})\to P_{r_v(j);v}$ whose value is $(h_0,e)$. Next define a map

$$\overline{\mathbf{S}}\to\mathcal{O}(r_v(j);v)$$

by taking the non-trivial simplex of $(\overline{\mathbf{S}}_k)_{\mathbf{n}}$ to $b_{j,k,\mathbf{n}}$.

Now the composite

$$\begin{aligned}\mathbb{A}(\mathbf{Y})=\bigvee_{j\ge 0}\mathbf{Y}^{\wedge j}&\cong\bigvee_{j\ge 0}\overline{\mathbf{S}}\,\overline{\wedge}\,\mathbf{Y}^{\wedge j}\\ &\to\bigvee_{j\ge 0}\Big(\bigvee_{r}\big(\mathcal{O}(r;v)\,\overline{\wedge}\,(\mathbf{X},\mathbf{Y})^{\wedge r}\Big)/\Sigma_j=\mathbb{P}_2(\mathbf{X},\mathbf{Y})\end{aligned}$$

is the desired map.

Part (ii) is an easy consequence of the definitions. $\square$

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## 18. Rectification

In this section, we prove theorem 1.3. The argument is analogous to that in §10.

First we consider a monad in $\mathcal{S}p_{mss} \times \mathcal{S}p_{mss}$ which is simpler than $\mathbb{P}$.

Definition 18.1. *(i) Define* $\mathbb{P}'(\mathbf{X}, \mathbf{Y})$ *to be*

$$(\bigvee_{j\geq 0} \mathbf{X}^{\wedge j}/\Sigma_j, \bigvee_{j\geq 0} (\bigvee_r (\mathbf{X}, \mathbf{Y})^{\wedge r})/\Sigma_j).$$

*(ii) For each* $j \geq 0$ *and each* $r : \{1, \ldots, j\} \to \{u, v\}$, *let*

$$\xi_j : \mathcal{O}(r_u(j); u) \to \overline{\mathbf{S}}$$

*and*

$$\zeta_r : \mathcal{O}(r; v) \to \overline{\mathbf{S}}$$

*be the maps which take each non-trivial simplex of the k-th object to the non-trivial simplex of* $\overline{S}_k$ *in the same multidegree. Define a natural transformation*

$$\Xi : \mathbb{P} \to \mathbb{P}'$$

*to be the pair* $(\Xi_1, \Xi_2)$, *where* $\Xi_1$ *is the wedge of the composites*

$$\big(\mathcal{O}(r_u(j); u) \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j \xrightarrow{\xi_j \overline{\wedge} 1} \big(\overline{\mathbf{S}} \,\overline{\wedge}\, \mathbf{X}^{\wedge j}\big)/\Sigma_j \xrightarrow{\cong} \mathbf{X}^{\wedge j}/\Sigma_j$$

*and* $\Xi_2$ *is the wedge of the composites*

$$\Big(\bigvee_r \mathcal{O}(r; v) \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r}\Big)/\Sigma_j \xrightarrow{\bigvee \zeta_r \overline{\wedge} 1} \Big(\bigvee_r \overline{\mathbf{S}} \,\overline{\wedge}\, (\mathbf{X}, \mathbf{Y})^{\wedge r}\Big)/\Sigma_j \xrightarrow{\cong} \Big(\bigvee_r (\mathbf{X}, \mathbf{Y})^{\wedge r}\Big)/\Sigma_j.$$

Proposition 18.2. *(i) An algebra over* $\mathbb{P}'$ *is the same thing as a pair of commutative monoids* $(\mathbf{X}, \mathbf{Y})$ *in* $\mathcal{S}p_{mss}$ *together with a monoid map* $\mathbf{X} \to \mathbf{Y}$.

*(ii)* $\Xi$ *is a map of monads.*

*(iii) Suppose that each* $X_k$ *and each* $Y_k$ *has compatible degeneracies (see definition 9.1). Let* $\mathbb{P}^q$ *denote the q-th iterate of* $\mathbb{P}$. *Then each map*

$$\Xi : \mathbb{P}^q(\mathbf{X}, \mathbf{Y}) \to \mathbb{P}'\mathbb{P}^{q-1}(\mathbf{X}, \mathbf{Y})$$

*is a weak equivalence.*

*Proof.* Part (i). Let $(\mathbf{X}, \mathbf{Y})$ be an algebra over $\mathbb{P}'$. The fact that $\mathbf{X}$ and $\mathbf{Y}$ are commutative monoids is immediate from the definitions. The map $f : \mathbf{X} \to \mathbf{Y}$ is constructed as in the proof of lemma 17.3(i).

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To show that $f$ is a monoid map, we first observe that there are two inclusions of $\mathbf{X}^{\wedge j}$ into $\mathbb{P}'_2\mathbb{P}'(\mathbf{X}, \mathbf{Y})$. Let $i_1$ be the composite

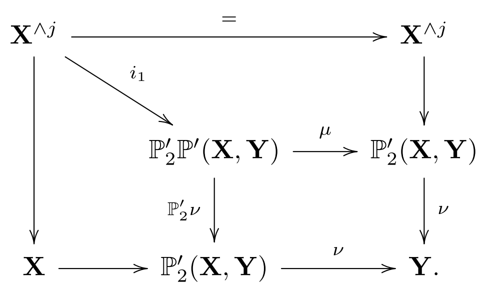
$$\mathbf{X}^{\wedge j} \hookrightarrow \mathbb{P}'_1(\mathbf{X}) \hookrightarrow \mathbb{P}'_2\mathbb{P}'(\mathbf{X}, \mathbf{Y}),$$

where the second arrow is the inclusion of the summand indexed by $j = 1, r = r_u(1)$. Let $i_2$ be the composite

$$\mathbf{X}^{\wedge j} \hookrightarrow \mathbb{P}'_2(\mathbf{X}, \mathbf{Y})^{\wedge j} \hookrightarrow \mathbb{P}'_2\mathbb{P}'(\mathbf{X}, \mathbf{Y}),$$

where the first arrow is the $j$-fold smash of the inclusion of the $r_u(1)$ summand, and the second arrow is the inclusion of the $r_v(j)$ summand.

Consider the commutative diagram

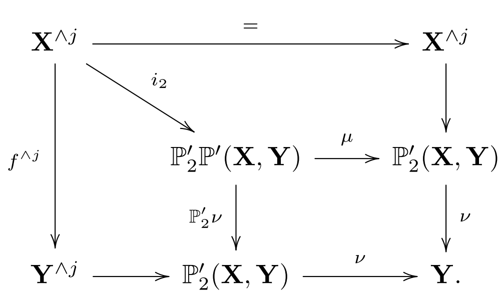



Let $H$ denote the composite of the right-hand vertical arrows. Then the diagram shows that the composite

$$\mathbf{X}^{\wedge j} \to \mathbf{X} \xrightarrow{f} \mathbf{Y} \tag{18.1}$$

is $H$.

Next consider the commutative diagram




This diagram shows that the composite

$$\mathbf{X}^{\wedge j} \xrightarrow{f^{\wedge j}} \mathbf{Y}^{\wedge j} \to \mathbf{Y} \tag{18.2}$$

is also $H$. Therefore the composites (18.1) and (18.2) are equal as required.

Part (ii) is immediate from the definitions, and the proof of part (iii) is the same as for proposition 10.2(iii) (but using remark 15.7(ii)). $\square$

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*Proof of theorem 1.3.* The proof follows the outline of the proof of theorem 1.1 (given in §10); we refer the reader to that proof for some of the details. We have a diagram of simplicial $\mathbb{P}$-algebras

$$(\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}})_\bullet \xleftarrow{\varepsilon} B_\bullet(\mathbb{P}, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}})) \xrightarrow{\Xi_\bullet} B_\bullet(\mathbb{P}', \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}})).$$

By lemma 17.3, this gives a diagram

$$\begin{array}{ccccc}
(\mathbf{R}_{e,*,1})_\bullet & \xleftarrow{\varepsilon} & B_\bullet(\mathbb{P}_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}})) & \xrightarrow{(\Xi_1)_\bullet} & B_\bullet(\mathbb{P}'_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}})) \\
{\scriptstyle \text{sig}_{\text{rel}}}\downarrow & & \downarrow & & \downarrow \\
(\mathbf{R}^{\mathbb{Z}}_{\text{rel}})_\bullet & \xleftarrow{\varepsilon} & B_\bullet(\mathbb{P}_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}})) & \xrightarrow{(\Xi_2)_\bullet} & B_\bullet(\mathbb{P}'_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}})),
\end{array} \tag{18.3}$$

in which all objects are simplicial monoids and all horizontal arrows are monoid maps. By Proposition 18.2(i), the right column is a simplicial monoid map between simplicial commutative monoids. Moreover, each map $\varepsilon$ is a homotopy equivalence of simplicial objects, and (using proposition 18.2(iii)) $(\Xi_1)_\bullet$ and $(\Xi_2)_\bullet$ are weak equivalences in each simplicial degree.

The objects of the diagram (18.3) are simplicial objects in $\mathcal{S}p_{mss}$. We obtain a diagram

$$\begin{array}{ccccc}
|(\mathbf{R}_{e,*,1})_\bullet| & \xleftarrow{|\varepsilon|} & |B_\bullet(\mathbb{P}_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}}))| & \xrightarrow{|(\Xi_1)_\bullet|} & |B_\bullet(\mathbb{P}'_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}}))| \\
{\scriptstyle |\text{sig}_{\text{rel}}|}\downarrow & & \downarrow & & \downarrow \\
|(\mathbf{R}^{\mathbb{Z}}_{\text{rel}})_\bullet| & \xleftarrow{|\varepsilon|} & |B_\bullet(\mathbb{P}_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}}))| & \xrightarrow{|(\Xi_2)_\bullet|} & |B_\bullet(\mathbb{P}'_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}}))|
\end{array} \tag{18.4}$$

of simplicial objects in $\mathcal{S}p$ (the category of symmetric spectra) by applying the geometric realization functor $\mathcal{S}p_{mss} \to \mathcal{S}p$ to the diagram (18.3) in each simplicial degree. All objects are simplicial monoids and all horizontal arrows are monoid maps, and the right column is a simplicial monoid map between simplicial commutative monoids. The maps $|\varepsilon|$ are homotopy equivalences of simplicial objects and the maps $|(\Xi_1) \bullet|$ and $|(\Xi_1) \bullet|$ are weak equivalences in each simplicial degree.

Finally, we apply geometric realization to the diagram (18.4). We define $\mathbf{A}$ to be $||B_\bullet(\mathbb{P}_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}}))||$, $\mathbf{B}$ to be $||B_\bullet(\mathbb{P}_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}}))||$, $\mathbf{C}$ to be $||B_\bullet(\mathbb{P}'_1, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}}))||$, and $\mathbf{D}$ to be $||B_\bullet(\mathbb{P}'_2, \mathbb{P}, (\mathbf{R}_{e,*,1}, \mathbf{R}^{\mathbb{Z}}_{\text{rel}}))||$. This gives the diagram of theorem 1.3. □

REMARK 18.3. The symmetric ring spectrum $\mathbf{C}$ is the same as the symmetric ring spectrum $\mathbf{M}^{\text{comm}}_{e,*,1}$ given by theorem 1.1. There is a ring map

$$(\mathbf{M}^{\mathbb{Z}}_{\text{rel}})^{\text{comm}} \to \mathbf{D}$$

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which is a weak equivalence (because there is a commutative diagram whose first row is diagram (10.3) and whose second row is the second row of the diagram in theorem 1.3).

## 19. Improved versions of geometric and symmetric Poincaré bordism

In order to state our next theorem, we need some background.

Let $\mathcal{S}p$ denote the category of symmetric spectra.

Recall (from [11, definition 13.2(i) and the second paragraph of §19]) the strict monoidal category $\mathcal{T}$ whose objects are the triples $(\pi, Z, w)$, where $\pi$ is a group, $Z$ is a simply connected free $\pi$-space, and $w$ is a homomorphism $\pi \to \{\pm 1\}$. There is a monoidal functor

$$\mathbf{M}_{\mathrm{geom}} : \mathcal{T} \to \mathcal{S}p$$

which takes $(\pi, Z, w)$ to $\mathbf{M}_{\pi,Z,w}$ [11, definition 19.1 and theorem 19.2].

Let $\mathcal{R}$ be the category of rings with involution. We like to say that there is a functor

$$\mathbf{M}_{\mathrm{sym}} : \mathcal{R} \to \mathcal{S}p$$

which takes $R$ to $\mathbf{M}^R_{\mathrm{rel}}$. As explained in Appendix C, the ad theory $\mathrm{ad}^R_{\mathrm{rel}}$ does not depend on $R$ in a functorial way. However, there is functorial refinement which is constructed in Appendix C and which we may use instead: there is an ad theory $\mathrm{ad}^R_{\mathrm{Rel,sch}}$ which depends on $R$ in a natural way. The same proof as in [11, theorem 19.2, theorem 18.5] then shows that its geometric realization $\mathbf{M}_{\mathrm{sym}}$ is monoidal.

There is a functor $\rho : \mathcal{T} \to \mathcal{R}$ which takes $(\pi, Z, w)$ to $\mathbb{Z}[\pi]$ with the $w$-twisted involution [11, definition 13.2(ii)]. In §14, we constructed a natural transformation

$$\mathrm{sig}_{\mathrm{rel}} : \mathbf{M}_{\mathrm{geom}} \to \mathbf{M}_{\mathrm{sym}} \circ \rho.$$

More precisely, in the notation of appendix $C$ for objects in $\mathrm{ad}^R_{\mathrm{Rel,sch}}$, we have

$$\mathrm{sig}_{\mathrm{rel}}(X, f, \xi, \Phi) = (S_*(\tilde{X}), (S_*(\tilde{X})^t \otimes S_*(\tilde{X}))^W, \gamma, \phi)$$

where $\gamma$ is the obvious map and $\phi \in (\mathbb{Z}^\omega \otimes_R (S_*(\tilde{X})^t \otimes S_*(\tilde{X}))^W)^{\mathbb{Z}/2}$ is induced by the Alexander–Whitney map. The lifting function $\Phi$ gives an isomorphism between $S_*(\tilde{X})$ and the free $R$-module on the set of all singular simplexes in $X$ (see [11, §10]). This means that $\mathrm{sig}_{\mathrm{rel}}(X, f, \xi, \Phi)$ refines to a schematic Relaxed quasi-symmetric complex in a functorial way (for the action of morphisms in $\mathcal{T}$ on the lifting structure see [11, §13]). The new map $\mathrm{sig}_{\mathrm{rel}}$ coincides with the map $\mathrm{sig}_{\mathrm{rel}}$ of §14 under the natural transformation from $\mathcal{A}^R_{\mathrm{Rel,sch}}$ to $\mathcal{A}^R_{\mathrm{rel}}$.

However, $\mathrm{sig}_{\mathrm{rel}}$ is not a monoidal transformation, and $\mathbf{M}_{\mathrm{geom}}$ and $\mathbf{M}_{\mathrm{sym}}$ are not symmetric monoidal functors (we recall the definitions of monoidal transformation and symmetric monoidal functor below). Our next result shows that there is a monoidal transformation between symmetric monoidal functors which is weakly equivalent to $\mathrm{sig}_{\mathrm{rel}}$.

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Theorem 19.1 *There are symmetric monoidal functors* $\mathbf{P} : \mathcal{T} \to \mathcal{S}p$, $\mathbf{L}_{\mathrm{sym}} : \mathcal{R} \to \mathcal{S}p$, *and a monoidal natural transformation* $\mathrm{sig} : \mathbf{P} \to \mathbf{L}_{\mathrm{sym}} \circ \rho$ *such that*

*(i)* $\mathbf{P}$ *is weakly equivalent as a monoidal functor to* $\mathbf{M}_{\mathrm{geom}}$*; specifically, there is a monoidal functor* $\mathbf{A} : \mathcal{T} \to \mathcal{S}p$ *and monoidal weak equivalences*

$$\mathbf{M}_{\mathrm{geom}} \leftarrow \mathbf{A} \to \mathbf{P}.$$

*(ii)* $\mathbf{L}_{\mathrm{sym}}$ *is weakly equivalent as a monoidal functor to* $\mathbf{M}_{\mathrm{sym}}$*; specifically, there is a monoidal functor* $\mathbf{B} : \mathcal{R} \to \mathcal{S}p$ *and monoidal weak equivalences*

$$\mathbf{M}_{\mathrm{sym}} \leftarrow \mathbf{B} \to \mathbf{L}_{\mathrm{sym}}.$$

*(iii) The natural transformations* $\mathrm{sig} : \mathbf{P} \to \mathbf{L}_{\mathrm{sym}} \circ \rho$ *and* $\mathrm{sig}_{\mathrm{rel}} : \mathbf{M}_{\mathrm{geom}} \to \mathbf{M}_{\mathrm{sym}} \circ \rho$ *are weakly equivalent in* $\mathcal{S}p$*; specifically, there is a natural transformation* $\mathbf{A} \to \mathbf{B} \circ \rho$ *which makes the following diagram strictly commute*

$$\begin{array}{ccccc}
\mathbf{M}_{\mathrm{geom}} & \longleftarrow & \mathbf{A} & \longrightarrow & \mathbf{P} \\
{\scriptstyle \mathrm{sig}_{\mathrm{rel}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{sig}} \\
\mathbf{M}_{\mathrm{sym}} \circ \rho & \longleftarrow & \mathbf{B} \circ \rho & \longrightarrow & \mathbf{L}_{\mathrm{sym}} \circ \rho.
\end{array}$$

Remark 19.2. (i) Theorem 19.1 implies that $\mathbf{L}_{\mathrm{sym}}(R)$ is a strictly commutative symmetric ring spectrum when $R$ is commutative. Also, $\mathbf{P}(e, *, 1)$ is a strictly commutative symmetric ring spectrum and $\mathrm{sig} : \mathbf{P}(e, *, 1) \to \mathbf{L}_{\mathrm{sym}}(\mathbb{Z})$ is a map of symmetric ring spectra. This is compatible with theorem 1.3: there is a commutative diagram

$$\begin{array}{ccc}
\mathbf{C} & \longrightarrow & \mathbf{P}(e, *, 1) \\
\downarrow & & {\scriptstyle \mathrm{sig}}\downarrow \\
\mathbf{D} & \longrightarrow & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z})
\end{array}$$

in which the horizontal arrows are ring maps, and they are weak equivalences by the argument given in remark 18.3.

(ii) The fact that sig is a monoidal functor is a spectrum-level version of Ranicki's multiplicativity formula for the symmetric signature [21, proposition 8.1(i)]. It seems likely that his multiplicativity formula for the surgery obstruction [21, proposition 8.1(ii)] can also be given a spectrum-level interpretation.

We recall the definitions of symmetric monoidal functor and monoidal transformation. The theorem says that $\mathbf{L}_{\mathrm{sym}}$ (and similarly $\mathbf{P}$) is a monoidal functor with the additional property that the diagram

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$$\begin{array}{ccc}
\mathbf{L}_{\mathrm{sym}}(R) \wedge \mathbf{L}_{\mathrm{sym}}(S) & \longrightarrow & \mathbf{L}_{\mathrm{sym}}(R \otimes S) \\
\downarrow & & \downarrow \\
\mathbf{L}_{\mathrm{sym}}(S) \wedge \mathbf{L}_{\mathrm{sym}}(R) & \longrightarrow & \mathbf{L}_{\mathrm{sym}}(S \otimes R)
\end{array}$$

strictly commutes. Moreover, sig has the property that the diagrams

$$\begin{array}{ccccc}
 & & \mathbf{S} & & \\
 & \swarrow & & \searrow & \\
\mathbf{P}(e, *, 1) & & \xrightarrow{\mathrm{sig}} & & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z})
\end{array}$$

and

$$\begin{array}{ccc}
\mathbf{P}(\pi, Z, w) \wedge \mathbf{P}(\pi', Z', w') & \xrightarrow{\mathrm{sig} \wedge \mathrm{sig}} & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z}[\pi]^w) \wedge \mathbf{L}_{\mathrm{sym}}(\mathbb{Z}[\pi']^{w'}) \\
 & & \downarrow \\
\downarrow & & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z}[\pi]^w \otimes \mathbb{Z}[\pi']^{w'}) \\
 & & \downarrow \\
\mathbf{P}(\pi \times \pi', Z \times Z', w \cdot w') & \xrightarrow{\mathrm{sig}} & \mathbf{L}_{\mathrm{sym}}(\mathbb{Z}[\pi \times \pi']^{w \cdot w'})
\end{array}$$

strictly commute.

## 20. Proof of theorem 19.1

The proof is a modification of the proof of theorem 1.3; the main difference is that we need more elaborate notation.

NOTATION 20.1. (i) For an object $x$ of $\mathcal{T}$ or $\mathcal{R}$, write $\mathcal{A}_x$ (meaning $\mathcal{A}^x_{\mathrm{Rel,sch}}$ for $x \in \mathcal{R}$) for the corresponding $\mathbb{Z}$-graded category and $\mathbf{R}_x$ for the associated object of $\mathcal{S}p_{mss}$.

(ii) Given a $j$-tuple $(x_1, \ldots, x_j)$, where each $x_i$ is an object of $\mathcal{T}$ or $\mathcal{R}$, write

$$[x_1, \ldots, x_j]$$

for $y_1 \otimes \cdots \otimes y_j$, where $y_i$ is $x_i$ if $x_i$ is an object of $\mathcal{R}$ and $\rho(x_i)$ if $x_i$ is an object of $\mathcal{T}$.

(iii) Given a $j$-tuple $(f_1, \ldots, f_j)$, where each $f_i$ is a morphism in $\mathcal{T}$ or $\mathcal{R}$, write

$$[f_1, \ldots, f_j]$$

for $g_1 \otimes \cdots \otimes g_j$, where $g_i$ is $f_i$ if $f_i$ is a morphism in $\mathcal{R}$ and $\rho(f_i)$ if $f_i$ is a morphism in $\mathcal{T}$.

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The reader should see proposition 20.11(i) for motivation for the following definitions.

DEFINITION 20.2. *(i) Let $y$ be an object of $\mathcal{T}$. An entity of type $(r_u(j), y)$ is a $j+1$-tuple $(x_1, \ldots, x_j, f)$, where $f$ is a morphism in $\mathcal{T}$ from $x_1 \boxtimes \cdots \boxtimes x_j$ to $y$.*

*(ii) Let $\mathcal{E}_{r_u(j),y}$ denote the set of entities of type $(r_u(j), y)$.*

*(iii) Let $z$ be an object of $\mathcal{R}$ and let $r : \{1, \ldots, j\} \to \{u, v\}$ be a function. An entity of type $(r, z)$ is a $j+1$-tuple $(x_1, \ldots, x_j, f)$, where each $x_i$ is an object of $\mathcal{T}$ or $\mathcal{R}$ and $f$ is a morphism in $\mathcal{R}$ from $[x_1, \ldots, x_j]$ to $z$.*

*(iv) Let $\mathcal{E}_{r,z}$ denote the set of entities of type $(r, z)$.*

NOTATION 20.3. (i) Let $\mathfrak{S}$ denote the union of the set of objects of $\mathcal{T}$ and the set of objects of $\mathcal{R}$.

(ii) Let $\Pi\mathcal{S}p_{mss}$ be the infinite product of copies of $\mathcal{S}p_{mss}$, indexed over $\mathfrak{S}$.

We will define a monad in $\Pi\mathcal{S}p_{mss}$.

First we need to define the relevant right $\Sigma_j$ actions. Recall definition 16.1(i).

DEFINITION 20.4. *Let $\{\mathbf{X}_x\}_{x\in\mathfrak{S}}$ be an object of $\Pi\mathcal{S}p_{mss}$ and let $j \geq 0$.*

*(i) Given an object $y$ of $\mathcal{T}$ and $\alpha \in \Sigma_j$, define a map $\bar{\alpha}$ from*

$$\bigvee_{(x_1,\ldots,x_j,f)\in\mathcal{E}_{r_u(j),y}} \mathcal{O}(r_u(j); u) \barwedge (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j})$$

*to itself to be the map which takes the summand indexed by $(x_1, \ldots, x_j, f)$ to the summand indexed by $(x_{\alpha(1)}, \ldots, x_{\alpha(1)}, f \circ \alpha)$ by means of the map*

$$\mathcal{O}(r_u(j); u) \barwedge (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j}) \xrightarrow{\alpha\barwedge\alpha} \mathcal{O}(r_u(j); u) \barwedge (\mathbf{X}_{x_{\alpha(1)}} \wedge \cdots \wedge \mathbf{X}_{x_{\alpha(j)}}).$$

*(ii) Given an object $z$ of $\mathcal{R}$ and $\alpha \in \Sigma_j$, define a map $\bar{\alpha}$ from*

$$\bigvee_r \bigvee_{(x_1,\ldots,x_j,f)\in\mathcal{E}_{(r,z)}} \mathcal{O}(r; v) \barwedge (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j})$$

*to itself to be the map which takes the summand indexed by $(x_1, \ldots, x_j, f) \in \mathcal{E}_{(r,z)}$ to the summand indexed by $(x_{\alpha(1)}, \ldots, x_{\alpha(1)}, f \circ \alpha) \in \mathcal{E}_{(r^\alpha,z)}$ by means of the map*

$$\mathcal{O}(r; v) \barwedge (\mathbf{X}_{x_1} \wedge \cdots \wedge \mathbf{X}_{x_j}) \xrightarrow{\tilde{\alpha}\barwedge\alpha} \mathcal{O}(r^\alpha; v) \barwedge (\mathbf{X}_{x_{\alpha(1)}} \wedge \cdots \wedge \mathbf{X}_{x_{\alpha(j)}}).$$

Note that this definition gives right $\Sigma_j$ actions on the objects mentioned.

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Definition 20.5. *Let* $\{\mathbf{X}_x\}_{x\in\mathfrak{S}}$ *be an object of* $\Pi\mathcal{S}p_{mss}$.

*(i) Given an object* $y$ *of* $\mathcal{T}$, *define*

$$\mathbb{P}_y(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

*to be*

$$\bigvee_{j\geq 0}\Big(\bigvee_{(x_1,\ldots,x_j,f)\in\mathcal{E}_{r_u(j),y}} \mathcal{O}(r_u(j);u)\,\overline{\wedge}\,(\mathbf{X}_{x_1}\wedge\cdots\wedge\mathbf{X}_{x_j})\Big)/\Sigma_j.$$

*(ii) Given an object* $z$ *of* $\mathcal{R}$, *define*

$$\mathbb{P}_z(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

*to be*

$$\bigvee_{j\geq 0}\Big(\bigvee_{r}\bigvee_{(x_1,\ldots,x_j,f)\in\mathcal{E}_{(r,z)}} \mathcal{O}(r;v)\,\overline{\wedge}\,(\mathbf{X}_{x_1}\wedge\cdots\wedge\mathbf{X}_{x_j})\Big)/\Sigma_j.$$

*(iii) Define* $\mathbb{P}:\Pi\mathcal{S}p_{mss}\to\Pi\mathcal{S}p_{mss}$ *to be the functor whose projection on the* $y$ *factor (where* $y$ *is an object of* $\mathcal{T}$*) is* $\mathbb{P}_y$ *and whose projection on the* $z$ *factor (where* $z$ *is an object of* $\mathcal{R}$*) is* $\mathbb{P}_z$.

Definition 20.6. *Let* $\{\mathbf{X}_x\}_{x\in\mathfrak{S}}$ *be an object of* $\Pi\mathcal{S}p_{mss}$.

*(i) For an object* $y$ *of* $\mathcal{T}$, *define*

$$\iota_y:\mathbf{X}_y\to\mathbb{P}_y(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

*to be the composite*

$$\mathbf{X}_y\cong\overline{\mathbf{S}}\,\overline{\wedge}\,\mathbf{X}_y=\mathcal{O}(r_u(1);u)\,\overline{\wedge}\,\mathbf{X}_y\hookrightarrow\mathbb{P}_y(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),$$

*where the last map is the inclusion of the summand corresponding to the entity* $(y,\mathrm{id})$.

*(ii) For an object* $z$ *of* $\mathcal{R}$, *define*

$$\iota_z:\mathbf{X}_z\to\mathbb{P}_z(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

*to be the composite*

$$\mathbf{X}_z\cong\overline{\mathbf{S}}\,\overline{\wedge}\,\mathbf{X}_z=\mathcal{O}(r_v(1);v)\,\overline{\wedge}\,\mathbf{X}_y\hookrightarrow\mathbb{P}_z(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),$$

*where the last map is the inclusion of the summand corresponding to the entity* $(z,\mathrm{id})$.

*(iii) Define*

$$\iota:\{\mathbf{X}_x\}_{x\in\mathfrak{S}}\to\mathbb{P}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

*to be the map whose projection on the* $y$ *factor (where* $y$ *is an object of* $\mathcal{T}$*) is* $\iota_y$ *and whose projection on the* $z$ *factor (where* $z$ *is an object of* $\mathcal{R}$*) is* $\iota_z$.

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In order to define the structure map $\mu : \mathbb{P}\mathbb{P} \to \mathbb{P}$ we need a composition operation on entities. For part (ii), we use notation 20.1(iii) and the notation of definition 16.3.

DEFINITION 20.7. *Let $i \geq 0$, and for each $l$ with $1 \leq l \leq i$ let $j_l \geq 0$.*

*(i) Let $y$ be an object of $\mathcal{T}$ and let*

$$\mathbf{e} = (x_1, \ldots, x_i, f) \in \mathcal{E}_{r_u(i),y}.$$

*For each $l$ with $1 \leq l \leq i$ let*

$$\mathbf{e}_l = (x_1^{(l)}, \ldots, x_{j_l}^{(l)}, f^{(l)}) \in \mathcal{E}_{r_u(j_l),x_l}.$$

*Define*

$$\mathbf{e} \circ (\mathbf{e}_1, \ldots, \mathbf{e}_i) \in \mathcal{E}_{r_u(j_1+\cdots+j_i),y}$$

*to be*

$$(x_1^{(1)}, \ldots, x_{j_i}^{(i)}, g),$$

*where $g$ is the composite*

$$x_1^{(1)} \boxtimes \cdots \boxtimes x_{j_i}^{(i)} \xrightarrow{f^{(1)}\boxtimes\cdots\boxtimes f^{(i)}} x_1 \boxtimes \cdots \boxtimes x_i \xrightarrow{f} y.$$

*(ii) Let $z$ be an object of $\mathcal{R}$, let $r : \{1, \ldots, i\} \to \{u, v\}$, and let*

$$\mathbf{e} = (x_1, \ldots, x_i, f) \in \mathcal{E}_{r,z}.$$

*For each $l$ with $1 \leq l \leq i$ let $r_l : \{1, \ldots, j_l\} \to \{u, v\}$; assume that if $r(l) = u$ then $r_l$ is $r_u(j_l)$. Let*

$$\mathbf{e}_l = (x_1^{(l)}, \ldots, x_{j_l}^{(l)}, f^{(l)}) \in \mathcal{E}_{r_l,x_l}.$$

*Define*

$$\mathbf{e} \circ (\mathbf{e}_1, \ldots, \mathbf{e}_i) \in \mathcal{E}_{R,z}$$

*to be*

$$(x_1^{(1)}, \ldots, x_{j_i}^{(i)}, g),$$

*where $g$ is the composite*

$$[x_1^{(1)}, \ldots, x_{j_i}^{(i)}] \xrightarrow{[f^{(1)},\ldots,f^{(i)}]} [x_1, \ldots, x_i] \xrightarrow{f} z.$$

Now we can define $\mu : \mathbb{P}\mathbb{P} \to \mathbb{P}$. We begin with the projection on the $y$-factor,

$$\mu_y : \mathbb{P}_y\mathbb{P} \to \mathbb{P}_y,$$

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where $y$ is an object of $\mathcal{T}$. A collection of entities $\mathbf{e}, \mathbf{e}_1, \ldots, \mathbf{e}_i$ as in definition 20.7(i) determines a summand

$$\mathcal{O}(r_u(i); u) \barwedge \Big( \big(\mathcal{O}(r_u(j_1); u) \barwedge (\mathbf{X}_{x_1^{(1)}} \wedge \cdots \wedge \mathbf{X}_{x_{j_1}^{(1)}})\big) \wedge \cdots \Big)$$

in $\mathbb{P}_y\mathbb{P}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$. We define the restriction of $\mu_y$ to this summand to be the map to the summand of $\mathbb{P}_y(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$ indexed by $\mathbf{e}\circ(\mathbf{e}_1, \ldots, \mathbf{e}_i)$ which is induced (after passage to quotients) by the composite

$$\begin{aligned} &\mathcal{O}(r_u(i); u) \barwedge \Big( \big(\mathcal{O}(r_u(j_1); u) \barwedge (\mathbf{X}_{x_1^{(1)}} \wedge \cdots \wedge \mathbf{X}_{x_{j_1}^{(1)}})\big) \wedge \cdots \Big) \\ &\qquad \to \Big( \mathcal{O}(r_u(i); u) \barwedge \big(\mathcal{O}(r_u(j_1); u) \otimes \cdots \big) \Big) \barwedge (\mathbf{X}_{x_1^{(1)}} \otimes \cdots \otimes \mathbf{X}_{x_{j_i}^{(i)}}) \\ &\qquad \xrightarrow{\gamma \barwedge 1} \mathcal{O}(r_u(j_1 + \cdots + j_i); u) \barwedge (\mathbf{X}_{x_1^{(1)}} \otimes \cdots \otimes \mathbf{X}_{x_{j_i}^{(i)}}). \end{aligned}$$

The projection of $\mu$ on the $z$ factor (where $z$ is an object of $\mathcal{R}$) is defined similarly (using definition 16.3).

Next we give the action of $\mathbb{P}$ on the object $\{\mathbf{R}_x\}_{x\in\mathfrak{S}}$. Let $y$ be an object of $\mathcal{T}$ and let $(x_1, \ldots, x_j, f)$ be an entity of type $(r_u(j), y)$. A slight modification of definition 6.9 gives a map

$$\mathcal{O}(r_u(j); u) \barwedge (\mathbf{R}_{x_1} \wedge \cdots \wedge \mathbf{R}_{x_j}) \to \mathbf{R}_{x_1 \boxtimes \cdots \boxtimes x_j},$$

and composing with the map induced by $f$ gives a map

$$\mathcal{O}(r_u(j); u) \barwedge (\mathbf{R}_{x_1} \wedge \cdots \wedge \mathbf{R}_{x_j}) \to \mathbf{R}_y. \tag{20.1}$$

We define

$$\nu_y : \mathbb{P}_y(\{\mathbf{R}_x\}_{x\in\mathfrak{S}}) \to \mathbf{R}_y$$

to be the map whose restriction to the summand indexed by $(x_1, \ldots, x_j, f)$ is the map (20.1). We define

$$\nu_z : \mathbb{P}_z(\{\mathbf{R}_x\}_{x\in\mathfrak{S}}) \to \mathbf{R}_z$$

similarly when $z$ is an object of $\mathcal{R}$ (using a slight modification of definition 16.7), and we define

$$\nu : \mathbb{P}(\{\mathbf{R}_x\}_{x\in\mathfrak{S}}) \to \{\mathbf{R}_x\}_{x\in\mathfrak{S}}$$

to be the map with projections $\nu_y$ and $\nu_z$.

LEMMA 20.8. *$\nu$ is an action of $\mathbb{P}$ on $\{\mathbf{R}_x\}_{x\in\mathfrak{S}}$.*

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Now we need the analogue of lemma 17.3. Let $\mathcal{C}$ be the category whose objects are triples $(\mathbf{F}, \mathbf{G}, \mathbf{t})$, where $\mathbf{F}$ is a monoidal functor $\mathcal{T} \to \mathcal{S}p_{mss}$, $\mathbf{G}$ is a monoidal functor $\mathcal{R} \to \mathcal{S}p_{mss}$, and $\mathbf{t}$ is a natural transformation $\mathbf{F} \to \mathbf{G} \circ \rho$ which is not required to be a monoidal transformation; the morphisms are commutative diagrams

$$\begin{array}{ccc} \mathbf{F} & \longrightarrow & \mathbf{F}' \\ \mathbf{t} \downarrow & & \downarrow \mathbf{t}' \\ \mathbf{G} \circ \rho & \longrightarrow & \mathbf{G}' \circ \rho, \end{array}$$

where the horizontal arrows are monoidal transformations.

Let us write $\mathbf{R}_{\text{geom}}$ (resp., $\mathbf{R}_{\text{sym}}$) for the functor $\mathcal{T} \to \mathcal{S}p_{mss}$ (resp., $\mathcal{R} \to \mathcal{S}p_{mss}$) which takes $x$ to $\mathbf{R}_x$.

Lemma 20.9. *(i) There is a functor $\Upsilon$ from $\mathbb{P}$ algebras to $\mathcal{C}$ which takes $\{\mathbf{X}_x\}_{x \in \mathfrak{S}}$ to a triple $(\mathbf{F}, \mathbf{G}, \mathbf{t})$ with $\mathbf{F}(y) = \mathbf{X}_y$ and $\mathbf{G}(z) = \mathbf{X}_z$.*

*(ii) $\Upsilon(\{\mathbf{R}_x\}_{x \in \mathfrak{S}})$ is the triple*

$$(\mathbf{R}_{\text{geom}}, \mathbf{R}_{\text{sym}}, \text{sig}_{\text{rel}}).$$

*Proof.* Part (i). Let $\mathbf{X} = \{\mathbf{X}_x\}_{x \in \mathfrak{S}}$ be a $\mathbb{P}$ algebra. Define a functor

$$\mathbf{F} : \mathcal{T} \to \mathcal{S}p_{mss}$$

on objects by $\mathbf{F}(y) = \mathbf{X}_y$ and on morphisms by letting $\mathbf{F}(f : y \to y')$ be the composite

$$\mathbf{X}_y \cong \mathcal{O}(r_u(1); u) \barwedge \mathbf{X}_y \hookrightarrow \mathbb{P}_{y'}(\{\mathbf{X}_x\}_{x \in \mathfrak{S}}) \xrightarrow{\nu_{y'}} \mathbf{X}_{y'},$$

where the unlabelled arrow is the inclusion of the summand indexed by the entity $(y, f)$. The functoriality of $\mathbf{F}$ follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{P}\mathbb{P}\mathbf{X} & \xrightarrow{\mu_{\mathbf{X}}} & \mathbb{P}\mathbf{X} \\ \downarrow \mathbb{P}\nu & & \downarrow \nu \\ \mathbb{P}\mathbf{X} & \xrightarrow{\nu} & \mathbf{X} \end{array}$$

and the definition of $\mu$ in definition 20.7. We define

$$\mathbf{G} : \mathcal{R} \to \mathcal{S}p_{mss}$$

similarly. The proof that $\mathbf{F}$ and $\mathbf{G}$ are monoidal functors is similar to the argument, in the proof in lemma 17.3(i), that $\mathbf{X}$ and $\mathbf{Y}$ are monoids. The functoriality is as above.

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It remains to give the natural transformation

$$\mathbf{t} : \mathbf{F} \to \mathbf{G} \circ \rho.$$

For an object $y$ of $\mathcal{T}$, let $\mathbf{t}_y$ be the composite

$$\mathbf{X}_y \cong \mathcal{O}(r_u(1); v) \barwedge \mathbf{X}_y \hookrightarrow \mathbb{P}_{\rho(y)}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \xrightarrow{\nu_{\rho(y)}} \mathbf{X}_{\rho(y)},$$

where the unlabelled arrow is the inclusion of the summand indexed by the entity $(y, \mathrm{id}) \in \mathcal{E}_{r_u(1),\rho(y)}$. To show that $\mathbf{t}$ is a natural transformation, let $f : y \to y'$ be a morphism in $\mathcal{T}$, and let $z = \rho(y)$, $z' = \rho(y')$. Let $i_1$ be the composite

$$\begin{aligned}\mathbf{X}_y \cong \mathcal{O}(r_u(1); u) \barwedge \mathbf{X}_y \hookrightarrow \mathbb{P}_{y'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) &\cong \mathcal{O}(r_u(1); v) \barwedge \mathbb{P}_{y'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \\ &\hookrightarrow \mathbb{P}_{z'}\mathbb{P}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),\end{aligned}$$

where the first arrow is the inclusion of the summand indexed by the entity $(y, f)$ and the second is the inclusion of the summand indexed by $(y', \mathrm{id})$. Let $j_1$ be the composite

$$\mathbf{X}_{y'} \cong \mathcal{O}(r_u(1); v) \barwedge \mathbf{X}_{y'} \hookrightarrow \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),$$

where the inclusion is indexed by $(y', \mathrm{id})$, and let $j_2$ be the composite

$$\mathbf{X}_y \cong \mathcal{O}(r_u(1); v) \barwedge \mathbf{X}_y \hookrightarrow \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),$$

where the inclusion is indexed by $(y, \rho(f))$.

Consider the commutative diagram

$$\begin{array}{ccccc}
\mathbf{X}_y & & \xrightarrow{\quad = \quad} & & \mathbf{X}_y \\
 & \searrow^{i_1} & & & \downarrow{\scriptstyle j_2} \\
\downarrow{\scriptstyle \mathbf{F}(f)} & & \mathbb{P}_{z'}\mathbb{P}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) & \xrightarrow{\mu} & \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \\
 & & \downarrow{\scriptstyle \mathbb{P}_{z'}\nu} & & \downarrow{\scriptstyle \nu} \\
\mathbf{X}_{y'} & \xrightarrow{j_1} & \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) & \xrightarrow{\nu} & \mathbf{X}_{z'}.
\end{array}$$

Let $H$ denote the composite of the right-hand vertical arrows. Then the diagram shows that the composite

$$\mathbf{X}_y \xrightarrow{\mathbf{F}(f)} \mathbf{X}_{y'} \xrightarrow{\mathbf{t}} \mathbf{X}_{z'} \tag{20.2}$$

is $H$.

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Let $i_2$ be the composite

$$\begin{aligned}\mathbf{X}_y \cong \mathcal{O}(r_u(1); v) \,\overline{\wedge}\, \mathbf{X}_y \hookrightarrow \mathbb{P}_z(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) &\cong \mathcal{O}(r_v(1); v) \,\overline{\wedge}\, \mathbb{P}_z(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \\ &\hookrightarrow \mathbb{P}_{z'}\mathbb{P}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),\end{aligned}$$

where the first inclusion is indexed by $(y, \mathrm{id})$ and the second is indexed by $(z, \rho(f))$. Let $j_2$ be as above and let $j_3$ be the composite

$$\mathbf{X}_z \cong \mathcal{O}(r_v(1); v) \,\overline{\wedge}\, \mathbf{X}_z \hookrightarrow \mathbb{P}_z(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),$$

where the inclusion is indexed by $(z, \rho(f))$.

Consider the commutative diagram

$$\begin{array}{ccccc}
\mathbf{X}_y & & \xrightarrow{\quad\quad =\quad\quad} & & \mathbf{X}_y \\
\Big\downarrow {\scriptstyle \mathbf{t}} & \searrow^{i_1} & & & \Big\downarrow {\scriptstyle j_2} \\
 & & \mathbb{P}_{z'}\mathbb{P}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) & \xrightarrow{\mu} & \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \\
 & & \Big\downarrow {\scriptstyle \mathbb{P}_{z'}\nu} & & \Big\downarrow {\scriptstyle \nu} \\
\mathbf{X}_z & \xrightarrow{j_3} & \mathbb{P}_{z'}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) & \xrightarrow{\nu} & \mathbf{X}_{z'}.
\end{array}$$

This diagram shows that the composite

$$\mathbf{X}_y \xrightarrow{\mathbf{t}} \mathbf{X}_z \xrightarrow{\mathbf{G}(f)} \mathbf{X}_{z'} \tag{20.3}$$

is also $H$, so the composites (20.2) and (20.3) are equal as required.

Part (ii) is an easy consequence of the definitions. □

Finally, we have the analogues of definition 18.1 and proposition 18.2.

DEFINITION 20.10. *Let* $\{\mathbf{X}_x\}_{x\in\mathfrak{S}}$ *be an object of* $\Pi\mathcal{S}p_{mss}$.

*(i) Given an object* $y$ *of* $\mathcal{T}$, *define*

$$\mathbb{P}'_y(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

*to be*

$$\bigvee_{j\geq 0}\Big(\bigvee_{((x_1,\ldots,x_j,f)\in\mathcal{E}_{r_u(j),y}} \mathbf{X}_{x_1}\wedge\cdots\wedge\mathbf{X}_{x_j}\Big)/\Sigma_j.$$

*(ii) Given an object* $z$ *of* $\mathcal{R}$, *define*

$$\mathbb{P}'_z(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

*to be*

$$\bigvee_{j\geq 0}\Big(\bigvee_{r}\ \bigvee_{(x_1,\ldots,x_j,f)\in\mathcal{E}_{(r,z)}} \mathbf{X}_{x_1}\wedge\cdots\wedge\mathbf{X}_{x_j}\Big)/\Sigma_j.$$

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*(iii) Define $\mathbb{P}' : \Pi\mathcal{S}p_{mss} \to \Pi\mathcal{S}p_{mss}$ to be the functor whose projection on the $y$ factor (where $y$ is an object of $\mathcal{T}$) is $\mathbb{P}'_y$ and whose projection on the $z$ factor (where $z$ is an object of $\mathcal{R}$) is $\mathbb{P}'_z$.*

A routine modification of definition 18.1(ii) gives a natural transformation

$$\Xi : \mathbb{P} \to \mathbb{P}'.$$

PROPOSITION 20.11. *(i) An algebra over $\mathbb{P}'$ is the same thing as a pair of symmetric monoidal functors $\mathbf{F}$ and $\mathbf{G}$ with a monoidal transformation $\mathbf{F} \to \mathbf{G} \circ \rho$.*

*(ii) $\Xi$ is a map of monads.*

*(iii) Suppose that each $(X_x)_k$ has compatible degeneracies (see definition 9.1). Let $\mathbb{P}^q$ denote the $q$-th iterate of $\mathbb{P}$. Then each map*

$$\Xi : \mathbb{P}^q(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \to \mathbb{P}'\mathbb{P}^{q-1}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}})$$

*is a weak equivalence.*

*Proof.* Part (i). Let $\{\mathbf{X}_x\}_{x\in\mathfrak{S}}$ be an algebra over $\mathbb{P}'$. The fact that $\mathbf{F}$ and $\mathbf{G}$ are symmetric monoidal functors is an easy consequence of the definitions. The natural transformation $\mathbf{t} : \mathbf{F} \to \mathbf{G} \circ \rho$ is constructed as in the proof of lemma 20.9(i). The proof that $\mathbf{t}$ is monoidal is similar to the proofs of proposition 18.2(i) and lemma 20.9(i), using the maps

$$i_1 : \mathbf{X}_{y_1} \wedge \cdots \wedge \mathbf{X}_{y_j} \hookrightarrow \mathbb{P}'_{y_1\boxtimes\cdots\boxtimes y_j}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \hookrightarrow \mathbb{P}'_{\rho(y_1\boxtimes\cdots\boxtimes y_j)}\mathbb{P}'(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),$$

(where the first inclusion is indexed by $(y_1, \dots, y_j, \mathrm{id})$ and the second by $(y_1 \boxtimes \cdots \boxtimes y_j, \mathrm{id})$), and

$$i_2 : \mathbf{X}_{y_1} \wedge \cdots \wedge \mathbf{X}_{y_j} \hookrightarrow \mathbb{P}'_{\rho(y_1)}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \wedge \cdots \wedge \mathbb{P}'_{\rho(y_j)}(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}) \hookrightarrow \mathbb{P}'_{\rho(y_1\boxtimes\cdots\boxtimes y_j)}\mathbb{P}'(\{\mathbf{X}_x\}_{x\in\mathfrak{S}}),$$

(where the first map is the smash product of the inclusions indexed by $(y_i, \mathrm{id})$ and the second is indexed by $(\rho(y_1), \dots, \rho(y_j), \rho(y_1) \otimes \cdots \otimes \rho(y_j) \to \rho(y_1 \boxtimes \cdots \boxtimes y_j)))$.

Part (ii) is immediate from the definitions, and the proof of part (iii) is the same as for proposition 10.2(iii) (but using remark 15.7(ii)). □

Now the proof of theorem 19.1 is the same as the proof of theorem 1.3 given in §18, with only the notation changed.

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## References


[1] R. Andrade. (mathoverflow.net/users/21095), *Multisimplicial geometric realization*, MathOverflow, http://mathoverflow.net/questions/124837 (version: 2013-03-19).

[2] N. A. Baas and G. Laures. Singularities and Quinn spectra. *MüNster J. Math.* **10** (2017), 1–17. MR 3624098.

[3] M. Banagl, G. Laures and J. E. McClure. The $L$-homology fundamental class for IP-spaces and the stratified Novikov conjecture, *Selecta Math. (N.S.)* **25** (2019), 104. MR 3908854.

[4] A. D. Elmendorf and M. A. Mandell. Rings, modules, and algebras in infinite loop space theory. *Adv. Math.* **205** (2006), 163–228. MR 2254311 (2007g:19001).

[5] M. H. Freedman and F. Quinn. *Topology of 4-manifolds*, Princeton Mathematical Series, Vol. 39, MR 1201584 (Princeton University Press, Princeton, NJ, 1990) (94b:57021).

[6] G. Friedman and J. McClure. *The symmetric signature of a Witt space*, http://arxiv.org/abs/1106.4798.

[7] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, Progress in Mathematics, Vol. 174 (Birkhäuser Verlag, Basel, 1999) MR 1711612 (2001d:55012).

[8] P. S. Hirschhorn. *Model categories and their localizations*, Mathematical Surveys and Monographs, Vol. 99 (American Mathematical Society, Providence, RI, 2003) MR 1944041 (2003j:18018).

[9] M. Hovey, B. Shipley and J. Smith. Symmetric spectra. *J. Amer. Math. Soc.* **13** (2000), 149–208. MR 1695653 (2000h:55016).

[10] P. Kuehl, T. Macko and A. Mole. *The total surgery obstruction revisited*, http://arxiv.org/abs/1104.5092.

[11] G. Laures and J. E. McClure. *Multiplicative properties of Quinn spectra. Forum Math.* **26** (2014), 1117–1185. MR 3228927.

[12] C. Leonard Reedy. *Homotopy theory of model categories*, available at http://www-math.mit.edu/∼psh.

[13] J. Lurie. *Algebraic L-theory and surgery*, notes for a course, available at http://www.math.harvard.edu/ lurie/287x.html.

[14] S. Mac Lane. Natural associativity and commutativity. *Rice Univ. Studies* **49** (1963), 28–46. MR 0170925 (30 #1160).

[15] I. Madsen and R. James Milgram. *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, Vol. 92, MR 548575 (Princeton University Press, Princeton, NJ, 1979) (81b:57014).

[16] J. E. McClure. *On semisimplicial objects satisfying the Kan condition*, http://arxiv.org/abs/1210.5650, to appear in Homology, Homotopy and Applications.

[17] J. Peter May. *The geometry of iterated loop spaces*, Lectures Notes in Mathematics, Vol. 271, MR 0420610 (Springer-Verlag, Berlin, 1972) (54 #8623b).

[18] J. Peter May. Simplicial Objects in Algebraic Topology, Chicago Lectures in Mathematics. MR 1206474 (University of Chicago Press, Chicago, IL, 1992) (93m:55025).

[19] F. Quinn. *Assembly maps in bordism-type theories, Novikov conjectures, index theorems and rigidity*, (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser., Vol. 226, 201–271. MR 1388303 (Cambridge University Press, Cambridge, 1995) (97h:57055).

[20] A. Ranicki. The algebraic theory of surgery. I. *Foundations, Proc. London Math. Soc. (3)* **40** (1980), 87–192. MR 560997 (82f:57024a).

[21] A. Ranicki. The algebraic theory of surgery. II. Applications to topology. *Proc. London Math. Soc. (3)* **40** (1980), 193–283. MR 566491 (82f:57024b).

[22] A. Ranicki. *Algebraic L-theory and topological manifolds*, Cambridge Tracts in Mathematics, Vol. 102, MR 1211640 (Cambridge University Press, Cambridge, 1992) (94i:57051).

[23] C. Schlichtkrull. Thom spectra that are symmetric spectra. *Doc. Math.* **14** (2009), 699–748. MR 2578805 (2011f:55017).

[24] G. Segal. Categories and cohomology theories. *Topology* **13** (1974), 293–312. MR 0353298 (50 #5782).

[25] D. P. Sullivan. *The 1970 MIT notes*, Edited and with a preface by Andrew Ranicki, $K$-Monographs in Mathematics, Vol. 8, MR 2162361 (Springer, Dordrecht, 2005) (2006m:55002).

[26] C. A. Weibel. *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, 1994).

## Appendix A. A property of the smash product of symmetric spectra.

For an object $\mathbf{X}$ of $\mathcal{S}p_{mss}$ let

$$\bar{\omega} : X_k \to X_{k+1}$$

be the map which takes $x$ to $\omega(s \wedge x)$ (where $s$ is the 1-simplex of $S^1$). More precisely, $\bar{\omega}$ is the $k$-fold multisemisimplicial map which takes $(X_k)_{n_1,\ldots,n_k}$ to $(X_k)_{1,n_1,\ldots,n_k}$ obtained from the suspension map of the spectrum $\mathbf{X}$.

DEFINITION A.1. *An object $\mathbf{X}$ of $\mathcal{S}p_{mss}$ is monomorphic if $\bar{\omega}$ is a monomorphism for every $k$. It is strongly monomorphic if it is monomorphic and has the following property: if $x \in X_k$ and $\alpha \in \Sigma_{k+1}$ with $\alpha(1) \neq 1$, and if $\alpha\bar{\omega}x$ is in the image of $\bar{\omega}$, then there is a $\beta \in \Sigma_k$ such that $\beta^{-1}(1) = \alpha^{-1}(1) - 1$ and $\beta x$ is in the image of $\bar{\omega}$.*

The purpose of this appendix is to prove the following fact, which is used in §10:

PROPOSITION A.2. *Let $\mathbf{X}$ be strongly monomorphic and suppose that the only element of $X_0$ is the basepoint. Then the $\Sigma_j$ action on $\mathbf{X}^{\wedge j}$ which permutes the factors is free away from the basepoints.*

REMARK A.3. (i) The main object $\mathbf{R}$ given in example 4.13 is strongly monomorphic: for an ad $F \in R_k$, the suspension map was defined by $\omega(s \wedge F) = \lambda^*(F)$ where $\lambda$ is the incidence-compatible isomorphism from $\mathcal{C}ell(\Delta^1 \times K, \partial\Delta^1 \times K)$ to $\mathcal{C}ell(K)$. Suppose we are given a multisimplex of the form $\alpha\bar{\omega}F$ which is in the image of $\bar{\omega}$ as above. Then it necessarily defines a functor with source

$$\mathcal{C}ell(\Delta^1 \times K_1 \times \Delta^1 \times K_2, (\partial\Delta^1 \times K_1 \times \Delta^1 \times K_2) \cup (\Delta^1 \times K_1 \times \partial\Delta^1 \times K_2))$$

for some products of simplices $K_1$,$K_2$. Thus a permutation $\beta$ which exchanges the factors $K_1$ and $\Delta^1$ has the property that $\beta F$ is in the image of $\bar{\omega}$.

(ii) The analogue of proposition A.2 for simplicial or topological symmetric spectra is also true, with essentially the same proof.

The proof of proposition A.2 will be given after the proof of our next result, which is the main ingredient in the proof of proposition A.2.

PROPOSITION A.4. *If $\mathbf{X}_1, \ldots, \mathbf{X}_j$ are strongly monomorphic then $\mathbf{X}_1 \wedge \cdots \wedge \mathbf{X}_j$ is monomorphic.*

We will give an example at the end of this appendix to show that propositions A.2 and A.4 both fail if 'strongly monomorphic' is replaced by 'monomorphic'.

Before we can give the proof of proposition A.4, we will need quite a bit of background, culminating with lemma A.15.

If $A$ is a multisemisimplicial set and $a \in A_{\mathbf{n}}$ for some $\mathbf{n}$, we will say *$a$ is a point of $A$*. If $\mathbf{X}$ is an object of $\mathcal{S}p_{mss}$ and $x$ is a point of $X_k$ for some $k$ we will say *$x$ is a point of* $\mathbf{X}$ and write $|x| = k$.

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We will use the fact that (by remark 4.7) points in the $k$-fold smash product $\mathbf{X}_1 \wedge \cdots \wedge \mathbf{X}_j$ are equivalence classes of symbols of the form

$$\theta \wedge x_1 \wedge \cdots \wedge x_j$$

where $x_i$ is a point of $\mathbf{X}_i$ for each $i$ and $\theta \in \Sigma_{|x_1|+\cdots+|x_j|}$. Our first task is to describe the equivalence relation $\sim$ explicitly, and for this we need the operations in definition A.7.

Notation A.5. (i) Let $\alpha \in \Sigma_l$, let $L \geq 1$, and let $k \geq L+l-1$. We will write $\alpha^{[L]}$ for the element of $\Sigma_k$ which permutes $L, L+1, \ldots, L+l-1$ in the same way that $\alpha$ permutes $1, \ldots, l$.

(ii) Let $\mathbf{X}$ be an object of $\mathcal{S}p_{mss}$. Let

$$\psi : \mathbf{X} \wedge \mathbf{S} \to \mathbf{X}$$

be the right action (that is, $\psi = \omega \circ \tau$, where $\tau$ is defined in Eq. (4.1)), and let

$$\bar{\psi} : X_k \to X_{k+1}$$

be the map that takes $x$ to $\psi(x \wedge s)$, where $s$ is the 1-simplex of $S^1$. Note that

$$\bar{\psi}x = \rho_{1,k}\bar{\omega}x, \tag{A.1}$$

where $\rho_{1,k}$ is defined after Eq. (4.1).

Remark A.6. (i) If $x$ is a point of $\mathbf{X}$ and $\alpha \in \Sigma_{|x|}$ then $\bar{\omega}\alpha x = \alpha^{[2]}\bar{\omega}x$ and $\bar{\psi}\alpha x = \alpha^{[1]}\bar{\psi}x$.

(ii) $\bar{\psi}$ commutes with $\bar{\omega}$.

(iii) Suppose that $\mathbf{X}$ is strongly monomorphic $\mathcal{S}p_{mss}$. Using Eq. A.1, we see that if $x$ is a point of $\mathbf{X}$ and $\gamma$ is an element of $\Sigma_{|x|+1}$ for which $\gamma(1) \neq |x|+1$ and $\gamma\bar{\omega}x$ is in the image of $\bar{\psi}$, then there is a $\delta \in \Sigma_{|x|}$ for which $\delta^{-1}(|x|) = \gamma^{-1}(|x|+1) - 1$ and $\delta x$ is in the image of $\bar{\psi}$.

Definition A.7. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_j$ be monomorphic objects of $\mathcal{S}p_{mss}$. Let $x_i$ be a point of $\mathbf{X}_i$ for $1 \leq i \leq j$, and let $\theta \in \Sigma_{|x_1|+\cdots+|x_j|}$.*

*(i) For $1 \leq m \leq j$ and $\alpha$ an element of $\Sigma_{|x_m|}$ other than the identity define*

$$A_{m,\alpha}(\theta \wedge x_1 \wedge \cdots \wedge x_j) = (\theta \circ (\alpha^{[1+\sum_{i<m}|x_i|]})^{-1}) \wedge y_1 \wedge \cdots \wedge y_j,$$

*where $\alpha^L$ was defined in notation A.5 and*

$$y_i = \begin{cases} x_i & \text{if } i \neq m \\ \alpha x_m & \text{if } i = m. \end{cases}$$

*(ii) Suppose that $m < j$ and that $x_m = \bar{\psi}z$ for some $z$ (in which case $z$ is uniquely determined since $\mathbf{X}_m$ is monomorphic) define*

$$B_m(\theta \wedge x_1 \wedge \cdots \wedge x_j) = \theta \wedge y_1 \wedge \cdots \wedge y_j,$$

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*where*

$$y_i = \begin{cases} x_i & \text{if } i \neq m, m+1 \\ z & \text{if } i = m \\ \bar{\omega} x_{m+1} & \text{if } i = m+1. \end{cases}$$

*(iii) If $m < j$ and $x_{m+1} = \bar{\omega} z$ for some $z$ define*

$$C_m(\theta \wedge x_1 \wedge \cdots \wedge x_j) = \theta \wedge y_1 \wedge \cdots \wedge y_j,$$

*where*

$$y_i = \begin{cases} x_i & \text{if } i \neq m, m+1 \\ \bar{\psi} x_m & \text{if } i = m \\ z & \text{if } i = m+1. \end{cases}$$

Now we can give an explicit description of the equivalence relation $\sim$, as follows: $\theta \wedge x_1 \wedge \cdots \wedge x_j \sim \kappa \wedge y_1 \wedge \cdots \wedge y_j$ if and only if there is a composable sequence $D_1, \ldots, D_n$ of operations of the types given in definition A.7 with

$$\kappa \wedge y_1 \wedge \cdots \wedge y_j = D_n \cdots D_1(\theta \wedge x_1 \wedge \cdots \wedge x_j).$$

We will only indicate why these operation generate: the operations $A_{m,\alpha}$ take care of the equivalences described in remark 4.7, the operations $B_m$ and $C_m$ generate the equivalences which come from the coequalizer diagram in definition 4.14.

In the situation of an equivalence, we will say that the $n$-tuple $P = (D_1, \ldots, D_n)$ is a *path* from $\theta \wedge x_1 \wedge \cdots \wedge x_j$ to $\kappa \wedge y_1 \wedge \cdots \wedge y_j$ and we will write

$$P(\theta \wedge x_1 \wedge \cdots \wedge x_j) = \kappa \wedge y_1 \wedge \cdots \wedge y_j.$$

The *length* of $P$ is the number of $A$-operations in $P$ plus twice the number of $B$-operations and twice the number of $C$-operations.

Notation A.8. If $\alpha$ is the identity element then $A_{m,\alpha}$ will be interpreted as the empty path.

We record some useful calculations in our next two lemmas. We define the *standard path of length 3p starting at $\theta \wedge \bar{\omega} x_1 \wedge x_2 \wedge \cdots \wedge x_j$* with $p < j$ to be the path

$$A_{1,\rho_{1,|x_1|}}, B_1, A_{2,\rho_{1,|x_2|}}, B_2, \ldots, A_{p,\rho_{1,|x_p|}}, B_p.$$

Note that the $B$-operations in this sequence are always possible because of Eq. (A.1) and that $A_{i,\rho_{1,|x_i|}}$ is interpreted as the empty path when $|x_i| = 0$.

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Lemma A.9. *Let $P$ be the standard path of length $3p$ starting at $\theta\wedge\bar\omega x_1\wedge x_2\wedge\cdots\wedge x_j$. Then*

$$P(\theta\wedge\bar\omega x_1\wedge\cdots\wedge x_j)=\theta(\rho^{[1]}_{1,|x_1|+\cdots+|x_p|})^{-1}\wedge x_1\wedge\cdots\wedge\bar\omega x_{p+1}\wedge\cdots.$$

*Proof.* This follows by induction on $p$ from Eq. A.1 and the equation $\rho^{[s+1]}_{1,t}\rho^{[1]}_{1,s}=\rho^{[1]}_{1,s+t}$ in the group $\Sigma_{s+t+1}$. □

Remark A.10. The operations in definition A.7 often commute with each other. Specifically, $A_{m,\alpha}$ commutes with $A_{n,\beta}$ for $m\neq n$ and with $B_n$ and $C_n$ for $m\neq n,n-1$; moreover, $B_m$ commutes with every $B_n$ and every $C_n$ and $C_m$ commutes with every $C_n$.

Lemma A.11. *Let $P$ be as in lemma A.9, and let $Q$ be any path starting at $\theta\wedge\bar\omega x_1\wedge x_2\wedge\cdots\wedge x_j$ of the form*

$$A_{1,\alpha_1},B_1,A_{2,\alpha_2},B_2,\dots,A_{p,\alpha_p},B_p$$

*where $\alpha_i(1)=|x_i|+1$ for each $i$. For each $i$ let $\beta_i$ be the element of $\Sigma_{|x_i|}$ with $\beta_i^{[2]}=\rho^{-1}_{1,|x_i|}\alpha_i$ (which exists because $\rho^{-1}_{1,|x_i|}\alpha_i$ takes 1 to 1). Then*

*(i) the path*

$$A_{1,\beta_1^{[2]}},A_{2,\beta_2},\dots,A_{p,\beta_p},P$$

*has the same effect on $\theta\wedge\bar\omega x_1\wedge x_2\wedge\cdots\wedge x_j$ as $Q$, and*

(ii) the path

$$P,A_{1,\beta_1},A_{2,\beta_2},\dots,A_{p,\beta_p}$$

also has the same effect on $\theta\wedge\bar\omega x_1\wedge x_2\wedge\cdots\wedge x_j$ as $Q$.

*Proof.* The proof in each case is by induction on $p$.

For part (i), the case $p=1$ is immediate. For $p>1$, we observe that

$$A_{1,\beta_1^{[2]}},A_{2,\beta_2},\dots,A_{p-1,\beta_{p-1}},P,A_{p,\alpha_p},B_p$$

has the same effect as

$$A_{1,\beta_1^{[2]}},A_{2,\beta_2},\dots,A_{p-1,\beta_{p-1}},P,A_{p,\beta_p^{[2]}},A_{p,\rho_{1,|x_p|}},B_p$$

and by lemma A.9 this has the same effect as

$$A_{1,\beta_1^{[2]}},A_{2,\beta_2},\dots,A_{p,\beta_p},P,A_{p,\rho_{1,|x_p|}},B_p$$

as required.

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For part (ii), we use the equation $\rho_{1,|x_p|}\beta_p^{[2]} = \beta_p^{[1]}\rho_{1,|x_p|}$. This shows that

$$P, A_{1,\beta_1}, A_{2,\beta_2}, \ldots, A_{p-1,\beta_{p-1}}, A_{p,\alpha_p}, B_p$$

has the same effect as

$$P, A_{1,\beta_1}, A_{2,\beta_2}, \ldots, A_{p-1,\beta_{p-1}}, A_{p,\rho_{1,|x_p|}}, A_{p,\beta_p^{[1]}}, B_p$$

which has the same effect as

$$P, A_{1,\beta_1}, A_{2,\beta_2}, \ldots, A_{p-1,\beta_{p-1}}, A_{p,\rho_{1,|x_p|}}, B_p, A_{p,\beta_p}$$

and by remark A.10 this has the same effect as

$$P, A_{p,\rho_{1,|x_p|}}, B_p, A_{1,\beta_1}, A_{2,\beta_2}, \ldots, A_{p,\beta_p}$$

as required. □

Notation A.12. Given a symbol $\mathbf{x} = \theta \wedge x_1 \wedge \cdots \wedge x_j$ as above, meaning a representative of the equivalence relation, we write $\bar{\omega}\mathbf{x}$ for $\theta^{[2]} \wedge \bar{\omega}x_1 \wedge \cdots \wedge x_j$.

Definition A.13. *Given symbols* $\mathbf{x}$ *and* $\mathbf{y}$*, and a path* $P$ *from* $\bar{\omega}\mathbf{x}$ *to* $\bar{\omega}\mathbf{y}$*, we will say that* $\mathbf{x}$ *and* $\mathbf{y}$ *are* $P$*-related.*

With this terminology, proposition A.4 is true if the following statement is true for all $P$:

$$\text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are } P\text{-related then } \mathbf{x} \sim \mathbf{y}. \qquad (*)$$

Before proving proposition A.4, we need two more lemmas.

Lemma A.14. *Let* $\mathbf{x} = \theta \wedge x_1 \wedge \cdots \wedge x_j$ *and let* $P$ *be a path starting at* $\mathbf{x}$*. Suppose that* $P$ *consists of a single operation* $D$ *and that*

*(i) if* $D = A_{1,\alpha}$ *then* $\alpha(1) = 1$*, and*

*(ii) if* $D = B_1$ *then* $|x_1| \neq 0$*.*

*Then there is a unique* $\mathbf{y}$ *with* $P(\bar{\omega}\mathbf{x}) = \bar{\omega}\mathbf{y}$*, and* $\mathbf{x} \sim \mathbf{y}$*. In particular, Statement (*) is true for* $P$*.*

*Proof.* If $D = A_{1,\alpha}$ with $\alpha(1) = 1$ then there is a $\beta$ with $\alpha = \beta^{[2]}$, and

$$P(\bar{\omega}\mathbf{x}) = P(\theta^{[2]} \wedge \bar{\omega}x_1 \wedge \cdots \wedge x_j) = \theta^{[2]}(\beta^{[2]})^{-1} \wedge \bar{\omega}\beta x_1 \wedge \cdots \wedge x_j,$$

so we can let

$$\mathbf{y} = \theta\beta^{-1} \wedge \beta x_1 \wedge \cdots \wedge x_j$$

which is the unique choice since $\mathbf{X}_1$ is monomorphic. We have $\mathbf{y} = A_{1,\beta}\mathbf{x}$, so $\mathbf{x} \sim \mathbf{y}$.

Next suppose $D = B_1$ and that $|x_1| \neq 0$. Since $B_1$ can be applied to $\mathbf{x}$ and we have (by definition of $B_1$) that $x_1 = \bar{\psi}z$, and so

$$\bar{\omega}x_1 = \bar{\omega}\bar{\psi}z = \bar{\psi}\bar{\omega}z \tag{A.2}$$

for some $z$, and then

$$P(\bar{\omega}\mathbf{x}) = \theta^{[2]} \wedge z \wedge \bar{\omega}x_2 \cdots .$$

Combining Eq. (A.2) and (A.1), we have $\rho_{1,|x_1|}\bar{\omega}z = \bar{\omega}x_1$, so since $\mathbf{X}_1$ is strongly monomorphic there is a $\beta \in \Sigma_{|x_1|}$ with $\beta^{-1}(1) = 1$ and

$$\beta z = \bar{\omega}w$$

for some $w$. There is a $\tilde{\beta} \in \Sigma_{|x_1|-1}$ with $\beta = \tilde{\beta}^{[2]}$ and we have

$$z = (\tilde{\beta}^{[2]})^{-1}\bar{\omega}w = \bar{\omega}\tilde{\beta}^{-1}w. \tag{A.3}$$

Now let

$$\mathbf{y} = \theta \wedge \tilde{\beta}^{-1}w \wedge \bar{\omega}x_2 \cdots .$$

We have $\bar{\omega}\mathbf{y} = P(\bar{\omega}\mathbf{x})$ by Eq. (A.3). We claim that

$$x_1 = \bar{\psi}\tilde{\beta}^{-1}w.$$

Assuming this for the moment, we have $\mathbf{y} = B_1\mathbf{x}$, so $\mathbf{x} \sim \mathbf{y}$ as required. Since $\mathbf{X}_1$ is monomorphic the claim follows from the equations

$$\bar{\omega}x_1 = \bar{\psi}z = \bar{\psi}\bar{\omega}\tilde{\beta}^{-1}w = \bar{\omega}\bar{\psi}\tilde{\beta}^{-1}w,$$

where we have used Eqs. (A.2) and (A.3) and remark A.6(ii).

The remaining cases are easy. □

Lemma A.15. *Let $Q$ be a path which can be written as a composite $Q_1, Q_2$, where Statement (*) is true for $Q_2$ and every operation in $Q_1$ satisfies the hypothesis of lemma A.14. Then Statement (*) is true for $Q$.*

*Proof.* By an iterated application of lemma A.14, $Q_1(\bar{\omega}\mathbf{x})$ has the form $\bar{\omega}\mathbf{z}$ for some $\mathbf{z}$ and $\mathbf{x} \sim \mathbf{z}$. But also $\mathbf{z} \sim \mathbf{y}$ since the symbols $\mathbf{z}$ and $\mathbf{y}$ are $Q_2$-related. □

*Proof of proposition A.4.* We will prove that Statement (*) holds for all $P$. So suppose that $P$ is a path of length $r$ from $\bar{\omega}\mathbf{x}$ to $\bar{\omega}\mathbf{y}$.

If $r = 0$ then $\mathbf{x} = \mathbf{y}$ since $\mathbf{X}_1$ is monomorphic.

Suppose that $r > 0$ and that the result holds for all paths of length $< r$.

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Let

$$\mathbf{x} = \theta \wedge x_1 \wedge \cdots \wedge x_j$$

and let

$$\mathbf{y} = \kappa \wedge y_1 \wedge \cdots \wedge y_j.$$

Let $p$ be the largest number for which $P$ begins with a path of the form

$$A_{1,\alpha_1}, B_1, A_{2,\alpha_2}, B_2, \ldots, A_{p,\alpha_p}, B_p$$

where $\alpha_i(1) = |x_i| + 1$ for each $i$. Denote this path by $P_1$ ($p$ is allowed to be 0 in which case $P_1$ is the empty path).

Lemmas A.11(i) and A.9 imply that $P \neq P_1$: if $|x_1| = 0$ this is because $x_1$ cannot equal $\bar{\omega} y_1$, and otherwise it's because $\theta^{[2]}(\beta_1^{[2]})^{-1} \cdots (\beta_p^{2+\sum_{i<p}|x_i|]})^{-1}(\rho^{[1]}_{1,\sum_{i\leq p}|x_i|})^{-1}$ does not take 1 to 1 and so cannot be equal to $\kappa^{[2]}$. Let $D$ be the next operation in $P$ and let $P_2$ be the part of $P$ after $D$. There are three cases.

**Case 1.** Suppose that $D = A_{m,\gamma}$ for some $\gamma$, with $m \neq p+1$.

If $m > p+1$ then $D$ commutes with all the earlier operations in $P$ (if any), so $P$ has the same effect as

$$D, P_1, P_2.$$

Since the length of $P_1, P_2$ is $r-1$ we are done by lemma A.15 (taking $Q_1 = D$ in that lemma).

If $m \leq p$, lemma A.11(ii) shows that $P_1, D$ has the same effect as

$$Q = (A_{1,\alpha'_1}, B_1, A_{2,\alpha'_2}, B_2, \ldots, A_{p,\alpha'_p}, B_p)$$

where

$$\alpha'_i = \begin{cases} \alpha_i & \text{if } i \neq m, \\ \rho_{1,|x_m|}\gamma^{[2]}\rho^{-1}_{1,|x_m|}\alpha_m & \text{if } i = m. \end{cases}$$

Then $P$ has the same effect as $Q, P_2$, which has length $r-1$.

**Case 2.** Suppose that $D$ is $B_m$ or $C_m$ for some $m$.

If $m > p$ then $D$ commutes with all earlier operations in $P$ (if any), so we are done by lemma A.15 (taking $Q_1 = D$; note that if $p = 0$ and $|x_1| = 0$ then we cannot have $D = B_1$ because of the way $p$ was chosen).

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Let $P_1'$ be the standard path of length $3p$ starting at $\bar{\omega}\mathbf{x}$ and let $Q$ be the path $P_1', D, P_2$. Lemma A.11(i) says that $P_1$ has the same effect as

$$A_{1,\beta_1^{[2]}}, A_{2,\beta_2}, \ldots, A_{p,\beta_p}, P_1'.$$

By lemma A.14, there is a symbol $\mathbf{z}$ such that the path

$$A_{1,\beta_1^{[2]}}, A_{2,\beta_2}, \ldots, A_{p,\beta_p}$$

takes $\bar{\omega}\mathbf{x}$ to $\bar{\omega}\mathbf{z}$ and $\mathbf{x} \sim \mathbf{z}$. Then $\mathbf{z}$ is $Q$-related to $\mathbf{y}$, and to complete Case 2 it suffices to show that $\mathbf{z} \sim \mathbf{y}$.

If $D$ is $B_m$ or $C_m$ with $m < p$, lemma A.9 shows that $P_1', D$ has the same effect as $D, P_1'$. Hence $Q$ has the same effect as $D, P_1', P_2$, and we are done by lemma A.15 since $P_1', P_2$ has length $< r$ (note that if $D = B_1$ in this situation then $|x_1|$ cannot be 0 since it would not be possible to apply $B_1$ to $P_1'(\mathbf{z})$).

If $D = C_p$ then $P_1', D$ has the same effect as

$$R = (A_{1,\rho_{1,|x_1|}}, B_1, A_{2,\rho_{1,|x_2|}}, B_2, \ldots, A_{p,\rho_{1,|x_p|}})$$

since $C_p$ and $B_p$ are inverses of each other. Then $Q$ has the same effect as $R, P_2$, which has length $r - 2$.

If $D = B_p$, lemma A.9 shows that $P_1', D$ has the same effect as $D, P_1', A_{p+1,\gamma}$, where $\gamma$ is the transposition (12). Hence $Q$ has the same effect as $D, P_1', A_{p+1,\gamma}, P_2$, and we are done by lemma A.15 since $P_1', A_{p+1,\gamma}, P_2$ has length $r - 1$ (this is why operations of type $B$ count for 2 in the definition of length); note that we cannot have $D = B_1$ and $|x_1| = 0$ in this situation since $p$ would be 1 and then $P_1$ would be $B_1$, and it would not be possible to apply the sequence $P_1, D$ to $\mathbf{x}$.

**Case 3.** Suppose that $D = A_{p+1,\gamma}$ for some $\gamma$.

First suppose $\gamma(1) = 1$, which implies $\gamma = \tilde{\gamma}^{[2]}$ for some $\tilde{\gamma}$. If $p = 0$ we're done by lemma A.14. If $p > 0$ lemmas A.11(i) and A.9 show that $P_1, D$ has the same effect as $A_{p+1,\gamma'}, P_1$, so $P$ has the same effect as $A_{p+1,\gamma'}, P_1, P_2$ and we're done by lemma A.15.

For the rest of Case 3 we suppose that

$$\gamma(1) \neq 1. \tag{A.4}$$

Lemmas A.11(i) and A.9 show that $P$ cannot equal $P_1, D$: if $|x_1| = 0$ this is because $x_1$ cannot equal $\bar{\omega}y_1$, and otherwise it's because the permutation

$$\theta^{[2]}(\beta_1^{[2]})^{-1} \cdots (\beta_p^{2+\sum_{i<p}|x_i|]})^{-1}(\rho^{[1]}_{1,\sum_{i\leq p}|x_i|})^{-1}(\gamma^{[1+\sum_{i\leq p}|x_i|]})^{-1}$$

does not take 1 to 1 and so cannot be equal to $\kappa^{[2]}$. Let $E$ be the next operation in $P$ and let $P_2'$ be the part of $P$ after $E$; then we have

$$P = (P_1, D, E, P_2'). \tag{A.5}$$

There are three cases to consider.

**Case 3.1.** $E = A_{m,\delta}$ for some $\delta$.

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If $m = p+1$ we can combine $D$ and $E$ into a single operation, which gives a path of length $r-1$ from $\mathbf{x}$ to $\mathbf{y}$.

Otherwise we can commute $E$ past $D$, which reduces to Case 1.

**Case 3.2.** $E = B_m$ for some $m$.

If $m \neq p, p+1$ we can commute $E$ past $D$, which reduces to Case 2.

If $m = p$ then $D, E$ has the same effect as $B_m, A_{p+1,\gamma^{[2]}}$, which reduces to Case 2.

So suppose $m = p+1$. Then

$$\gamma(1) \neq |x_{p+1}| + 1, \tag{A.6}$$

because of the way $p$ was chosen. Denote $P_1(\mathbf{x})$ by $\mathbf{z}$, and let

$$\mathbf{z} = \lambda \wedge z_1 \wedge \cdots \wedge z_j. \tag{A.7}$$

Then

$$z_{p+1} = \bar{\omega} x_{p+1} \tag{A.8}$$

by lemmas A.11 and A.9.

Because the operation $B_{p+1}$ can be applied to $A_{p+1,\gamma}(\mathbf{z})$, we have that

$$\gamma z_{p+1} = \bar{\psi} w \tag{A.9}$$

for some $w$, and then

$$ED(\mathbf{z}) = \lambda(\gamma^{[|z_1|+\cdots+|z_p|+1]})^{-1} \wedge \cdots w \wedge \bar{\omega} z_{p+2} \cdots . \tag{A.10}$$

Equations (A.6), (A.8), and (A.9) allow us to apply remark A.6(iii) to get a $\delta \in \Sigma_{|x_{p+1}|}$ with the properties that $\delta^{-1}(|x_{p+1}|) = \gamma^{-1}(|x_{p+1}|+1) - 1$ and

$$\delta x_{p+1} = \bar{\psi} v \tag{A.11}$$

for some $v$. Let

$$\varepsilon = \gamma(\delta^{[2]})^{-1}. \tag{A.12}$$

Then $\varepsilon$ takes $|x_{p+1}|+1$ to itself, so there is an $\tilde{\varepsilon} \in \Sigma_{|x_{p+1}|}$ with $\varepsilon = \tilde{\varepsilon}^{[1]}$.

We claim that the operation $B_{p+1}$ can be applied to $A_{p+1,\delta^{[2]}}(\mathbf{z})$ and that $A_{p+1,\delta^{[2]}}, B_{p+1}, A_{p+1,\tilde{\varepsilon}}$ has the same effect on $\mathbf{z}$ as $D, E$. Let us assume this for the moment. If $p = 0$ then (by Eq. (A.5)) $P$ has the same effect as $A_{1,\delta^{[2]}}, B_1, A_{1,\tilde{\varepsilon}}, P_2'$; since the length of $A_{1,\tilde{\varepsilon}}, P_2'$ is $r-2$ we're done by lemma A.15 (note that $|x_1| \neq 0$ because of Eq. A.4). If $p \neq 0$ then $P$ has the same effect as $P_1, A_{p+1,\delta^{[2]}}, B_{p+1}, A_{p+1,\tilde{\varepsilon}}, P_2'$, which (by lemmas A.11 and A.9) has the same effect as $A_{p+1,\delta}, P_1, B_{p+1}, A_{p+1,\tilde{\varepsilon}}, P_2'$. Case 2 applies to part of this path after the first operation, so we're done by lemma A.15.

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It remains to verify the claim. By Eqs. (A.8) and (A.11) and remark A.6(ii), we have

$$\delta^{[2]} z_{p+1} = \bar{\omega}\delta x_{p+1} = \bar{\omega}\bar{\psi}v = \bar{\psi}\bar{\omega}v,$$

so (using Eq. (A.7)) $B_{p+1}$ can be applied to $A_{p+1,\delta^{[2]}}(\mathbf{z})$ and we have

$$A_{p+1,\tilde{\varepsilon}} B_{p+1} A_{p+1,\delta^{[2]}}(\mathbf{z}) \\ = \lambda((\delta^{[2]})^{-1}(\tilde{\varepsilon}^{[1]})^{-1})^{[|z_1|+\cdots+|z_p|+1]} \wedge \cdots \tilde{\varepsilon}\bar{\omega}v \wedge \bar{\omega}z_{p+2} \cdots .$$

By Eq. (A.10), it suffices to show $\tilde{\varepsilon}^{[1]}\delta^{[2]} = \gamma$ (which follows from Eq. (A.12)) and $w = \tilde{\varepsilon}\bar{\omega}v$. Because $\mathbf{X}_{p+1}$ is monomorphic, for the latter equation, it suffices to show $\bar{\psi}w = \bar{\psi}\tilde{\varepsilon}^{[1]}\bar{\omega}v$, and this in turn follows from

$$\bar{\psi}w = \gamma z_{p+1} = \gamma\bar{\omega}x_{p+1} = \varepsilon\delta^{[2]}\bar{\omega}x_{p+1} \\ = \varepsilon\bar{\omega}\delta x_{p+1} = \varepsilon\bar{\omega}\bar{\psi}v = \varepsilon\bar{\psi}\bar{\omega}v = \bar{\psi}\tilde{\varepsilon}^{[1]}\bar{\omega}v,$$

where we have used (in this order) Eqs. (A.9), (A.8), (A.12), remark A.6(i), Eq. A.11, and remark A.6(ii).

**Case 3.3.** $E = C_m$ for some $m$.

If $m \neq p, p+1$ we can commute $E$ past $D$, which reduces to Case 2.

If $m = p+1$ then $D, E$ has the same effect as $C_m, A_{p+1,\gamma^{[1]}}$, which reduces to Case 2.

So suppose $m = p$ (which implies $p \neq 0$). Let $P_0$ be $A_{1,\alpha_1}, B_1, A_{2,\alpha_2}, B_2, \ldots, A_{p-1,\alpha_p}, B_{p-1}$ if $p > 1$ and the empty path if $p = 1$; note that

$$P = (P_0, A_{p,\alpha_p}, B_p, D, E, P_2'). \tag{A.13}$$

We denote $A_{p,\alpha_p}P_0(\mathbf{x})$ by $\mathbf{z}$ and let

$$\mathbf{z} = \lambda \wedge z_1 \wedge \cdots \wedge z_j. \tag{A.14}$$

Because the operation $B_p$ can be applied to $\mathbf{z}$, we have

$$z_p = \bar{\psi}w \tag{A.15}$$

for some $w$. Because $C_p$ can be applied to $A_{p+1,\gamma}B_p\mathbf{z}$, we have

$$\gamma\bar{\omega}z_{p+1} = \bar{\omega}v \tag{A.16}$$

for some $v$, and

$$EDB_p(\mathbf{z}) = \lambda(\gamma^{-1})^{[|z_1|+\cdots+|z_p|+1]}) \wedge \cdots \bar{\psi}w \wedge v \cdots . \tag{A.17}$$

Since $\mathbf{X}_{p+1}$ is strongly monomorphic, Eqs. (A.4) and (A.16) imply that there is a $\delta \in \Sigma_{|x_{p+1}|}$ with the properties that $\delta^{-1}(1) = \gamma^{-1}(1) - 1$ and

$$\delta z_{p+1} = \bar{\omega}u \tag{A.18}$$

for some $u$.

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Let $\varepsilon$ be the transposition (12). Let

$$\eta = \gamma(\delta^{[2]})^{-1}(\varepsilon^{[1]})^{-1}. \tag{A.19}$$

Then $\eta$ takes 1 to 1, so there is an $\tilde{\eta} \in \Sigma_{|x_{p+1}|}$ with $\eta = \tilde{\eta}^{[2]}$. We claim that the sequence of operations

$$A_{p+1,\delta}, C_p, A_{p,\varepsilon^{[|z_p|+1]}}, B_p, A_{p+1,\tilde{\eta}}$$

can be applied to $\mathbf{z}$ and that it has the same effect as $B_p, D, E$. Assuming this for the moment, we see (using Eq. (A.13)) that $P$ has the same effect as

$$P_0, A_{p,\alpha_p}, A_{p+1,\delta}, C_p, A_{p,\varepsilon^{[|z_p|+1]}}, B_p, A_{p+1,\tilde{\eta}}, P_2'$$

which (by commuting $A_{p+1,\delta}$ past the earlier operations) has the same effect as

$$A_{p+1,\delta}, P_0, A_{p,\alpha_p}, C_p, A_{p,\varepsilon^{[|z_p|+1]}}, B_p, A_{p+1,\tilde{\eta}}, P_2'$$

which (moving $C_p$ past $A_{p,\alpha_p}$) has the same effect as

$$A_{p+1,\delta}, P_0, C_p, A_{p,\alpha_p^{[1]}}, A_{p,\varepsilon^{[|z_p|+1]}}, B_p, A_{p+1,\tilde{\eta}}, P_2'$$

which (since $C_p$ commutes with the operations in $P_0$) has the same effect as

$$A_{p+1,\delta}, C_p, P_0, A_{p,\alpha_p^{[1]}}, A_{p,\varepsilon^{[|z_p|+1]}}, B_p, A_{p+1,\tilde{\eta}}, P_2'.$$

The part of this path after the first two operations has length $r-1$, so we're done by lemma A.15.

It remains to verify the claim. Equations (A.14), (A.15), and (A.18) give

$$A_{p+1,\delta}(\mathbf{z}) = \lambda((\delta^{[2]})^{-1})^{[|z_1|+\cdots+|z_p|+1]} \wedge \cdots \psi w \wedge \bar{\omega} u \cdots \tag{A.20}$$

so $C_p$ can be applied to $A_{p+1,\delta}(\mathbf{z})$. Now $\varepsilon^{[|z_p|+1]}(\bar{\psi}\bar{\psi}w) = \bar{\psi}\bar{\psi}w$, so Eq. (A.20) gives

$$\begin{aligned} &A_{p,\varepsilon^{[|z_p|+1]}} C_p A_{p+1,\delta}(\mathbf{z}) \\ &\quad = \lambda((\delta^{[2]})^{-1}(\varepsilon^{[1]})^{-1})^{[|z_1|+\cdots+|z_p|+1]} \wedge \cdots \bar{\psi}\bar{\psi}w \wedge u \cdots . \end{aligned}$$

Then $B_p$ can be applied to this and we have

$$\begin{aligned} &A_{p+1,\tilde{\eta}} B_p A_{p,\varepsilon^{[|z_p|+1]}} C_p A_{p+1,\delta}(\mathbf{z}) \\ &\quad = \lambda((\delta^{[2]})^{-1}(\varepsilon^{[1]})^{-1}(\tilde{\eta}^{[2]})^{-1})^{[|z_1|+\cdots+|z_p|+1]} \wedge \cdots \bar{\psi}w \wedge \tilde{\eta}\bar{\omega}u \cdots . \end{aligned}$$

Comparing this to Eq. (A.17), we see that it suffices to show $\gamma = \tilde{\eta}^{[2]}\varepsilon^{[1]}\delta^{[2]}$ (which follows from Eq. (A.19)) and $\tilde{\eta}\bar{\omega}u = v$. Since $\mathbf{X}_{p+1}$ is monomorphic, the latter equation follows from the equations

$$\bar{\omega}\tilde{\eta}\bar{\omega}u = \eta\bar{\omega}\bar{\omega}u = \gamma\bar{\omega}\delta^{-1}\bar{\omega}u = \gamma\bar{\omega}z_{p+1} = \bar{\omega}v$$

where we have used remark A.6(i) and Eqs. A.19, A.18, and A.16.

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This completes the proof of proposition A.4 □

*Proof of proposition A.2.* First we need some notation to distinguish the action of $\Sigma_j$ on $\mathbf{X}^{\wedge j}$ from the action of $\Sigma_k$ on the $k$th object of $\mathbf{X}^{\wedge j}$: given $\nu \in \Sigma_j$, $\eta \in \Sigma_k$ and a point $\mathfrak{x}$ of $\mathbf{X}^{\wedge j}$ with $|\mathfrak{x}| = k$, we write $\mathfrak{x}^\nu$ (resp., $\eta\mathfrak{x}$) for the point obtained by applying $\nu$ (resp., $\eta$) to $\mathfrak{x}$.

Now suppose there is a point $\mathfrak{x}$ in $\mathbf{X}^{\wedge j}$ which is not a basepoint and a non-trivial $\nu \in \Sigma_j$ such that $\mathfrak{x}^\nu = \mathfrak{x}$. Let $\mathfrak{x}$ be represented by

$$\mathbf{x} = \theta \wedge x_1 \wedge \cdots \wedge x_j$$

and note that $|x_i| > 0$ for all $i$ by our hypothesis on $X_0$. Then $\mathfrak{x}^\nu$ is represented by

$$\mathbf{y} = \theta\eta^{-1} \wedge x_{\nu^{-1}(1)} \wedge \cdots \wedge x_{\nu^{-1}(j)},$$

where $\eta$ permutes blocks of size $|x_1|, \ldots, |x_j|$ in the same way that $\nu$ permutes $1, \ldots, j$. There must be a path from $\mathbf{x}$ to $\mathbf{y}$, and it cannot consist entirely of operations of type $A$, since $\eta$ is not in $\Sigma_{|x_1|} \times \cdots \times \Sigma_{|x_j|}$. Thus we may assume without loss of generality that some $x_i$ is in the image of $\bar{\omega}$, and this implies (using operations of type A and C) that there is a point $\mathfrak{w}$ of $\mathbf{X}^{\wedge j}$ and a $\zeta \in \Sigma_{|\mathfrak{x}|}$ with $\mathfrak{x} = \zeta\bar{\omega}\mathfrak{w}$. Now $\nu$ commutes with both $\zeta$ and $\bar{\omega}$, so the fact that $\mathfrak{x}^\nu = \mathfrak{x}$ implies that

$$\zeta\bar{\omega}(\mathfrak{w}^\nu) = \zeta\bar{\omega}\mathfrak{w}.$$

Using proposition A.4, we see that $\mathfrak{w}^\nu = \mathfrak{w}$, and thus $\mathfrak{w}$ is a non-trivial fixed point of the $\Sigma_j$ action with $|\mathfrak{w}| < |\mathfrak{x}|$. Continuing in this way would give a non-trivial fixed point in the 0-th object of $\mathbf{X}^{\wedge j}$, which is impossible by our hypothesis on $X_0$. □

We conclude with an example which shows that propositions A.2 and A.4 fail if 'strongly monomorphic' is replaced by 'monomorphic'.

Recall definition 4.10. Let $\mathbf{X}$ be the subobject of $\mathbf{S}$ with $X_0 = *$ and $X_k = S^k$ for $k > 0$.

We will denote the point of $\mathbf{X} \wedge \mathbf{X}$ represented by a symbol $\theta \wedge x \wedge y$ by $[\theta \wedge x \wedge y]$. Let $x$ be the non-trivial simplex in $X_1$. Note that $\mathbf{X}$ is not strongly monomorphic because the transposition $(12)$ takes $\bar{\omega}x$ to itself, so $[(12)\bar{\omega}x]$ is in the image of $\bar{\omega}$, but $x$ is not in the image of $\bar{\omega}$.

Let $\tau$ be the operation which switches the two factors of $\mathbf{X} \wedge \mathbf{X}$. Now we claim that

$$([\bar{\omega}x \wedge x])^\tau = [\bar{\omega}x \wedge x],$$

which gives a counterexample for proposition A.2. We have

$$\begin{aligned}([\bar{\omega}x \wedge x])^\tau &= [\rho_{1,2} \wedge x \wedge \bar{\omega}x] = [(23)(12) \wedge x \wedge \bar{\omega}x] = [(23)(12) \wedge \bar{\psi}x \wedge x] \\ &= [(23) \wedge \bar{\psi}x \wedge x] \text{because } (12) \text{acts trivially on } (X_2)_{1,1} \\ &= [(23) \wedge x \wedge \bar{\omega}x] = [x \wedge \bar{\omega}x] = [\bar{\psi}x \wedge x] = [\bar{\omega}x \wedge x].\end{aligned}$$

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Next we observe that $[(12) \wedge x \wedge x] \neq [x \wedge x]$, because there is no non-trivial path beginning at $(12) \wedge x \wedge x$. But we claim that

$$\bar{\omega}[((12) \wedge x \wedge x)] = \bar{\omega}([x \wedge x]),$$

which gives a counterexample for proposition A.4. We have

$$\begin{aligned}\bar{\omega}([(12) \wedge x \wedge x]) &= [(23) \wedge \bar{\omega}x \wedge x] = [(23) \wedge \bar{\psi}x \wedge x] \\ &= [(23) \wedge x \wedge \bar{\omega}x] = [x \wedge \bar{\omega}x] = [\bar{\psi}x \wedge x] \\ &= [\bar{\omega}x \wedge x] = \bar{\omega}([x \wedge x]).\end{aligned}$$

## Appendix B. The proof of lemma 11.3

Let $\mathbf{W}$ denote the multisemisimplicial spectrum whose $k$-th object is $S_\bullet^{k-\mathrm{multi},\pitchfork}(T(\mathrm{STop}(k)))$, so that $|\mathbf{W}| = \mathbf{Y}$. We begin by showing that the monad $\mathbb{P}$ of definition 17.2 acts on the pair $(\mathbf{W}, \mathbf{R}_{\mathrm{STop}})$.

Let us define a $\mathbb{Z}$-graded category $\mathcal{B}$ as follows. The objects of $\mathcal{B}$ are pairs

$$(g : \Delta^{\mathbf{n}} \to T(\mathrm{STop}(k)), o),$$

where both $\mathbf{n}$ and $k$ are allowed to vary and $o$ is an orientation of $\Delta^{\mathbf{n}}$; the grading is given by $d(g, o) = \dim(\Delta^{\mathbf{n}}) - k$. We assume that the preimage $g^{-1}S$ of the zero section $S$ is a topological manifold and write $\mathrm{Inv}(g, o)$ for the resulting oriented manifold in $\mathcal{A}_{\mathrm{STop}}$. The morphisms are commutative diagrams

$$\begin{array}{ccc} \Delta^{\mathbf{n}} & \xrightarrow{g} & T(\mathrm{STop}(k)) \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \alpha} \\ \Delta^{\mathbf{n}'} & \xrightarrow{g'} & T(\mathrm{STop}(k)) \end{array}$$

in which $\phi$ is a composite of coface maps and permutations of the factors and $\alpha$ is a permutation; we require $\phi$ to be orientation preserving if the dimensions are equal. $\mathcal{B}$ is a symmetric monoidal $\mathbb{Z}$-graded category with product $\square$, where

$$(g, o) \,\square\, (g', o')$$

is the pair consisting of the composite

$$\Delta^{\mathbf{n}} \times \Delta^{\mathbf{n}'} \xrightarrow{g \times g'} T(\mathrm{STop}(k)) \times T(\mathrm{STop}(k')) \to T(\mathrm{STop}(k + k'))$$

and the orientation $o \times o'$. The symmetry isomorphism $\gamma$ is
where $\phi$ and $\alpha$ are the evident permutations.

In the construction of definition 15.4, if we replace $\mathcal{A}_{e,*,1}$ by $\mathcal{B}$, $\mathcal{A}^{\mathbb{Z}}_{\mathrm{rel}}$ by $\mathcal{A}_{\mathrm{STop}}$, $\boxtimes$ by $\square$ and $\mathrm{sig}_{\mathrm{rel}}$ by Inv we obtain a functor

$$\mathbf{d}_{\blacksquare} : \mathcal{A}_1 \times \cdots \times \mathcal{A}_j \to \mathcal{A}_{\mathrm{STop}},$$

for each datum $\mathbf{d}$, where $\mathcal{A}_i$ denotes $\mathcal{B}$ if $r(i) = u$ and $\mathcal{A}_{\mathrm{STop}}$ if $r(i) = v$.

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$$\begin{array}{ccccc} \Delta^{\mathbf{n}} \times \Delta^{\mathbf{n}'} & \xrightarrow{g \times g'} & T(\mathrm{STop}(k)) \times T(\mathrm{STop}(k')) & \longrightarrow & T(\mathrm{STop}(k+k')) \\ \downarrow \phi & & & & \downarrow \alpha \\ \Delta^{\mathbf{n}'} \times \Delta^{\mathbf{n}} & \xrightarrow{g' \times g} & T(\mathrm{STop}(k')) \times T(\mathrm{STop}(k)) & \longrightarrow & T(\mathrm{STop}(k'+k)) \end{array}$$

Next we have the analogue of definition 16.5.

Definition B.1. *Let $k_1, \dots, k_j$ be non-negative integers and let $\mathbf{n}_i$ be a $k_i$-fold multi-index for $1 \le i \le j$. Let $r : \{1, \dots, j\} \to \{u, v\}$, and for $1 \le i \le j$ let $\mathbf{Z}_i$ denote $\mathbf{W}$ if $r(i) = u$ and $\mathbf{R}_{\mathrm{STop}}$ if $r(i) = v$. Then for each map of preorders*

$$a : U(\Delta^{\mathbf{n}_1} \times \cdots \times \Delta^{\mathbf{n}_j}) \to P_{r;v}$$

*we define*

$$a_* : ((\mathbf{Z}_1)_{k_1})_{\mathbf{n}_1} \times \cdots \times ((\mathbf{Z}_j)_{k_j})_{\mathbf{n}_j} \to ((\mathbf{R}_{\mathrm{STop}})_{k_1+\cdots+k_j})_{(\mathbf{n}_1,\dots,\mathbf{n}_j)}$$

*by*

$$\begin{aligned} &a_*(z_1, \dots, z_j)(\sigma_1 \times \cdots \times \sigma_j, o_1 \times \cdots \times o_j) \\ &\quad = i^{\epsilon(\zeta)} a(\sigma_1 \times \cdots \times \sigma_j)_{\blacksquare}(z_1(\sigma_1, o_1), \dots, z_j(\sigma_j, o_j)), \end{aligned}$$

*where*

- *if $r(i) = u$ then $z_i(\sigma_i, o_i))$ denotes $(z_i|_{\sigma_i}, o_i)$, and*
- *$\zeta$ is the block permutation that takes blocks $\mathbf{b}_1, \dots, \mathbf{b}_j, \mathbf{c}_1, \dots, \mathbf{c}_j$ of size $k_1, \dots, k_j, \dim \sigma_1, \dots, \dim \sigma_j$ into the order $\mathbf{b}_1, \mathbf{c}_1, \dots, \mathbf{b}_j, \mathbf{c}_j$.*

As in §17, this definition leads to a map

$$\mathbb{P}_2(\mathbf{W}, \mathbf{R}_{\mathrm{STop}}) \to \mathbf{R}_{\mathrm{STop}}. \tag{B.1}$$

Since $\mathbf{W}$ is a commutative multisemisimplicial symmetric ring spectrum, we have a map

$$\mathbb{P}_1(\mathbf{W}) \xrightarrow{\Xi_1} \bigvee_{j \ge 0} \mathbf{W}^{\wedge j}/\Sigma_j \to \mathbf{W}, \tag{B.2}$$

where $\Xi_1$ is given in definition 18.1(ii).

The maps (B.1) and (B.2) give the required action of $\mathbb{P}$ on $(\mathbf{W}, \mathbf{R}_{\mathrm{STop}})$. Now the proof of theorem 1.3 (given in §18) gives a map of commutative symmetric ring

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spectra

$$|B_\bullet(\mathbb{P}'_1, \mathbb{P}, (\mathbf{W}, \mathbf{R}_{\text{STop}}))| \to |B_\bullet(\mathbb{P}'_2, \mathbb{P}, (\mathbf{W}, \mathbf{R}_{\text{STop}}))|$$

which is a weak equivalence by lemma 11.1. As in remark 18.3, there is a weak equivalence of commutative symmetric ring spectra

$$(\mathbf{M}_{\text{STop}})^{\text{comm}} \to |B_\bullet(\mathbb{P}'_2, \mathbb{P}, (\mathbf{W}, \mathbf{R}_{\text{STop}}))|.$$

To complete the proof, we observe that there is a weak equivalence of commutative symmetric ring spectra

$$|B_\bullet(\mathbb{P}'_1, \mathbb{P}, (\mathbf{W}, \mathbf{R}_{\text{STop}}))| = |B_\bullet(\mathbb{P}'_1, \mathbb{P}_1, \mathbf{W})| \to |B_\bullet(\mathbb{P}'_1, \mathbb{P}'_1, \mathbf{W})| \to |\mathbf{W}| = \mathbf{Y},$$

where the first arrow is a weak equivalence by proposition 18.2(iii) and the second by [17, proposition 9.8 and corollary 11.9].

## Appendix C. A functorial version of $\text{ad}^R_{\text{rel}}$

In [11, §13], we explained why $\text{ad}^R$ (as defined in [11, §9], which is the definition we have used in the present article) is not a functor of $R$, and how to modify the definition of $\text{ad}^R$ to make it a functor (unfortunately, in [11] we also denoted the modified version by $\text{ad}^R$; in this appendix, we will be more careful with the notation). Our goal in this appendix is to give a functorial version of $\text{ad}^R_{\text{rel}}$, which is needed in theorem 19.1.

Unfortunately, it seems that we cannot just adapt the method of [11, §13] to this situation, because $\text{ad}^R_{\text{rel}}$ isn't even 'approximately' functorial: given a ring homomorphism $R \to S$ and an object $(C, D, \beta, \varphi)$ of $\mathcal{A}^R_{\text{rel}}$, there does not seem to be a reasonable way to create an object $(C', D', \beta', \varphi')$ of $\mathcal{A}^S_{\text{rel}}$ from this data (we could let $C' = S \otimes_R C$, but the obvious candidate for $D'$ does not come with a quasi-isomorphism). So in §C.2 (after some preliminary motivation in §C.1), we give a variant of $\text{ad}^R_{\text{rel}}$, which we denote by $\text{ad}^R_{\text{Rel}}$, and in §C.3 we show that $\text{ad}^R_{\text{Rel}}$ is approximately functorial (i.e., functorial up to isomorphism). In §C.4, we show that the ad theories $\text{ad}^R$ and $\text{ad}^R_{\text{Rel}}$ are equivalent, that is, there is a morphism of ad theories from $\text{ad}^R_{\text{Rel}}$ to $\text{ad}^R_{\text{rel}}$ which induces an isomorphism of bordism groups. In §C.5, we give an enhanced version of the material in [11, §13], and in §C.6 we use this to create a variant of $\text{ad}^R_{\text{Rel}}$, which we denote by $\text{ad}^R_{\text{Rel,sch}}$. In §C.7, we show (using §C.3) that $\text{ad}^R_{\text{Rel,sch}}$ is a functor of $R$.

Remark C.1. We could have used $\text{ad}^R_{\text{Rel}}$ throughout this article instead of $\text{ad}^R_{\text{rel}}$, but that would have added extra complexity and functoriality is only an issue at the end of the article in theorem 19.1.

### C.1. Background

As motivation for the definition of $\text{ad}^R_{\text{Rel}}$, we need

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Lemma C.2. *Let $M$ be a right $R$ module and $N$ a left $R$ module, and let $R^{\mathrm{op}} \otimes R$ act on $R$ on the right in the usual way. Then the map*

$$a : R \otimes_{R^{\mathrm{op}} \otimes R} (M \otimes N) \to M \otimes_R N$$

*given by $a(r \otimes m \otimes n) = mr \otimes n$ is an isomorphism.*

*Proof.* This follows immediately from the isomorphisms

$$R \otimes_{R^{\mathrm{op}} \otimes R} (M \otimes N) \cong M \otimes_R R \otimes_R N \cong M \otimes_R N.$$

□

Now fix a ring $R$ with involution.

Definition C.3. *Let $P$ and $Q$ be left $R^{\mathrm{op}} \otimes R$ modules. A map*

$$b : P \to Q$$

*is quasi-linear if $b((r \otimes s)p) = (\bar{s} \otimes \bar{r})b(p)$.*

Lemma C.4. *Let $b : P \to Q$ be a quasi-linear map of left $R^{\mathrm{op}} \otimes R$ modules. Then the map*

$$\hat{b} : R \otimes_{R^{\mathrm{op}} \otimes R} P \to R \otimes_{R^{\mathrm{op}} \otimes R} Q$$

*given by $\hat{b}(r \otimes p) = \bar{r} \otimes b(p)$ is well-defined.*

## C.2. The ad theory $\mathrm{ad}^R_{\mathbf{Rel}}$

Recall the definition of homotopy finite [11, definition 9.2(iv)].

Definition C.5. *A Relaxed quasi-symmetric complex of dimension $n$ is a quadruple $(C, E, \gamma, \phi)$, where $C$ is a homotopy finite* [2.] *chain complex over $R$, $E$ is a homotopy finite chain complex over $R^{\mathrm{op}} \otimes R$ with a $\mathbb{Z}/2$ action for which the generator acts quasi-linearly, $\gamma$ is a $\mathbb{Z}/2$ equivariant $R^{\mathrm{op}} \otimes R$-linear quasi-isomorphism $C^t \otimes C \to E$, and $\phi$ is an $n$-dimensional element of $(R \otimes_{R^{\mathrm{op}} \otimes R} E)^{\mathbb{Z}/2}$ (where the $\mathbb{Z}/2$ action is given by lemma C.4).*

For the following example, note that if $A$ is a left $R^{\mathrm{op}} \otimes R$ module which is nonzero in only finitely many dimensions then $A^W$ is (additively) a direct sum of copies of $A$, and hence the natural map

$$R \otimes_{R^{\mathrm{op}} \otimes R} (A^W) \to (R \otimes_{R^{\mathrm{op}} \otimes R} A)^W$$

is an isomorphism because tensor product preserves direct sums.

Example C.6. If $(C, \varphi)$ is a quasi-symmetric complex as defined in [11, definition 9.3], and if $C$ is nonzero in only finitely many dimensions, then the quadruple

[2.] In [11, §9], we also required $C$ to be free, but that turns out not to be necessary; see remark C.18.

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$(C,(C^t\otimes C)^W,\gamma,\phi)$ is a Relaxed quasi-symmetric complex, where $\gamma: C^t\otimes C\to (C^t\otimes C)^W$ is induced by the augmentation $W\to\mathbb{Z}$ and $\phi$ is the image of $\varphi$ under the composite

$$((C^t\otimes_R C)^W)^{\mathbb{Z}/2}\cong((R\otimes_{R^{\mathrm{op}}\otimes R}(C^t\otimes C))^W)^{\mathbb{Z}/2}\cong(R\otimes_{R^{\mathrm{op}}\otimes R}((C^t\otimes C)^W))^{\mathbb{Z}/2}$$

(where the first isomorphism is lemma C.2).

Definition C.7. *We define a category $\mathcal{A}^R_{Rel}$ as follows. The objects of $\mathcal{A}^R_{Rel}$ are the Relaxed quasi-symmetric complexes. A morphism $(C,E,\gamma,\phi)\to(C',E',\gamma',\phi')$ is a pair $(f: C\to C', g: E\to E')$, where $f$ is an $R$-linear chain map and $g$ is a $\mathbb{Z}/2$ equivariant $R^{\mathrm{op}}\otimes R$-linear chain map, such that $g\gamma=\gamma'(f\otimes f)$, and (if $\dim\phi=\dim\phi'$) $(1\otimes g)_*(\phi)=\phi'$.*

$\mathcal{A}^R_{\mathrm{Rel}}$ is a balanced [11, definition 5.1] $\mathbb{Z}$-graded category, where $i$ takes $(C,E,\gamma,\phi)$ to $(C,E,\gamma,-\phi)$ and $\emptyset_n$ is the $n$-dimensional object for which $C$ and $E$ are zero in all degrees.

Remark C.8. There is a morphism $\mathcal{A}^R_{\mathrm{Rel}}\to\mathcal{A}^R_{\mathrm{rel}}$ of $\mathbb{Z}$-graded categories which takes $(C,E,\gamma,\phi)$ to $(C,R\otimes_{R^{\mathrm{op}}\otimes R}E,\beta,\phi)$, where $\beta$ is the composite

$$C^t\otimes_R C\cong R\otimes_{R^{\mathrm{op}}\otimes R}(C^t\otimes C)\xrightarrow{1\otimes\gamma}R\otimes_{R^{\mathrm{op}}\otimes R}E.$$

Remark C.9. Let $\mathcal{A}^R_{\mathrm{fin}}$ be the full subcategory of $\mathcal{A}^R$ consisting of objects $(C,\varphi)$ with $C$ finite (not just homotopy finite). Let $\mathcal{A}^R_{\mathrm{Rel,fin}}$ be the full subcategory of $\mathcal{A}^R_{\mathrm{Rel}}$ consisting of objects $(C,E,\gamma,\phi)$ with $C$ and $E$ finite. The construction of example C.6 gives a morphism

$$\mathcal{A}^R_{\mathrm{fin}}\to\mathcal{A}^R_{\mathrm{Rel,fin}}$$

of $\mathbb{Z}$-graded categories.

Next we must say what the $K$-ads with values in $\mathcal{A}^R_{\mathrm{Rel}}$ are. For a balanced pre $K$-ad $F$, we will use the notation

$$F(\sigma,o)=(C_\sigma,E_\sigma,\gamma_\sigma,\phi_{\sigma,o}).$$

Recall [11, definition 9.7].

Definition C.10. *A balanced pre $K$-ad $F$ is well-behaved if $C$ and $E$ are well-behaved.*

Definition C.11. *(i) A balanced $K$-ad is a pre $K$-ad with the following properties.*

*(a) It is balanced, closed, and well-behaved, and*

*(b) the composite of $F$ with the morphism of remark C.8 satisfies part (i)(b) of definition 12.10.*

*(ii) A $K$-ad is a pre $K$-ad which is naturally isomorphic to a balanced $K$-ad.*

We write $\mathrm{ad}^R_{\mathrm{Rel}}(K)$ for the set of $K$-ads with values in $\mathcal{A}^R_{\mathrm{Rel}}$.

Theorem C.12 $\mathrm{ad}^R_{\mathrm{Rel}}$ *is an ad theory.*

This follows from the proof of theorem 12.12 with minor changes.

Remark C.13. The morphisms of remarks C.8 and C.9 take ads to ads.

## C.3. $\mathrm{ad}^R_{\mathrm{Rel}}$ is approximately functorial

The result in this subsection will be used in §C.7.

Definition C.14. *Let $h : R \to S$ be a ring homomorphism. Define a functor*

$$h_{\mathrm{Rel}} : \mathcal{A}^R_{\mathrm{Rel}} \to \mathcal{A}^S_{\mathrm{Rel}}$$

*as follows. For an object $(C, E, \gamma, \phi)$ of $\mathcal{A}^R_{\mathrm{Rel}}$, let*

$$h_{\mathrm{Rel}}(C, E, \gamma, \phi) = (S \otimes_R C, (S^{\mathrm{op}} \otimes S) \otimes_{R^{\mathrm{op}} \otimes R} E, \gamma', \phi'),$$

*where $\gamma'$ is the composite*

$$(S \otimes_R C)^t \otimes (S \otimes_R C) \cong (S^{\mathrm{op}} \otimes S) \otimes_{R^{\mathrm{op}} \otimes R} (C^t \otimes C) \xrightarrow{1 \otimes \gamma} (S^{\mathrm{op}} \otimes S) \otimes_{R^{\mathrm{op}} \otimes R} E$$

*(which is a quasi-isomorphism by the Künneth spectral sequence [26, theorem 5.6.4], using the fact that $C$ and $E$ are homotopy finite) and $\phi'$ is the image of $\phi$ under the composite*

$$(R \otimes_{R^{\mathrm{op}} \otimes R} E)^{\mathbb{Z}/2} \to (S \otimes_{R^{\mathrm{op}} \otimes R} E)^{\mathbb{Z}/2} \cong (S \otimes_{S^{\mathrm{op}} \otimes S} (S^{\mathrm{op}} \otimes S) \otimes_{R^{\mathrm{op}} \otimes R} E)^{\mathbb{Z}/2}.$$

The reader can check that if $k : S \to T$ is another ring homomorphism then $(kh)_{\mathrm{Rel}}(C, E, \gamma, \phi)$ is isomorphic to but not equal to $k_{\mathrm{Rel}} h_{\mathrm{Rel}}(C, E, \gamma, \phi)$.

Proposition C.15. *$h_{\mathrm{Rel}}$ takes ads to ads.*

*Proof.* Let $F \in \mathrm{ad}^R_{\mathrm{Rel}}(K)$. We may assume that $F$ is balanced. Write

$$F(\sigma, o) = (C_\sigma, E_\sigma, \gamma_\sigma, \phi_{\sigma,o}).$$

Let $\Psi$ be the functor of remark C.8. Let $G = \Psi \circ F$; then $G$ is an element of $\mathrm{ad}^R_{\mathrm{rel}}(K)$. Write

$$G(\sigma, o) = (C_\sigma, D_\sigma, \beta_\sigma, \phi_{\sigma,o}).$$

Let $H = \Psi \circ h_{\mathrm{Rel}} \circ F$; we need to show that $H$ is an ad. It's immediate that $H$ is balanced, well-behaved and closed, so it only remains to show that it satisfies part

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(ii) of definition 12.10. Write

$$H(\sigma, o) = (S \otimes_R C_\sigma, D^S_\sigma, \beta^S_\sigma, \psi_{\sigma,o})$$

and fix an oriented cell $(\sigma, o)$ of $K$. Recall notation 12.9 and let

$$k_\sigma : ((S \otimes_R C)^t \otimes_S (S \otimes_R C))_\sigma / ((S \otimes_R C)^t \otimes_S (S \otimes_R C))_{\partial\sigma} \\ \to ((S \otimes_R C_\sigma)/(S \otimes_R C_{\partial\sigma})) \otimes_S (S \otimes_R C_\sigma)$$

be the analogous map. We need to show that the slant product with $(k_\sigma)_*(\beta^S_\sigma)^{-1}_*([\psi_{\sigma,o}])$ is an isomorphism

$$H^*(\mathrm{Hom}_S(S \otimes_R C_\sigma, S)) \to H_{\dim\sigma - \deg F - *}(S \otimes_R C_\sigma / S \otimes_R C_{\partial\sigma}).$$

First we observe that (using the definition of $h_{\mathrm{Rel}}$) the image of $(j_\sigma)_*(\beta_\sigma)^{-1}_*([\phi_{\sigma,o}])$ in $H_*((S \otimes_R C_\sigma)/(S \otimes_R C)_{\partial\sigma}) \otimes S \otimes_R C_\sigma$ is $(k_\sigma)_*(\beta^S_\sigma)^{-1}_*([\psi_{\sigma,o}])$. Now the desired isomorphism follows from our next lemma. □

LEMMA C.16. *Let $h : R \to S$ be a homomorphism of rings with involution. Let $A$ and $B$ be homotopy finite chain complexes over $R$. Let $x$ be a cycle in $A^t \otimes_R B$ with the property that the slant product with $x$ is an isomorphism*

$$H^*(\mathrm{Hom}_R(B, R)) \to H_{\dim x - *}A^t.$$

*Let $y$ be the image of $x$ under the map*

$$A^t \otimes_R B \to (S \otimes_R A)^t \otimes_S (S \otimes_R B).$$

*Then the slant product with $y$ is an isomorphism*

$$H^*(\mathrm{Hom}_S(S \otimes_R B, S)) \to H_{\dim x - *}(S \otimes_R A)^t.$$

*Proof of lemma C.16.* By naturality of the slant product, we may assume that $A$ and $B$ are finite. Because $B$ is additively a direct sum of finitely many copies of $R$, the map

$$\Upsilon : \mathrm{Hom}_R(B, R) \otimes_R S \to \mathrm{Hom}_S(S \otimes_R B, S)$$

defined by $\Upsilon(f \otimes s)(t \otimes b) = tf(b)s$ is an isomorphism.

Consider the diagram

$$\begin{array}{ccc} H^*(\mathrm{Hom}_S(S\otimes_R B, S)) & \xrightarrow{\backslash y} & H_{\dim x-*}(S\otimes_R A)^t \\ \Upsilon \uparrow \cong & & \uparrow \cong \\ H^*(\mathrm{Hom}_R(B,R)\otimes_R S) & \longrightarrow & H_{\dim x-*}(A^t\otimes_R S) \end{array}$$

where the bottom arrow is induced by the chain map $(\backslash x)\otimes 1$. It's straightforward to check that the diagram commutes, and the lower arrow is an isomorphism by the Künneth spectral sequence [26, theorem 5.6.4]. ☐

## C.4. Comparison of $\mathrm{ad}^R_{\mathrm{Rel}}$ and $\mathrm{ad}^R_{\mathrm{rel}}$

PROPOSITION C.17. *The morphism* $\mathrm{ad}^R_{\mathrm{Rel}} \to \mathrm{ad}^R_{\mathrm{rel}}$ *induces an isomorphism of bordism groups.*

*Proof.* Let $\mathcal{A}^R_{\mathrm{Rel,fin}}$ be the full subcategory of $\mathcal{A}^R_{\mathrm{Rel}}$ consisting of objects $(C, E, \gamma, \phi)$ for which $C$ is finite. Consider the diagram

$$\begin{array}{ccc} \mathcal{A}^R_{\mathrm{Rel}} & \longrightarrow & \mathcal{A}^R_{\mathrm{rel}} \\ a\uparrow & & \\ \mathcal{A}^R_{\mathrm{Rel,fin}} & & \uparrow d \\ b\uparrow & & \\ \mathcal{A}^R_{\mathrm{fin}} & \xrightarrow{c} & \mathcal{A}^R \end{array}$$

where $a$ is induced by the inclusion of categories, $b$ is given by remark C.9, $c$ is induced by the inclusion of categories, and $d$ is given by remark 12.5. This diagram commutes up to natural isomorphism, so it induces a commutative diagram of bordism groups:

$$\begin{array}{ccc} (\Omega^R_{\mathrm{Rel}})_* & \longrightarrow & (\Omega^R_{\mathrm{rel}})_* \\ \Omega^a_*\uparrow & & \\ (\Omega^R_{\mathrm{Rel,fin}})_* & & \uparrow \Omega^d_* \\ \Omega^b_*\uparrow & & \\ (\Omega^R_{\mathrm{fin}})_* & \xrightarrow{\Omega^c_*} & (\Omega^R)_* \end{array}$$

$\Omega^d_*$ is an isomorphism by proposition 13.3, and the proof of that proposition, with minor modifications, shows that $\Omega^b_*$ is an isomorphism.

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To see that $\Omega^a_*$ is onto, let $(C, E, \gamma, \phi)$ represent an element of $\mathrm{ad}^R_{\mathrm{Rel}}(*)$. Since $C$ is homotopy finite, there is a chain homotopy equivalence $f : C' \to C$ with $C^{'}$ finite. Let $\gamma' = \gamma \circ (f \otimes f)$. Then $(C', E, \gamma', \phi)$ is an element of $\mathrm{ad}^R_{\mathrm{Rel,fin}}(*)$, and $(f, \mathrm{id})$ is a morphism from $(C', E, \gamma', \phi)$ to $(C, E, \gamma, \phi)$. Let $F$ be the cylinder object of $(C, E, \gamma, \phi)$ [11, definition 3.10(g)]. Let $0, 1, \iota$ denote the three cells of the unit interval $I$, with their standard orientations. Replacing $F(0)$ with $(C', E, \gamma', \phi)$ gives an $I$-ad which is a bordism between $(C, E, \gamma, \phi)$ and $(C', E, \gamma', \phi)$.

To complete the proof it suffices to show that $\Omega^c_*$ is an isomorphism, since this will imply that $\Omega^a_*$ is a monomorphism.

To see that $\Omega^c_*$ is onto, let $(C, \varphi)$ represent an element of $\mathrm{ad}^R(*)$. There is a chain homotopy equivalence $f : C \to C'$ with $C^{'}$ finite. Then $(C', (f \otimes f) \circ \varphi)$ represents an element of $\mathrm{ad}^R_{\mathrm{fin}}(*)$, and $(f, \phi)$ is a morphism from $(C, \varphi)$ to $(C', (f \otimes f) \circ \varphi)$. Let $F$ be the cylinder object of $(C', (f \otimes f) \circ \varphi)$. Replacing $F(0)$ with $(C, \varphi)$ gives a bordism between $(C, \varphi)$ and $(C', (f \otimes f) \circ \varphi)$.

To see that $\Omega^c_*$ is a monomorphism, let $F \in \mathrm{ad}^R(I)$ with $F(0)$ and $F(1)$ in $\mathcal{A}^R_{\mathrm{fin}}$. Write

$$F(0) = (C_0, \varphi_0), \quad F(1) = (C_1, \varphi_1), \quad F(\iota) = (C_\iota, \varphi_\iota).$$

Since $C_\iota$ is homotopy finite, there is a chain homotopy equivalence $f : C_\iota \to B$ with $B$ finite. Let $g_0$ be the composite $C_0 \to C_\iota \xrightarrow{f} B$ and similarly for $g_1 : C_1 \to B$. Let $\mathrm{cl}(I)$ be the cellular chain complex of $I$ and let $j_0$ (resp., $j_1$) be the composite $\mathbb{Z} \cong \mathrm{cl}(0) \to \mathrm{cl}(I)$ (resp., $\mathbb{Z} \cong \mathrm{cl}(1) \to \mathrm{cl}(I)$, where the second map is the inclusion. Let $B^{'}$ be the colimit

$$\begin{array}{ccccccc} & & C_0 & & & C_1 & & \\ & \swarrow^{j_1 \otimes \mathrm{id}} & & \searrow^{g_0} & \swarrow^{g_1} & & \searrow^{j_1 \otimes \mathrm{id}} & \\ \mathrm{cl}(I) \otimes C_0 & & & B & & & \mathrm{cl}(I) \otimes C_1 \end{array}$$

The composites $C_0 \xrightarrow{j_0 \otimes \mathrm{id}} \mathrm{cl}(I) \otimes C_0 \to B'$ and $C_1 \xrightarrow{j_0 \otimes \mathrm{id}} \mathrm{cl}(I) \otimes C_1 \to B'$ are strong monomorphisms [11, definition 9.6]. Let $n$ be the degree of $F$ and let $\psi \in B'_{1-n}$ be the image of $\iota \otimes \varphi_0 + f_*(\phi_\iota) - \iota \otimes \varphi_1$.

Define an $I$-ad $G$ by

$$G(0) = F(0), \quad G(1) = F(1), \quad G(\iota) = (B', \psi).$$

Then $G$ is the desired bordism. $\square$

Remark C.18. The proof that $\Omega^c_*$ is an isomorphism also shows that the requirement in [11, §9] that $C$ should be free over $R$ is not needed. That is, if we define $\mathrm{ad}^R$ as in [11, §9] and $(\mathrm{ad}^R)^\dagger$ by requiring only that $C$ be homotopy finite then the forgetful map $\mathrm{ad}^R \to (\mathrm{ad}^R)^\dagger$ induces an isomorphism of bordism groups.

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### C.5. Enhanced version of [11, §13]

In [11, §13], we gave a model for the category of free $R$ modules which is functorial in $R$. In this subsection give a similar model for the category of all $R$ modules. Our terminology and notation will be different from [11, §13].

We define the category of *schematic free $R$ modules* as follows. An object is a set $\mathbb{M}$. This should be thought of as representing the free $R$ module generated by $\mathbb{M}$, which we denote by $R\langle\mathbb{M}\rangle$. We define a map $\mathbb{M} \to \mathbb{M}'$ to be a map of $R$-modules $R\langle\mathbb{M}\rangle \to R\langle\mathbb{M}'\rangle$.

We define the category of *schematic $R$ modules* as follows. An object of this category is a triple $(\mathbb{M}, \mathbb{N}, T)$, where $\mathbb{M}$ and $\mathbb{N}$ are schematic free $R$ modules and $T$ is a map $\mathbb{N} \to \mathbb{M}$. Such a triple should be thought of as representing the quotient of $R\langle\mathbb{M}\rangle$ by the image of $T$; we write $R\langle(\mathbb{M}, \mathbb{N}, T)\rangle$ for this quotient. A map $(\mathbb{M}, \mathbb{N}, T) \to (\mathbb{M}', \mathbb{N}', T')$ is defined to be an $R$-module map $R\langle(\mathbb{M}, \mathbb{N}, T)\rangle \to R\langle(\mathbb{M}', \mathbb{N}', T')\rangle$.

Lemma C.19. *The functor from schematic $R$ modules to $R$ modules which takes $(\mathbb{M}, \mathbb{N}, T)$ to $R\langle(\mathbb{M}, \mathbb{N}, T)\rangle$ is an equivalence of categories.*

*Proof.* The functor is the identity on morphism sets, so it's only necessary to show that every $R$ module $P$ is isomorphic to one of the form $R\langle(\mathbb{M}, \mathbb{N}, T)\rangle$. Choose an exact sequence $Q_1 \to Q_2 \to P \to 0$ where $Q_1$ and $Q_2$ are free, let $\mathbb{M}$ and $\mathbb{N}$ be bases for $Q_1$ and $Q_2$, and let $T$ be the map induced by $Q_1 \to Q_2$. □

A *schematic chain complex* $\mathbb{C}$ over $R$ is a sequence of schematic $R$ modules and maps, and we write $R\langle\mathbb{C}\rangle$ for the corresponding sequence of $R$ modules and maps. A map $\mathbb{C} \to \mathbb{C}'$ of schematic chain complexes is a map of $R$ chain complexes $R\langle\mathbb{C}\rangle \to R\langle\mathbb{C}'\rangle$.

Let $h : R_1 \to R_2$ be a homomorphism. For a schematic free $R_1$ module $\mathbb{M}$, we write $h_{\mathrm{sch}}\mathbb{M}$ for $\mathbb{M}$ thought of as a schematic free $R_2$ module. There is a canonical isomorphism

$$R_2\langle h_{\mathrm{sch}}\mathbb{M}\rangle \cong R_2 \otimes_{R_1} R_1\langle\mathbb{M}\rangle \tag{C.1}$$

which takes an element $m$ of $\mathbb{M}$ to $1 \otimes m$. For a map $T : \mathbb{M} \to \mathbb{N}$ we write $h_{\mathrm{sch}}T$ for the map $h_{\mathrm{sch}}\mathbb{M} \to h_{\mathrm{sch}}\mathbb{N}$ defined by the following diagram.

$$\begin{array}{ccc} R_2\langle h_{\mathrm{sch}}\mathbb{M}\rangle & \xrightarrow{h_{\mathrm{sch}}T} & R_2\langle h_{\mathrm{sch}}\mathbb{N}\rangle \\ \cong\downarrow & & \cong\downarrow \\ R_2 \otimes_{R_1} R_1\langle\mathbb{M}\rangle & \xrightarrow{1\otimes T} & R_2 \otimes_{R_1} R_1\langle\mathbb{N}\rangle \end{array}$$

For a schematic $R_1$ module $(\mathbb{M}, \mathbb{N}, T)$, we define $h_{\mathrm{sch}}(\mathbb{M}, \mathbb{N}, T)$ to be the schematic $R_2$ module $(h_{\mathrm{sch}}\mathbb{M}, h_{\mathrm{sch}}\mathbb{N}, h_{\mathrm{sch}}T)$. The isomorphism C.1 induces a canonical isomorphism

$$R_2\langle h_{\mathrm{sch}}(\mathbb{M}, \mathbb{N}, T)\rangle \cong R_2 \otimes_{R_1} R_1\langle(\mathbb{M}, \mathbb{N}, T)\rangle. \tag{C.2}$$

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For a map of $R_1$ modules $U : (\mathbb{M}, \mathbb{N}, T) \to (\mathbb{M}', \mathbb{N}', T')$, we define $h_{\rm sch}U$ to be the map defined by the following diagram.

$$\begin{array}{ccc} R_2\langle h_{\rm sch}(\mathbb{M},\mathbb{N},T)\rangle & \xrightarrow{h_{\rm sch}U} & R_2\langle h_{\rm sch}(\mathbb{M}',\mathbb{N}',T')\rangle \\ \cong\downarrow & & \cong\downarrow \\ R_2\otimes_{R_1} R_1\langle(\mathbb{M},\mathbb{N},T)\rangle & \xrightarrow{1\otimes U} & R_2\otimes_{R_1} R_1\langle(\mathbb{M}',\mathbb{N}',T')\rangle \end{array}$$

This gives a functor $h_{\rm sch}$ from schematic $R_1$ modules to schematic $R_2$ modules. If $h' : R_2 \to R_3$ is a homomorphism we have $(h' \circ h)_{\rm sch} = h'_{\rm sch} \circ h_{\rm sch}$, and thus the category of schematic $R$ bf modules is a functor of $R$.

## C.6. The ad theory $\mathrm{ad}^R_{\rm Rel,sch}$

First we translate definition C.5 into the language of schematic modules:

Definition C.20. *(i) A schematic Relaxed quasi-symmetric complex of dimension $n$ is a quadruple $(\mathbb{C}, \mathbb{E}, \gamma, \phi)$, where $\mathbb{C}$ is a schematic $R$ chain complex, $\mathbb{E}$ is a schematic $(R^{\rm op} \otimes R)$ chain complex, $R\langle\mathbb{C}\rangle$ is homotopy finite, $(R^{\rm op} \otimes R)\langle\mathbb{E}\rangle$ is a homotopy finite chain complex over $R^{\rm op} \otimes R$ with a $\mathbb{Z}/2$ action for which the generator acts quasi-linearly, $\gamma$ is a $\mathbb{Z}/2$ equivariant $R^{\rm op} \otimes R$-linear quasi-isomorphism $(R\langle\mathbb{C}\rangle)^t \otimes R\langle\mathbb{C}\rangle \to (R^{\rm op} \otimes R)\langle\mathbb{E}\rangle$, and $\phi$ is an $n$-dimensional element of $(R \otimes_{R^{\rm op}\otimes R} (R^{\rm op} \otimes R)\langle\mathbb{E}\rangle)^{\mathbb{Z}/2}$ (where the $\mathbb{Z}/2$ action is given by lemma C.4).*

*(ii) We define a category $\mathcal{A}^R_{\rm Rel,sch}$ as follows. The objects of $\mathcal{A}^R_{\rm Rel,sch}$ are the schematic Relaxed quasi-symmetric complexes. A morphism $(\mathbb{C}, \mathbb{E}, \gamma, \phi) \to (\mathbb{C}', \mathbb{E}', \gamma', \phi')$ is a pair $(f : R\langle\mathbb{C}\rangle \to R\langle\mathbb{C}'\rangle, g : (R^{\rm op} \otimes R)\langle\mathbb{E}\rangle \to (R^{\rm op} \otimes R)\langle\mathbb{E}'\rangle)$, where $f$ is an $R$-linear chain map and $g$ is a $\mathbb{Z}/2$ equivariant $R^{\rm op} \otimes R$-linear chain map, such that $g\gamma = \gamma'(f \otimes f)$, and (if $\dim\phi = \dim\phi'$) $(1 \otimes g)_*(\phi) = \phi'$.*

$\mathcal{A}^R_{\rm Rel,sch}$ is a balanced $\mathbb{Z}$-graded category, where $i$ takes $(\mathbb{C}, \mathbb{E}, \gamma, \phi)$ to $(\mathbb{C}, \mathbb{E}, \gamma, -\phi)$.

There is a morphism

$$\Lambda : \mathcal{A}^R_{\rm Rel,sch} \to \mathcal{A}^R_{\rm Rel}$$

of $\mathbb{Z}$-graded categories which takes $(\mathbb{C}, \mathbb{E}, \gamma, \phi)$ to $(R\langle\mathbb{C}\rangle, (R^{\rm op} \otimes R)\langle\mathbb{E}\rangle, \gamma, \phi)$; this is an equivalence of categories.

Definition C.21. *A $K$-ad with values in $\mathcal{A}^R_{\rm Rel,sch}$ is a pre $K$-ad $F$ for which $\Lambda \circ F$ is a $K$-ad.*

We write $\mathrm{ad}^R_{\rm Rel,sch}(K)$ for the set of $K$-ads with values in $\mathcal{A}^R_{\rm Rel,sch}$.

Proposition C.22. *(i) $\mathrm{ad}^R_{\rm Rel,sch}$ is an ad theory.*

*(ii) $\Lambda$ induces a morphism of ad theories which is an isomorphism on bordism groups.*

This is an easy consequence of theorem C.12 and the following lemma.

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Lemma C.23. *Let $K$ be a ball complex and $L$ a subcomplex. Given a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}ell(L) & \xrightarrow{F} & \mathcal{A} \\ \big\downarrow & & \big\downarrow{\scriptstyle I} \\ \mathcal{C}ell(K) & \xrightarrow{G} & \mathcal{A}' \end{array}$$

*in which $I$ is an equivalence of categories, there is a functor $H : \mathcal{C}ell(K) \to \mathcal{A}$ such that $H|_{\mathcal{C}ell(L)} = F$ and $I \circ H$ is naturally isomorphic to $G$.*

## C.7. $\mathrm{ad}^R_{\mathrm{Rel,sch}}$ is a functor of $R$.

Let $h : R \to S$ be a homomorphism of rings with involution.

Definition C.24. *Define a functor*

$$h_{\mathrm{Rel,sch}} : \mathcal{A}^R_{\mathrm{Rel,sch}} \to \mathcal{A}^S_{\mathrm{Rel,sch}}$$

*as follows. For an object $(\mathbb{C}, \mathbb{E}, \gamma, \phi)$ of $\mathcal{A}^R_{\mathrm{Rel,sch}}$, Let*

$$h_{\mathrm{Rel}}(C, E, \gamma, \phi) = (h_{\mathrm{sch}}\mathbb{C}, (h \otimes h)_{\mathrm{sch}}\mathbb{E}, \delta, \psi),$$

*where $h_{\mathrm{sch}}$ and $(h \otimes h)_{\mathrm{sch}}$ are given in §C.5, and (letting $C = R\langle \mathbb{C} \rangle$ and $E = (R^{\mathrm{op}} \otimes R)\langle \mathbb{E} \rangle$, and using the notation of definition C.14 and the isomorphism of Eq. (C.2)), $\delta$ is defined by the diagram*

$$\begin{array}{ccc} S\langle h_{\mathrm{sch}}\mathbb{C}\rangle^t \otimes S\langle h_{\mathrm{sch}}\mathbb{C}\rangle & \xrightarrow{\delta} & (S^{\mathrm{op}} \otimes S)\langle (h \otimes h)_{\mathrm{sch}}\mathbb{E}\rangle \\ {\scriptstyle \cong}\big\downarrow & & {\scriptstyle \cong}\big\downarrow \\ (S \otimes_R C)^t \otimes (S \otimes_R C) & \xrightarrow{\gamma'} & (S^{\mathrm{op}} \otimes S) \otimes_{R^{\mathrm{op}} \otimes R} E \end{array}$$

*and $\psi$ is the image of $\phi'$ under the isomorphism*

$$(S \otimes_{S^{\mathrm{op}} \otimes S} (S^{\mathrm{op}} \otimes S) \otimes_{R^{\mathrm{op}} \otimes R} E)^{\mathbb{Z}/2} \cong (S \otimes_{S^{\mathrm{op}} \otimes S} (S^{\mathrm{op}} \otimes S)\langle (h \otimes h)_{\mathrm{sch}}\mathbb{E}\rangle)^{\mathbb{Z}/2}.$$

Proposition C.25. *$h_{\mathrm{Rel,sch}}$ takes ads to ads.*

*Proof.* Let $K$ be a ball complex and let $F \in \mathrm{ad}^R_{\mathrm{Rel,sch}}(K)$. By definition C.21, we only need to show that $\Lambda \circ h_{\mathrm{Rel,sch}} \circ F$ is in $\mathrm{ad}^R_{\mathrm{Rel}}(K)$. But $\Lambda \circ h_{\mathrm{Rel,sch}} \circ F$ is naturally isomorphic to $h_{\mathrm{Rel}} \circ \Lambda \circ F$, which is in $\mathrm{ad}^R_{\mathrm{Rel}}(K)$ by definition C.21 and proposition C.15. □

The reader can check that if $k : S \to T$ is another ring homomorphism then $(kh)_{\mathrm{Rel,sch}}$ is equal to $k_{\mathrm{Rel,sch}} h_{\mathrm{Rel,sch}}$, so $\mathrm{ad}^R_{\mathrm{Rel,sch}}$ is a functor of $R$ as required.