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LOCAL BOUNDARY BEHAVIOR OF BOUNDED HOLOMORPHIC FUNCTIONS

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1. Introduction and statement of results. Let $D \subset\subset \mathbf{C}^n$ be a bounded domain with smooth boundary ∂D, and let F be a bounded holomorphic function on D. A generalization of the classical theorem of Fatou says that the set E of points on ∂D at which F fails to have nontangential limits satisfies the condition $\sigma(E) = 0$, where σ denotes surface area measure. We show in the present paper that this result remains true when σ is replaced by 1-dimensional Lebesgue measure on *certain* smooth curves γ in ∂D. The condition that γ must satisfy is that its tangents avoid certain directions.

We now describe the setting of our theorems in more detail.

1.1. *The domains under consideration.* To say that a bounded open set $D \subset \mathbf{C}^n$ has C^k-boundary means that there is an open set $W \supset \partial D$ and a k times continuously differentiable function ρ: $W \to R$ (i.e., $\rho \in C^k$) such that

$$D \cap W = \{w \in W : \rho(w) < 0\}$$

and such that the vector

$$(1) \quad N(\zeta) = \left(\frac{\partial \rho}{\partial \bar{w}_1}(\zeta), \ldots, \frac{\partial \rho}{\partial \bar{w}_n}(\zeta)\right)$$

is different from 0 at every $\zeta \in \partial D$.

If $\rho \in C^2$ and if there is a constant $\beta > 0$ such that the inequality

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial w_j \partial \bar{w}_k}(w) z_j \bar{z}_k \geqq \beta |z|^2$$

holds for all $z \in \mathbf{C}^n$ and $w \in W$, then D is said to be *strictly pseudoconvex*.

(As usual, $|z|^2 = \langle z, z\rangle^{1/2}$, where $\langle z, w\rangle = \sum z_j \bar{w}_j$ for $z \in \mathbf{C}^n$, $w \in \mathbf{C}^n$.)

1.2. *Tangent spaces.* If D has C^1-boundary and $\zeta \in \partial D$, the tangent space to ∂D at ζ is

$$(2) \quad T_\zeta = \{w \in \mathbf{C}^n: \text{Re}\, \langle w, N(\zeta)\rangle = 0\}.$$

Its maximal complex subspace is

$$(3) \quad P_\zeta = \{w \in \mathbf{C}^n: \langle w, N(\zeta)\rangle = 0\}.$$

The directional condition mentioned in the opening paragraph is that

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for no $\zeta \in \gamma$ *should the tangent to* γ *lie in* P_ζ. To put this into different form, let

$$\varphi: [0, 1] \to \partial D$$

be a C^1-parametrization of a curve γ in ∂D, with $\varphi'(t) \neq 0$ for $0 \leqq t \leqq 1$. Then $\varphi'(t)$ is tangent to γ at $\varphi(t)$, and hence (2) shows that

(4) $\quad \operatorname{Re} \langle \varphi'(t), N(\varphi(t)) \rangle = 0 \quad (0 \leqq t \leqq 1).$

According to (3), the tangent to γ at $\varphi(t)$ belongs to $P_{\varphi(t)}$ if and only if (4) is replaced by the stronger condition

(5) $\quad \langle \varphi'(t), N(\varphi(t)) \rangle = 0.$

1.3. *Nontangential and admissible limits.* If D has C^1-boundary and $\zeta \in \partial D$, the unit outward normal at ζ is the vector

$$\nu(\zeta) = N(\zeta)/|N(\zeta)|.$$

Following Stein [**10**] and Čirka [**2**] we let $\delta_\zeta(w)$ be the minimum of the distances from w to ∂D and from w to the affine tangent plane $\zeta + T_\zeta$. For $\alpha > 0$, we define

(6) $\quad \Gamma_\alpha(\zeta) = \{w \in D: |w - \zeta| < (1 + \alpha)\delta_\zeta(w)\}$

and we let $\mathscr{A}_\alpha(\zeta)$ be the set of all $w \in D$ such that

$$|\langle w - \zeta, \nu(\zeta)\rangle| < (1 + \alpha)\delta_\zeta(w)$$

and $|w - \zeta|^2 < \alpha\delta_\zeta(w)$.

Since $|\operatorname{Re} \langle \zeta - w, \nu(\zeta)\rangle|$ is the distance from w to $\zeta + T_\zeta$, we see that, for a sufficiently small neighborhood V of ζ, $V \cap \Gamma_\alpha(\zeta)$ lies in the cone

$$K_\alpha(\zeta) = \{w \in \mathbf{C}^n: |w - \zeta| < (1 + \alpha) \operatorname{Re} \langle \zeta - w, \nu(\zeta)\rangle\},$$

and that $V \cap \Gamma_\alpha(\zeta) \supset V \cap K_\beta(\zeta)$ for some $\beta < \alpha$. Thus $\Gamma_\alpha(\zeta)$ is a *nontangential* approach region to ζ, and $\mathscr{A}_\alpha(\zeta)$ is a so-called *admissible* approach region to ζ which contains

$$\Gamma_\alpha(\zeta) \cap \{w: |w - \zeta| < \alpha/1 + \alpha\}$$

but which also contains sequences that approach ζ tangentially. (See [**10**, Chapter II]).

A function $f: D \to \mathbf{C}$ is said to have a *nontangential limit* (resp. *admissible limit*) at $\zeta \in \partial D$ if, for all $\alpha > 0$, $\lim f(w)$ exists as $w \to \zeta$ within $\Gamma_\alpha(\zeta)$ (resp. within $\mathscr{A}_\alpha(\zeta)$).

We let $E_\Gamma(f)$ be the set of all $\zeta \in \partial D$ at which f *fails* to have a nontangential limit, and we write $E_{\mathscr{A}}(f)$ for the set where f *fails* to have an admissible limit. Obviously, $E_\Gamma(f) \subset E_{\mathscr{A}}(f)$.

1.4. *The Fatou theorem of Korányi and Stein.* This is the theorem (proved by Korányi for the ball [**6**] and by Stein in general [**10**]) that we referred to in the opening paragraph:

THEOREM. *If D has C^2-boundary and if $f \in H^\infty(D)$ then* $\sigma(E_{\mathscr{A}}(f)) = 0$. *Hence also* $\sigma(E_\Gamma(f)) = 0$.

(As usual, $H^\infty(D)$ is the space of all bounded holomorphic functions $f : D \to \mathbf{C}$, with sup-norm $||f||_\infty$.)

1.5. *Statement of results.* If γ is a curve in ∂D, parametrized by φ as in § 1.2, we can define a measure μ on ∂D by setting

$$(7)\qquad \int f d\mu = \int_0^1 f(\varphi(t))dt$$

for every continuous $f : \partial D \to \mathbf{C}$. Then μ is supported by γ, and μ depends of course on the particular parametrization φ that is chosen. But the collection of sets of μ-measure 0 depends only on γ itself, and in this sense we may speak of a property holding almost everywhere on γ.

We recall that γ is said to belong to the class $\Lambda_{1+\alpha}$ if γ has a C^1-parametrization φ whose derivative φ' satisfies a uniform Lipschitz condition of order α; here $0 < \alpha < 1$.

THEOREM 1. *Suppose D has C^1-boundary, γ is a curve in ∂D, $\gamma \in \Lambda_{1+\alpha}$ for some $\alpha > 0$, and*

$$(8)\qquad \langle \varphi'(t), N(\varphi(t))\rangle \neq 0$$

for every $t \in [0, 1]$. Then $\mu(E_\Gamma(F)) = 0$ for every $F \in H^\infty(D)$.

In other words, if the tangent to γ belongs nowhere to P_ζ (see § 1.2) then every $F \in H^\infty(D)$ has nontangential limits almost everywhere on γ.

Here is what happens when (8) is violated:

THEOREM 2. *Suppose D is strictly pseudoconvex, with C^2-boundary, and suppose $\varphi : [0, 1] \to \partial D$ parametrizes a C^1-curve γ. If*

$$(9)\qquad \langle \varphi'(t), N(\varphi(t))\rangle = 0$$

for every $t \in [0, 1]$, then there exists an $F \in H^\infty(D)$ which has no limit along any curve in D that ends on γ. In particular, $\gamma \subset E_\Gamma(F)$.

In our next theorem, we specialize D to be the unit ball

$$B_2 = \{z \in \mathbf{C}^2 : |z| < 1\}.$$

THEOREM 3. *There exists an $F \in H^\infty(B_2)$ that has no admissible limit at any point of the circle*

$$(10)\qquad \gamma = \{(e^{i\theta}, 0) : 0 \leqq \theta \leqq 2\pi\}.$$

Thus $\gamma \subset E_{\mathscr{A}}(F)$.

Note that the curve (10) satisfies (8). Theorem 3 shows therefore that the conclusion of Theorem 1 cannot be strengthened to give $\mu(E_{\mathscr{A}}(F)) = 0$ for every $F \in H^\infty(D)$.

Our proof of Theorem 1 uses a one-variable theorem which extends the classical Fatou theorem in yet another way:

THEOREM 4. *Let the segment* $(0, 1) \subset \mathbf{R}$ *be one edge of an open rectangle* Q *in the upper half of* $\mathbf{C}$. *Suppose*

(a) $f : Q \to \mathbf{C}$ *is a bounded* C^1*-function, and*

(b) $\partial f/\partial \bar{z} \in L^p(Q)$ *for some* $p > 1$.

Then $\lim f(x + iy)$ *exists for almost all* $x \in (0, 1)$, *as* $y \to 0$.

Here, and later, L^p refers to Lebesgue measure on $\mathbf{C}$. Note that (b) represents a considerable weakening of the classical hypothesis that $f \in H^\infty(Q)$, i.e. that $\partial f/\partial \bar{z} = 0$.

2. Proof of Theorem 4. For $1 \leqq k \leqq \infty$, we shall write C_c^k for the class of all $f : \mathbf{C} \to \mathbf{C}$ that are k times continuously differentiable and have compact support.

2.1. LEMMA. *To every* p, $1 < p < \infty$, *corresponds a constant* $A_p < \infty$ *such that the inequality*

$$\left\| \frac{\partial f}{\partial y} \right\|_p \leqq A_p \left\| \frac{\partial f}{\partial \bar{z}} \right\|_p$$

holds for all $f \in C_c^1$.

This follows from the L^p-boundedness (for $1 < p < \infty$) of the Riesz transforms on $\mathbf{R}^2$. A proof is given on p. 60 of [**11**].

2.2. LEMMA. *Suppose* Ω *is a bounded open set in* $\mathbf{C}$, $1 < p < \infty$, *and* $g \in L^p(\Omega)$. *If* $G \in C^1(\Omega)$ *and if*

$$G(z) = \frac{1}{2\pi i} \int_\Omega \frac{g(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta} \tag{11}$$

for almost all $z \in \Omega$, *then* $\partial G/\partial y \in L^p(\Omega)$.

Proof. Regard g as a member of $L^p(\mathbf{C})$ which is 0 off Ω. Put $k(z) = 1/\pi z$. Then k is locally L^1, and the convolution $H = g * k$, defined by

$$H(z) = \int_{\mathbf{C}} g(\zeta)k(z - \zeta)d\xi d\eta \quad (\zeta = \xi + i\eta) \tag{12}$$

exists for almost all $z \in \mathbf{C}$, as a Lebesgue integral. Moreover, comparison of (11) and (12) shows that the C^1-function G coincides with H a.e. in Ω.

Choose $\chi \in C_c^\infty$ so that $\chi = 1$ on Ω. Choose $\psi \in C_c^\infty$, $\psi \geqq 0$, so that $\int_{\mathbf{C}} \psi = 1$. For $1 \leqq t < \infty$, define $\psi_t(z) = t^2\psi(tz)$.

There is a disc $D \subset \mathbf{C}$, of radius r, that contains the supports of χ and of

$|g| * \psi_t$ for all $t \in [1, \infty)$. It is easily seen that

$$(13) \quad \int_D |k(z - \zeta)| d\xi d\eta \leqq 2r$$

for all $z \in \mathbf{C}$.

Define $H_t = H * \psi_t$. Since $H = g * k$, we have

$$(14) \quad H_t = (g * \psi_t) * k.$$

Since $||\psi_t||_1 = 1$ for all t, Hölder's inequality and (13) lead from (14) to

$$(15) \quad \left\{ \int_D |H_t(z)|^p dxdy \right\}^{1/p} \leqq 2r||g||_p \quad (1 \leqq t < \infty).$$

Since $g * \psi_t \in C_c^\infty$, Theorem 1.2.2 of [**5**] can be applied to (14) and shows that

$$(16) \quad \partial H_t / \partial \bar{z} = g * \psi_t.$$

Hence

$$(17) \quad \frac{\partial(\chi H_t)}{\partial \bar{z}} = H_t \cdot \frac{\partial \chi}{\partial \bar{z}} + \chi \cdot (g * \psi_t)$$

so that (15) gives the estimate

$$(18) \quad \left\| \frac{\partial(\chi H_t)}{\partial \bar{z}} \right\|_p \leqq M||g||_p$$

in which M is a real number that depends only on χ and r. It now follows from Lemma 2.1 that

$$(19) \quad \left\| \frac{\partial(\chi H_t)}{\partial y} \right\|_p \leqq A_p M||g||_p \quad (1 \leqq t < \infty).$$

To every compact $K \subset \Omega$ corresponds a $t(K)$ such that $K - \operatorname{supp} \psi_t \subset \Omega$ for all $t > t(K)$. Since $G = H$ a.e. in Ω, we have

$$(20) \quad H_t(z) = \int G(z - \zeta) \psi_t(\zeta) d\xi d\eta \quad (z \in K)$$

if $t > t(K)$. Since $G \in C^1(\Omega)$ and $\chi = 1$ in Ω, it follows that

$$(21) \quad \left(\frac{\partial G}{\partial y} \right)(z) = \lim_{t \to \infty} \frac{\partial(\chi H_t)}{\partial y}(z) \quad (z \in \Omega).$$

By (19) and (21), Fatou's lemma shows that $\partial G/\partial y \in L^p(\Omega)$.

Remark. Lemma 2.2 would become false if, instead of (11), we merely assumed that $\partial G/\partial \bar{z} = g \in L^p(\Omega)$. To see this, take G holomorphic in Ω, so that $\partial G/\partial \bar{z} = 0$, but of sufficiently rapid growth near some boundary point to have $\partial G/\partial y \notin L^p(\Omega)$.

2.3. *Proof of Theorem* 4. To fix the notation, assume that $Q = (0, 1) \times (0, h)$. Since f is bounded, there is a sequence $y_n \searrow 0$ such that the functions $x \to f(x + iy_n)$ converge weak* in $L^\infty(0, 1)$, to some $\varphi \in L^\infty(0, 1)$. Extend f to $(0, 1) \times [0, h)$ by setting $f(x, 0) = \varphi(x)$.

Choose a small $\epsilon > 0$ and define

$$Q_\epsilon = [\epsilon, 1 - \epsilon] \times (0, h - \epsilon], \quad Q_{\epsilon,n} = [\epsilon, 1 - \epsilon] \times [y_n, h - \epsilon].$$

For z interior to Q_ϵ and for n sufficiently large, the fact that f is C^1 on the compact set $Q_{\epsilon,n}$ shows (Theorem 1.2.1 in [**5**]) that

$$f(z) = \frac{1}{2\pi i} \int_{\partial Q_{\epsilon,n}} f(\zeta) \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{Q_{\epsilon,n}} \frac{\partial f}{\partial \bar{\zeta}} (\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

The above-mentioned weak*-convergence, combined with the fact that $\partial f / \partial \bar{\zeta} \in L^1(Q)$, shows that we can let $n \to \infty$, to obtain

$$f(z) = \frac{1}{2\pi i} \int_{\partial Q_\epsilon} f(\zeta) \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{Q_\epsilon} \frac{\partial f}{\partial \bar{\zeta}} (\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

$$= H(z) + G(z).$$

Since H is the Cauchy integral of a bounded function, it is classical (see, for instance, Lemma 2.6 in Chap. V of [**12**]) that $\lim H(x + iy)$ exists, as $y \to 0$, for almost all x in $(\epsilon, 1 - \epsilon)$.

Since $G = f - H$, $G \in C^1(Q_\epsilon)$. By Lemma 2.2, $\partial G / \partial y \in L^p(Q_\epsilon)$. Setting

$$M(x) = \int_0^{h-\epsilon} \left| \frac{\partial G}{\partial y} (x, y) \right|^p dy.$$

it follows that $M(x) < \infty$ a.e. in $(\epsilon, 1 - \epsilon)$. If $0 < y_0 < y_1 < h - \epsilon$, Hölder's inequality gives

$$|G(x + iy_1) - G(x + iy_0)| \leqq M(x)^{1/p} |y_1 - y_0|^{1-1/p}.$$

Hence $\lim G(x + iy)$ exists, as $y \to 0$, for almost all x in $(\epsilon, 1 - \epsilon)$. The arbitrariness of ϵ shows that the proof is complete.

3. Proof of Theorem 1. Referring to §1.1, we may of course assume that the gradient of ρ is bounded in W.

We are given $\varphi : [0, 1] \to \partial D$, $\varphi \in C^1$, $\varphi' \in \Lambda_\alpha$ for some $\alpha \in (0, 1)$. Since (8) is assumed to hold, we may assume, without loss of generality, that there is a constant $\eta > 0$ such that

(22) $\quad \operatorname{Im} \langle \varphi'(x), N(\varphi(x)) \rangle \geqq \eta > 0 \quad (0 \leqq x \leqq 1).$

The proof proceeds in several steps. We extend φ to a map Φ of a rectangle Q into D, in such a way that each point $\varphi(x) \in \gamma$ is an end point of a nontangential curve ψ_x lying in $\Phi(Q)$. We then show that $F \circ \Phi$ and Q satisfy the hypotheses of Theorem 4, and that F therefore has limits along almost all of the curves ψ_x. The desired conclusion follows then from Čirka's recent extension of Lindelöf's theorem to n variables.

Step 1. *The map* Φ. Extend φ' to be a ($\mathbf{C}^n$-valued) function on $\mathbf{R}$, with compact support, of class Λ_α. Let $u(x, y)$ be the Poisson integral of φ', for $y \geqq 0$, and define

(23) $\quad \Phi(x + iy) = \varphi(x) + iyu(x, y) \quad (0 \leqq x \leqq 1, y \geqq 0).$

Since $(\partial\Phi/\partial y)(x) = iu(x, 0) = i\varphi'(x)$, we have, by (22),

$$\left[\frac{\partial}{\partial y}(\rho \circ \Phi)\right](x) = 2 \operatorname{Re} \sum_{j=1}^{n} \frac{\partial \rho}{\partial w_j}(\Phi(x))\frac{\partial \Phi_j}{\partial y}(x)$$
$$= -2 \operatorname{Im} \langle \varphi'(x), N(\varphi(x))\rangle \leqq -2\eta.$$

If $h > 0$ is small enough, it follows that Φ maps the rectangle $Q = (0, 1) \times (0, h)$ into $W \cap D$, and that $(\partial/\partial y)(\rho \circ \Phi)(x + iy) \leqq -\eta$, hence

(24) $\quad \rho(\Phi(x + iy)) \leqq -\eta y \quad (x + iy \in Q).$

Standard estimates of the Poisson integral show that $|y(\partial u/\partial x)|$, $|y(\partial u/\partial y)|$, and $|u(x, y) - \varphi'(x)|$ are dominated by Cy^α, where C depends on α and on the Lipschitz constant of φ'. Hence differentiation of (23) yields

(25) $\quad \left|\frac{\partial \Phi}{\partial \bar{z}}(x + iy)\right| \leqq C_1 y^\alpha \quad (0 \leqq x \leqq 1, y > 0).$

Step 2. *The curves* ψ_x. For $0 \leqq x \leqq 1$, $y \geqq 0$, define

(26) $\quad \psi_x(y) = \Phi(x + iy).$

We claim that $\psi_x(y)$ tends *nontangentially* to $\psi_x(0) = \varphi(x) \in \gamma$ when $y \searrow 0$.

Setting $\zeta = \varphi(x)$, $w = \psi_x(y)$, (23) shows that

(27) $\quad \zeta - w = -iyu(x, y).$

Thus $|\zeta - w| \leqq cy$ and, by (22)

$$\operatorname{Re}\langle \zeta - w, N(\zeta)\rangle = y \operatorname{Im}\langle u(x, y), N(\varphi(x))\rangle \geqq \tfrac{1}{2}y\eta \geqq (2c)^{-1}\eta|\zeta - w|$$

as soon as y is small enough. Thus $\psi_x(y)$ lies in some cone $K_\alpha(\zeta)$ (see § 1.3) for all sufficiently small y.

Step 3. Now let $F \in H^\infty(D)$. Define $f : Q \to \mathbf{C}$ by $f(z) = F(\Phi(z))$. Fix $z \in Q$, for the moment. Then $w = \Phi(z)$ is the center of a ball in D whose radius is at least $|\rho(w)|/C_2$, where C_2 is an upper bound for the gradient of ρ in W. The one-variable Schwarz lemma, applied to restrictions of F to complex lines through w, shows therefore that

(28) $\quad \left|\frac{\partial F}{\partial w_j}(w)\right| \leqq C_2|\rho(w)|^{-1}||F||_\infty \leqq C_2||F||_\infty \eta^{-1}y^{-1},$

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by (24), since $w = \Phi(x + iy)$. We now conclude from (25), (28), and the formula

$$(29)\quad \frac{\partial f}{\partial \bar{z}}(z) = \sum_{j=1}^{n} \frac{\partial F}{\partial w_j}\left(\Phi(z)\right)\frac{\partial \Phi_j}{\partial \bar{z}}(z)$$

that $|(\partial f/\partial \bar{z})(x + iy)| \leqq C_3 y^{\alpha-1}$, so that $f \in L^p(Q)$ for some $p > 1$. [Observe that (29) depends on the fact that F is holomorphic.] The other hypotheses of Theorem 4 are obviously satisfied.

It follows that $\lim f(x + iy)$ exists, as $y \searrow 0$, for every x in a set $E \subset (0, 1)$ whose complement has measure 0. In other words, F has a limit along the *nontangential* curve ψ_x that ends at $\varphi(x)$, for every $x \in E$. Since $F \in H^\infty(D)$, Čirka's Lindelöf theorem (Theorem 1 in [**2**]) asserts that F has a nontangential limit at $\varphi(x) \in \gamma$, for every $x \in E$.

This completes the proof of Theorem 1.

Remark. The technique of mapping the rectangle Q into D by a map Φ that satisfies $\partial\Phi/\partial\bar{z} = 0$ on the real axis has been used by Henkin and Tumanov to study peak sets for the algebra $A(D)$. (These will be defined in the section that follows.)

4. Proof of Theorem 2.

4.1 *Definitions.* Let $D \subset\subset \mathbf{C}^n$ be a domain. Let $A(D)$ denote the algebra of all continuous complex functions on $\bar{D}$ that are holomorphic in D. A function $G \in A(D)$ is said to *peak on the set* $K \subset \partial D$ if $G(w) = 1$ for every $w \in K$ but $|G(w)| < 1$ for all other $w \in \bar{D}$. If K is such that some $G \in A(D)$ peaks on K, then K is a *peak set* for $A(D)$.

4.2. Lemma. *If $D \subset\subset \mathbf{C}^n$ is a domain and if $K \subset \partial D$ is a peak set for $A(D)$, then there exists an $F \in H^\infty(D)$ which has no limit along any curve in D that ends at a point of K.*

Proof. Let $G \in A(D)$ peak on K. Then $\mathrm{Re}\,(1 - G(w)) > 0$ if $w \in \bar{D}\backslash K$. Hence there is a well defined branch of $\log\,(1 - G(w))$, holomorphic on D and continuous on $\bar{D}\backslash K$. Moreover,

$$\mathrm{Re}\,[\log\,(1 - G(w))] = \log|1 - G(w)| \to -\infty$$

as $w \to K$, and

$$|\mathrm{Im}\,[\log\,(1 - G(w))]| = |\arg\,(1 - G(w))| \leqq \pi/2.$$

Setting $F(w) = \exp\,[i \log\,(1 - G(w))]$, F has the desired properties.

4.3. Corollary. *Let $D \subset\subset \mathbf{C}^n$ have C^1-boundary. If $K \subset \partial D$ is a peak set for $A(D)$, if γ (parametrized by φ) satisfies the hypotheses of Theorem 1, and if μ is the measure on γ defined by (7), then $\mu(K \cap \gamma) = 0$.*

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Proof. By Lemma 4.2, some $F \in H^\infty(D)$ has no limit along any curve in D that ends on K. Thus $K \subset E_\Gamma(F)$. By Theorem 1, $\mu(E_\Gamma(F)) = 0$.

Remark. This corollary has been proved for C^2-curves by Henkin and Tumanov (in an as yet unpublished paper) by different methods. A third proof, for C^2-curves in the boundary of the unit ball in $\mathbf{C}^n$, appears in [**8**].

4.4. *Proof of Theorem* 2. The hypotheses of Theorem 2 show, by a theorem of Davie and Øksendal [**3**], that the range of φ is a peak set for $A(D)$. Thus Theorem 2 follows from Lemma 4.2.

Remark. If $D \subset\subset \mathbf{C}^n$ is strictly pseudoconvex with C^2-boundary, and if M is a real C^1-submanifold of ∂D whose tangent space lies in P_ζ for every $\zeta \in M$, it follows from [**9**] that every compact $K \subset M$ is a peak set for $A(D)$. (The same result was obtained earlier, under stronger regularity assumptions, in [**1**; **4** and **7**].) We understand that Nils Øvrelid has proved that the boundary of every C^2-strictly pseudo-convex domain contains such manifolds of real dimension $n - 1$. In conjunction with Lemma 4.2, this implies that there exists an $F \in H^\infty(D)$ such that the set $E_\Gamma(F)$ (where F has no nontangential limit) contains a manifold of real dimension $n - 1$.

5. Proof of Theorem 3. We change notation slightly, and let

$$B = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 < 1\}.$$

Put $n_k = (k!)^2$ and define

$$F(z, w) = w^2 \sum_{k=1}^{\infty} (n_k - n_{k-1})z^{n_k}. \tag{30}$$

We will show that $F \in H^\infty(B)$ and that F does *not* have an admissible limit at any point $(e^{i\theta}, 0)$, although $F(z, 0) = 0$ for all z with $|z| < 1$.

Put $g_k(z) = (n_k - n_{k-1})z^{n_k}$. Then $|g_k(z)| \leqq \sum |z|^m$, where m ranges over the integers that satisfy $n_{k-1} < m \leqq n_k$. Hence $\sum_1^\infty |g_k(z)| \leqq (1 - |z|)^{-1}$. Since $|w|^2 < 1 - |z|^2$ in B, we have $|F(z, w)| < 2$. Thus $F \in H^\infty(B)$.

Put $r_k = 1 - (1/n_k)$ for $k \geqq 2$. Since $(r_k)^{n_k}$ increases to the limit $1/e$ as $k \to \infty$, and since $n_k/n_{k-1} = k^2$, we obtain the following estimates for $z = r_k e^{i\theta}$:

$$|g_k(z)| = (1 - k^{-2})n_k(r_k)^{n_k} > n_k/3$$

for large k,

$$\sum_{s=1}^{k-1} |g_s(z)| \leqq \sum_{s=1}^{k-1} n_s < kn_{k-1} = k^{-1}n_k,$$

and

$$\sum_{s=k+1}^{\infty} |g_s(z)| \leqq \sum_{k+1}^{\infty} n_s(r_k)^{n_s} < \sum_{k+1}^{\infty} n_s(r_{s-1})^{n_s} < \sum_{k+1}^{\infty} n_s e^{-s^2}.$$

The ratio test shows that the last sum is the tail end of a convergent series, hence it tends to 0 as $k \to \infty$. It follows that there is a k_0 such that

$$(31) \quad \left| \sum_{s=1}^{\infty} g_s(r_k e^{i\theta}) \right| \geqq \frac{n_k}{4} = \frac{1}{4(1-r_k)} \quad (0 \leqq \theta \leqq 2\pi)$$

for all $k \geqq k_0$.

Now fix c, $0 < c < 1$. For $k \geqq k_0$ it follows from (30) and (31) that

$$(32) \quad |F(r_k e^{i\theta}, c\sqrt{1 - r_k^2})| \geqq c^2/4 \quad (0 \leqq \theta \leqq 2\pi).$$

But note also that $(r_k e^{i\theta}, c\sqrt{1 - r_k^2})$ tends to $(e^{i\theta}, 0)$ within an admissible approach region. In fact, setting $\zeta = (e^{i\theta}, 0)$, a little computation shows that the points in question lie in $\mathscr{A}_\alpha(\zeta)$ if $\alpha > 4/(1 - c^2)$. (See § 1.3.) Since $F(re^{i\theta}, 0) = 0$ for $0 < r < 1$, (32) shows that F does not have an admissible limit at $(e^{i\theta}, 0)$.

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