NOTES ON LOCALLY COMPACT CONNECTED TOPOLOGICAL LATTICES

TAE HO CHOE

It was shown in (**2**) that if

(1) L is a locally compact connected topological lattice and if

(2) L is topologically contained in R^2, the Euclidean plane,

then each compact subset of L has an upper bound and a lower bound in L. It was also asked whether this result could be proved without assuming condition (2). In this note, we show that this result continues to hold if condition (2) is weakened to: L is finite-dimensional.

In (**11**), it was shown that the centre of a compact topological lattice is totally disconnected. We shall prove that this result is also true even in a locally compact, locally convex topological lattice with 0 and 1. This yields that any locally compact topological Boolean algebra is totally disconnected.

Finally, we shall give a necessary and sufficient condition for a topological lattice to admit enough continuous lattice homomorphisms into I, the closed unit interval, to distinguish points.

The terminology and notation used in this note is the same as in (**1**; **2**; **5**). It is well known that any locally compact connected topological lattice is chain-wise connected, which means that for any pair a, b with $a < b$ there exists a compact connected chain from a to b.

THEOREM 1. *If L is a locally compact connected topological lattice of finite dimension, then each compact subset of L is bounded.*

Proof. We recall from (**1**) that a locally compact connected topological lattice is locally convex, and from (**9**) that its codimension is not less than its breadth. It was also shown in (**6**) that if L is a locally compact and locally convex topological lattice of finite breadth, then for a neighbourhood U of a point p in L, there exist a neighbourhood V of p and a closed interval $[s, t]$ ($= s \vee (t \wedge L)$ with $s \leqq t$) such that $V \subset [s, t] \subset U$. Now let us begin the proof of the theorem. Let A be a compact subset of L. For every $a \in L$, consider L as a neighbourhood of a. Choose a neighbourhood $V(a)$ of a and a closed interval $[s(a), t(a)]$ such that $V(a) \subset [s(a), t(a)]$. Clearly

$$\{V(a) \mid a \in A\}$$

Received September 26, 1968 and in revised form, June 12, 1969. This research was supported in part by the Department of University Affairs, Ontario Government of Canada, and by the National Research Council of Canada, under Grant A4809.

is also an open covering of A. Having a finite open sub-covering $\{V(a_i)\}$ of $\{V(a)\}$, we can easily see that A is bounded by the elements $\inf s(a_i)$ and $\sup t(a_i)$ in L.

The following corollaries are immediate.

COROLLARY 1. *If L is a locally compact and locally convex topological lattice of finite breadth, then each compact subset is bounded.*

COROLLARY 2. *If L is a locally compact connected metric topological lattice of finite dimension, then L is simply connected i.e., the fundamental group π_1 of L is trivial.* (See **2**, Theorem 5.)

An element a of a lattice L is *neutral* if and only if every triple $\{a, x, y\}$ in L generates a distributive sublattice of L. An element is in the *centre* of a lattice with 0 and 1 if and only if it is neutral and complemented. It is well known that the centre of a lattice with 0 and 1 forms a Boolean lattice. It is also well known that the connected component of an element in a topological lattice L forms a sublattice of L.

THEOREM 2. *If L is a locally compact, locally convex topological lattice with* 0 *and* 1, *then the centre of L is totally disconnected.*

Proof. Let C be the centre of L and let E be the connected component of C containing an element c. Choose neighbourhoods U, V, and W of c, in L, such that U^* (where * denotes topological closure) is compact, V is convex and $W \vee W \subset V \subset U^*$, $W \wedge W \subset V \subset U^*$. We now show that $W \cap E = \{c\}$. Assume that there is an element $d' \neq c$ in $W \cap E$. There are two cases to consider.

Case 1. $c \vee d' \neq c$. Let $d = c \vee d'$, thus $c < d$. Furthermore,

$$d \in W \vee W \subset V,$$

and $d \in E$, since E is a sublattice. Hence $d \in V \cap E$. Since V is convex, $[c, d] \subset V$.

Case 2. $c \vee d' = c$. In this case let $d = d'$, so that $d < c$. Then

$$d \in W \cap E \subset V \cap E,$$

and $[d, c] \subset V$. The cases are entirely analogous and thus we shall only consider the first case. Note that $[c, d]$ is a compact sublattice since it is a closed subset of U^*. Further, a little verification shows that since C is a sublattice of L, we have

$$\{c \vee (d \wedge L)\} \cap C = c \vee (d \wedge C) \supset c \vee (d \wedge E).$$

Note that $c \vee (d \wedge C)$ is a connected sub-Boolean topological lattice containing more than one point. Further, it is contained in the centre of the compact lattice $[c, d]$ $(= c \vee (d \wedge L))$. In fact, every element in $c \vee (d \wedge C)$ is relatively complemented in the closed interval $[c, d]$ and is a neutral element

https://doi.org/10.4153/CJM-1969-168-x Published online by Cambridge University Press

of $[c, d]$ since it is neutral in L. Now we recall that the centre of a compact topological lattice is totally disconnected (**11**); this contradiction completes the proof.

We note that compactness implies local convexity in a topological lattice, and hence Theorem 2 yields the theorem in (**11**).

COROLLARY 3. *If L is a locally compact topological Boolean lattice, then L is totally disconnected.*

Proof. It suffices to show that the connected component E of zero 0 in L is $\{0\}$ itself. Suppose that E contains an element a other than 0. Then we have $a \wedge E = [0, a]$, since if $x \in [0, a]$, then $x = x \wedge a \in x \wedge E$. Since $x \wedge E$ is connected, and contains 0, we have $x \in E$, hence $x \in a \wedge E$. Thus $a \wedge E$ is a locally compact connected non-degenerate topological Boolean lattice in its relative topology. Therefore it is locally convex. It must be totally disconnected by Theorem 2. This is a contradiction.

A non-degenerate closed interval $K = [a, b]$ (i.e., $a \neq b$) in a topological lattice L is called I-reducible if and only if there exists at least one non-constant continuous lattice homomorphism of K into the closed unit interval I with the usual lattice operations and the usual topology. Let $\mathrm{Hom}(L, I)$ denote the collection of continuous lattice homomorphisms of a topological lattice L into I.

LEMMA 1. *Let L be a topological lattice. Then* $\mathrm{Hom}(L, I)$ *distinguishes points if and only if L is distributive and each non-degenerate closed interval in L is I-reducible.*

Proof. Suppose that $\mathrm{Hom}(L, I)$ distinguishes points and let $[a, b]$ be a non-degenerate interval of L. Choose $\phi \in \mathrm{Hom}(L, I)$ such that $\phi(a) \neq \phi(b)$. Then clearly the restriction of ϕ on $[a, b]$ is non-constant, and it is in $\mathrm{Hom}([a, b], I)$. Thus the interval $[a, b]$ is I-reducible. We now show that L is distributive. Since $\mathrm{Hom}(L, I)$ distinguishes points, the evaluation mapping: $L \to I^{\mathrm{Hom}(L,I)}$ is a lattice monomorphism. Thus L must be distributive.

Conversely, for a, $b \in L$ with $a \neq b$, either $[a, a \vee b]$ or $[a \wedge b, a]$ is non-degenerate. Assume that $[a, a \vee b]$ is I-reducible. Let $a \vee b = c$. For a non-constant mapping $f \in \mathrm{Hom}([a, c], I)$ we define $F: L \to I$ by

$$F(x) = f(a \vee (c \wedge x));$$

then $F \in \mathrm{Hom}(L, I)$. Further, $F(a) \neq F(b)$ since $F(a) \neq F(c)$.

Recently Lawson (**10**) gave an example of a compact connected metrizable distributive topological lattice L with $\mathrm{Hom}(L, I)$ consisting of constant mappings only, i.e., $\mathrm{Hom}(L, I)$ does not separate points.

THEOREM 3. *If L is a locally compact connected distributive topological lattice, and if each non-degenerate closed interval in L has a finite-dimensional non-degenerate closed subinterval, then* $\mathrm{Hom}(L, I)$ *distinguishes points.*

Proof. It is enough to show that any finite-dimensional non-degenerate closed interval $K = [a, b]$ in L is an I-reducible interval. We recall (**3**; **5**) that $\mathrm{Ca}([a, b]) \leqq$ the breadth of $[a, b] \leqq$ the dimension of $[a, b]$ $(= n > 0)$, where $\mathrm{Ca}([a, b])$ denotes the number of atoms of the centre of $[a, b]$. Now consider the set $\{m \mid \mathrm{Ca}(x, y) = m$ for some $[x, y] \subset [a, b]\}$. Let m be the maximal positive integer of the set and let $m = \mathrm{Ca}([x, y])$ for some $[x, y] \subset [a, b]$. Then the interval $[x, y]$ is iseomorphic with a Cartesian product of m non-degenerate compact connected chains (see **7**). Therefore, since $m \geqq 1$, $[x, y]$ contains a non-degenerate compact connected chain, which is also a closed interval of L. It was established in (**12**) that for any compact connected chain C, $\mathrm{Hom}(C, I)$ is point-separating. Thus C is, of course, I-reducible if C is non-degenerate.

From the proof of Theorem 3, the following corollary is immediate.

COROLLARY 4 (L. W. Anderson). *If L is a locally compact connected distributive topological lattice of finite breadth, then* $\mathrm{Hom}(L, I)$ *separates points.*

References

1. L. W. Anderson, *One-dimensional topological lattices*, Proc. Amer. Math. Soc. *10* (1959), 715–720.

2. ——— *On the distributivity and simple connectivity of plane topological lattices*, Trans. Amer. Math. Soc. *91* (1959), 102–112.

3. ——— *On the breadth and co-dimension of a topological lattice*, Pacific J. Math. *9* (1959), 327–333.

4. ——— *The existence of continuous lattice homomorphisms*, J. London Math. Soc. *37* (1962), 60–62.

5. T. H. Choe, *On compact topological lattices of finite dimension*, Trans. Amer. Math. Soc. *140* (1969), 2023–2037.

6. ——— *Intrinsic topologies in a topological lattice*, Pacific J. Math. *28* (1969), 49–52.

7. ——— *The breadth and dimension of a topological lattice*, Proc. Amer. Math. Soc. *23* (1969), 82–84.

8. E. Dyer and A. Shields, *Connectivity of topological lattices*, Pacific J. Math. *9* (1959), 443–448.

9. J. D. Lawson, *Vietoris mappings and embeddings of topological semi-lattices*, Dissertation, University of Tennessee, Knoxville, Tennessee, 1967.

10. ——— *Lattices with no interval homomorphisms* (to appear).

11. A. D. Wallace, *The center of a compact lattice is totally disconnected*, Pacific J. Math. *7* (1957), 1237–1238.

12. L. W. Ward, Jr., *Binary relations in topological spaces*, An. Acad. Brazil. Ci. *26* (1954), 357–373.

McMaster University,
Hamilton, Ontario