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REGULARITY OF LOCALLY CONVEX SURFACES

FRIEDMAR SCHULZ

Interior estimates are derived for the $C^{2,\mu}$-Hölder norm of the radius vector $X \in C^{1,1}(\Omega)$ of a locally convex surface Σ in terms of the first fundamental form I_Σ, the Gauss curvature K and the integral $\int |H|\,d\sigma$. Here H is the mean curvature of Σ. The coefficients g_{ij} of I_Σ are assumed to belong to the Hölder class $C^{2,\mu}(\Omega)$ for some μ, $0 < \mu < 1$. A boundary condition is discussed which ensures an estimate for $\int |H|\,d\sigma$.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let Ω be a domain in the $u = (u^1, u^2)$-plane. Consider a differential geometric, locally convex surface Σ, which is given by a radius vector X of class $C^{1,1}(\Omega, \mathrm{R}^3)$ such that the unit normal

$$\nu = \frac{D_1X \wedge D_2X}{|D_1X \wedge D_2X|}$$

exists.

ASSUMPTION (A). Suppose that the coefficients g_{ij} of the first fundamental form

$$\begin{aligned} I_\Sigma &= D_iX \cdot D_jX\, du^i du^j \\ &= g_{ij} du^i du^j \end{aligned}$$

belong to the Hölder class $C^{2,\mu}(\Omega)$ for some μ, $0 < \mu < 1$, such that

$$\|g_{ij}\|_{C^{2,\mu}(\Omega)} \leqslant a,$$

and

$$g, K \geqslant \frac{1}{c}.$$

Here

$$g = \det I_\Sigma = |D_1X \wedge D_2X|^2,$$

and

$$K = \frac{\det II_\Sigma}{\det I_\Sigma} = \frac{h}{g}$$

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is the Gauss curvature of Σ, which, by the *theorema egregium*, depends only on the coefficients of I_Σ and their first and second derivatives.

$$II_\Sigma = D_{ij}X \cdot \nu \, du^i du^j$$
$$= h_{ij} du^i du^j$$

is the second fundamental form, which is defined almost everywhere.

ASSUMPTION(B). Suppose that

$$\int_\Sigma |H| \, d\sigma \leqslant M,$$

where
$$H = \frac{h_{ij} g^{ij}}{2}, \qquad [g^{ij}] = [g_{k\ell}]^{-1},$$

is the mean curvature, and

$$d\sigma = \sqrt{g} \, du$$

is the area element of Σ.

The main result of this note then reads as the following:

THEOREM 1. *The radius vector X belongs to the Hölder class $C^{2,\mu}_{loc}(\Omega)$. For each subset Ω', which is compactly contained in Ω, there is an estimate of the form*

$$\|D^2 X\|_{C^\mu(\Omega')} \leqslant C, \tag{1}$$

where the constant C only depends on μ, a, c, M, and dist$(\Omega', \partial\Omega)$.

The regularity part of Theorem 1 follows from the regularity theory for elliptic Monge–Ampère equations [**27**] via the Darboux equation (5) (see for example, Nirenberg [**18**]). The $C^{2,\mu}_{loc}$-estimates follow from [**24, 26**], if Σ is a graph or a closed surface. These results can also be derived from the prescribed Gauss curvature equation

$$\det D^2 z = K\left(1 + |Dz|^2\right)^2 \tag{2}$$

(compare Sabitov [**20**] for the regularity and [**22, 23**] for the *a priori* estimates for graphs and closed surfaces). The prescribed Gauss curvature equation (2) is particularly useful when the regularity requirements regarding I_Σ are weakened to the extent that the Gauss curvature K is only pinched between two positive numbers (see Heinz [**6**], Nikolaev and Shefel' [**16, 17**]).

The regularity statement can be considered a variation of regularity theorems of Alexandrow [**1**] and Pogorelov [**19**]. That it is sharp follows from Sabitov and Shefel' [**21**], who investigated the connections between the regularity of a surface and its metric.

The case of closed surfaces is of particular interest because of Weyl's embedding problem (see Weyl [**29**], Lewy [**14**], Nirenberg [**18**], Heinz [**5, 22, 23**]).

The purpose of the present note however is to provide the stated local $C^{2,\mu}$-estimates for the radius vector X of a locally convex surface, thus improving those of Heinz [**7**], which require additional regularity assumptions regarding both the radius vector $X(u)$ and the coefficients g_{ij} of the first fundamental form. The approach, which is due to Heinz [**7**], consists of introducing conjugate isothermal parameters, that is, of constructing a conformal map $x = x(u)$ with respect to the second fundamental form of Σ.

The present estimates rest on sharp estimates for the Jacobian of the Darboux system (8), which is satisfied by the inverse mapping $u = u(x)$. These estimates were derived in [**25**], generalising classical theorems of Lewy [**12, 13**] (see also Efimow [4]) and Heinz [**7, 8**].

The *a priori* constant in (1) does depend on the integral $\int |H|\, d\sigma$, because suitable Riemannian metrics on the unit disc with positive Gauss–Kronecker curvature can be embedded in Euclidean 3-space such that $\int |H|\, d\sigma$ is arbitrarily large (see Theorem 3 of Heinz [**9**]).

If Σ is a closed convex surface, then the integral $\int |H|\, d\sigma$ can be estimated because of Minkowski's integral formula

$$\int_\Sigma H\, d\sigma = -\int_\Sigma K\nu \cdot X\, d\sigma, \tag{3}$$

which holds for orientable closed surfaces (Minkowski [**15**], Herglotz [**10**], Efimow [4], Heinz [**7**]). A careful investigation of the proof of (3), (which we take from Klingenberg [**11**], p.106, instead of proving (3) like in [**7**]) shows that an *a priori* estimate for $\int |H|\, d\sigma$ can also be derived if Σ is attached to the unit sphere S^2 of order 1:

Proposition 2. *Suppose that $X \in C^{1,1}(\overline{B}) \cap C^2_{loc}(B)$, $B = \{|u| < 1\}$, is the radius vector of a locally convex surface, which satisfies the boundary condition*

$$|X| = 1, \quad \frac{\partial |X|}{\partial n} = 0 \text{ for } |u| = 1. \tag{4}$$

Here n is the outward pointing normal to $\partial B = S^1$. Then there is an estimate of the form

$$\int_\Sigma |H|\, d\sigma \leqslant C(a, \kappa),$$

where

$$|g_{ij}| \leqslant a, \quad K \leqslant \kappa.$$

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2. The Darboux equation and the regularity proof

Let

$$\rho = \rho(u) = X \cdot X_0, \quad X_0 = \nu(u_0),$$

for some $u_0 \in \Omega$. The Gauss equations

$$D_{ij}X = \Gamma_{ij}^k D_k X + h_{ij}\nu,$$
$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}(D_j g_{i\ell} + D_i g_{j\ell} - D_\ell g_{ij}),$$

then imply that

$$\begin{aligned} \det[D_{ij}\rho - \Gamma_{ij}^k D_k \rho] &= h(\nu \cdot X_0)^2 \\ &= K(D_1 X D_2 X X_0)^2 \\ &= K\left[g - \frac{g^{ij}}{g} D_i\rho D_j\rho\right]. \end{aligned} \tag{5}$$

(5) is the Darboux equation, which is elliptic in a neighbourhood $\mathcal{N}$ of $u_0 \in \Omega$, because Σ is then a convex graph over a plane perpendicular to $\nu(u_0)$. The regularity theory for elliptic Monge–Ampère equations, in particular Theorem 1 of [**27**], yields the regularity $\rho \in C^{2,\mu}_{loc}(\mathcal{N})$. To translate this into the regularity $X \in C^{2,\mu}_{loc}(\mathcal{N}, \mathbb{R}^3)$, consider the three 3×3-systems

$$X_0 \cdot D_{ij}X = D_{ij}\rho,$$
$$D_k X \cdot D_{ij} X = \frac{1}{2}(D_j g_{ik} + D_i g_{jk} - D_k g_{ij})$$

(which can easily be derived from the Gauss equations). By (5), the determinant of the coefficient matrix is

$$X_0 D_1 X D_2 X = \sqrt{g - \frac{g^{ij}}{g} D_i\rho D_j\rho} \neq 0,$$

and the statement $X \in C^{2,\mu}(\Omega, \mathbb{R}^3)$ of Theorem 1 follows from Cramer's rule. □

3. Conjugate isothermal parameters and the Darboux system

Lemma 3. *Let a_{ij} be functions of class $C^1(\Omega)$ such that*

$$\Delta = \det[a_{ij}] > 0.$$

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Let $\overline{B}_R = \overline{B}_R(u_0) \subset \Omega$. *Then there exists a homeomorphism* $u = u(x)$ *from* $\overline{B} = \{|x| \leqslant 1\}$ *onto* $\overline{B}_R$ *of class* $C^{1,\mu}_{loc}(B)$ *with* $u(0) = u_0$, *which satisfies the system*

$$D_\alpha\left(\sqrt{\Delta}D_\alpha u^k\right) = D_\alpha(\Delta a^{\alpha k})Du^1 \wedge Du^2. \tag{6}$$

Furthermore $$Du^1 \wedge Du^2 = D_1u^1D_2u^2 - D_2u^1D_1u^2 \neq 0,$$

and $$\sqrt{\Delta}a^{ij} = \frac{Du^i \cdot Du^j}{Du^1 \wedge Du^2}.$$

This lemma is contained in Lemma 2 of [**26**], which in turn is an improved version of Lemma 2 of Heinz [**5**]. The proof is by mapping the disc $\overline{B}_R(u_0)$ conformally with respect to the metric

$$ds^2 = a_{ij}du^i du^j$$

onto the unit disc $\overline{B} = \{|x| \leqslant 1\}$.

PROPOSITION 4. *Suppose that* Σ *is a locally convex surface with radius vector* $X \in C^{2,\mu}(\Omega, \mathbb{R}^3)$ *for some* μ, $0 < \mu < 1$, *and let* $\overline{B}_R = \overline{B}_R(u_0) \subset \Omega$. *Then there exist conjugate isothermal parameters* $x = (x^1, x^2)$, *that is, there exists a homeomorphism* $u = u(x)$ *from* $\overline{B} = \{|x| \leqslant 1\}$ *onto* $\overline{B}_R$ *of class* $C^{1,\mu}_{loc}(B)$ *with* $u(0) = u_0$, *and*

$$Du^1 \wedge Du^2 = D_1u^1D_2u^2 - D_2u^1D_1u^2 > 0$$

such that the following conformality relations hold:

$$\sqrt{gK}h^{ij} = \frac{Du^i \cdot Du^j}{Du^1 \wedge Du^2}. \tag{7}$$

Furthermore u *satisfies the Darboux system*

$$\Delta_K u^k = D_\alpha\left(\sqrt{K}D_\alpha u^k\right) + \sqrt{K}\Gamma^k_{ij}Du^i \cdot Du^j = 0 \qquad (k = 1, 2). \tag{8}$$

PROOF: Assume first that $X \in C^3(\Omega, \mathbb{R}^3)$ so that $II_\Sigma \in C^1(\Omega, \mathbb{R}^3)$ and consider the differential form

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{g}}II_\Sigma \\ &= \frac{h_{ij}}{\sqrt{g}}du^i du^j. \end{aligned}$$

Lemma 3 yields the existence of the parameters $x = (x^1, x^2)$ which satisfy the conformality relations (7). According to (6),

$$D_\alpha\left(\sqrt{K}D_\alpha u^k\right) = D_\alpha\left(\sqrt{g}K\, h^{\alpha k}\right)Du^1 \wedge Du^2,$$

that is, $$D_\alpha\left(\sqrt{K}D_\alpha u^1\right) = \left[D_1\left[\frac{h_{22}}{\sqrt{g}}\right] - D_2\left[\frac{h_{12}}{\sqrt{g}}\right]\right] Du^1 \wedge Du^2,$$

$$D_\alpha\left(\sqrt{K}D_\alpha u^2\right) = \left[D_2\left[\frac{h_{11}}{\sqrt{g}}\right] - D_1\left[\frac{h_{12}}{\sqrt{g}}\right]\right] Du^1 \wedge Du^2.$$

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By invoking the Codazzi–Mainardi equations

$$D_j h_{ik} - D_i h_{jk} = \Gamma^{\ell}_{jk} h_{i\ell} - \Gamma^{\ell}_{ik} h_{j\ell}$$

and

$$D_k g = 2g\left(\Gamma^1_{1k} + \Gamma^2_{2k}\right),$$

which follow from

$$D_k g_{ij} = \Gamma^{\ell}_{ik} g_{\ell j} + \Gamma^{\ell}_{jk} g_{\ell i},$$

one computes

$$\begin{aligned} D_1\left[\frac{h_{22}}{\sqrt{g}}\right] - D_2\left[\frac{h_{12}}{\sqrt{g}}\right] &= \frac{1}{\sqrt{g}}\left(\Gamma^{\ell}_{12} h_{2\ell} - \Gamma^{\ell}_{22} h_{1\ell} - h_{22}\left(\Gamma^1_{11} + \Gamma^2_{21}\right) + h_{12}\left(\Gamma^1_{12} + \Gamma^2_{22}\right)\right) \\ &= \frac{1}{\sqrt{g}}\left(-\Gamma^1_{11} h_{22} + 2\Gamma^1_{12} h_{12} - \Gamma^1_{22} h_{11}\right) \\ &= -\sqrt{K}\,\Gamma^1_{ij}\frac{Du^i \cdot Du^j}{Du^1 \wedge Du^2}, \end{aligned}$$

and

$$\begin{aligned} D_2\left[\frac{h_{11}}{\sqrt{g}}\right] - D_1\left[\frac{h_{12}}{\sqrt{g}}\right] &= \frac{1}{\sqrt{g}}\left(\Gamma^{\ell}_{21} h_{1\ell} - \Gamma^{\ell}_{11} h_{2\ell} - h_{11}\left(\Gamma^1_{12} + \Gamma^2_{22}\right) + h_{12}\left(\Gamma^1_{11} + \Gamma^2_{21}\right)\right) \\ &= \frac{1}{\sqrt{g}}\left(-\Gamma^2_{11} h_{22} + \Gamma^2_{21} h_{12} - \Gamma^2_{22} h_{11}\right) \\ &= -\sqrt{K}\,\Gamma^2_{ij}\frac{Du^i \cdot Du^j}{Du^1 \wedge Du^2}. \end{aligned}$$

The statement remains true if $X \in C^{2,\mu}(\Omega, \mathbf{R}^3)$. This is seen by essentially repeating the approximation argument in the proof of Lemma 2 of [**26**]: Let $\{X^{(n)}\}_{n=1}^{\infty}$ be $C^3(\Omega, \mathbf{R}^3)$-mappings which approximate the radius vector X and its first and second derivatives uniformly in $\overline{B}_R$. The regularity theory for linear equations (see [**25**]) yields local $C^{1,\mu}$-estimates for the approximating mappings $\{u^{(n)}\}_{n=1}^{\infty}$, because $K = h/g \in C^{\mu}(\Omega)$, and since the conformality relations for $u^{(n)}$ imply that

$$\begin{aligned} \int_B \left|Du^{(n)}\right|^2 dx &\leqslant C\int_B Du^1 \wedge Du^2\, dx \\ &= C\iint_{B_R} du \\ &= CR^2. \end{aligned}$$

Hence there exists a limit mapping $u = u(x)$, which is univalent because the inverses $x^{(n)} = x^{(n)}(u)$ are equicontinuous in $\overline{B}_R$ by the Courant–Lebesgue lemma.

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This is true because the conformality relations for $u^{(n)}$ also imply that

$$\begin{aligned}\int_{B_R}\left|Dx^{(n)}\right|^2 du &\leqslant C\int_{B_R} Dx^1\wedge Dx^2 du\\ &= C\int_B dx\\ &= C.\end{aligned}$$

In order to conclude that $u = u(x)$ is a diffeomorphism from B onto B_R, consider the integrability conditions for the inverses $x^{(n)} = x^{(n)}(u)$, the elliptic system

$$D_j\left(h^{ij(n)}D_i x^{(n)}\right) = 0,$$

which has Hölder continuous coefficients. Then there are $C^{1,\mu}_{loc}$-estimates for $\{(x)^{(n)}\}_{n=1}^{\infty}$, and the limit mapping $x = x(u)$ is therefore of class $C^{1,\mu}_{loc}(B_R)\cap C^0(\overline{B}_R)$, which is the inverse of $u = u(x)$. This in turn implies the nonvanishing of $Du^1\wedge Du^2$ and the relations (7) are therefore satisfied. $\square$

4. A PRIORI ESTIMATES FOR LOCALLY CONVEX SURFACES

LEMMA 5. *Let Σ be a locally convex surface with radius vector $X\in C^{2,\mu}(\Omega, \mathbb{R}^3)$ for some μ, $0<\mu<1$. Suppose that*

$$\begin{aligned}|g_{ij}| &\leqslant a,\\ g,\ K &\geqslant \frac{1}{c}.\end{aligned}$$

Then the mapping $u = u(x)$, $x\in B$, from Proposition 4, satisfies the estimate

$$\int_B |Du|^2 dx \leqslant C(a, c)\int_\Sigma |H|\, d\sigma. \tag{9}$$

PROOF: The mean curvature H of Σ can be estimated by the conformality relations (7):

$$\begin{aligned}|H| &= \left|\frac{h_{ij}g^{ij}}{2}\right|\\ &= \left|\frac{h}{2g}g_{ij}h^{ij}\right|\\ &= \frac{1}{2}\sqrt{\frac{K}{g}}g_{ij}\frac{Du^i\cdot Du^j}{|Du^1\wedge Du^2|}\\ &\geqslant \frac{1}{2}\sqrt{\frac{K}{g}}\frac{g}{2a}\frac{|Du|^2}{|Du^1\wedge Du^2|}\\ &\geqslant \frac{1}{4ac}\frac{|Du|^2}{|Du^1\wedge Du^2|}.\end{aligned}$$

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Therefore

$$\int_B |Du|^2\,dx \leqslant 4ac \int_{B_R(u_0)} |H|\,du$$
$$\leqslant 4a\sqrt{c}\int_\Sigma |H|\,d\sigma.$$

□

PROOF OF THEOREM 1 (of the *a priori* estimates): Let $\overline{B}_R = \overline{B}_R(u_0) \subset \Omega$. Consider the homeomorphism $u = u(x)$ from $\overline{B}$ onto $\overline{B}_R$ from Proposition 4. Now u is of class $C^{1,\mu}_{loc}(B)$ and its Dirichlet integral is estimated by (9). Furthermore

$$\sqrt{g}K\,h^{ij} = \frac{Du^i \cdot Du^j}{Du^1 \wedge Du^2}, \tag{10}$$

$$\Delta_K u = 0. \tag{11}$$

Suppose now, without loss of generality, that $B_R(u_0) = B = \{|u| < 1\}$ (otherwise consider the mapping $\frac{1}{R}u(x) - u_0$). The Main Theorem of [25] can then be applied to the system (11) to give the following estimates in any disc $B_\rho = \{|x| < \rho\}$, $0 < \rho < 1$:

$$\|u\|_{C^{1,\mu}(B_\rho)} \leqslant C(\ldots, \rho), \tag{12}$$

$$Du^1 \wedge Du^2 \geqslant c(\ldots, \rho) > 0. \tag{13}$$

By taking $\rho = 1/2$, and then also taking into account that we assumed that $B_R(u_0) = B$, the relations (10) yield the bounds

$$|h_{ij}(u_0)| \leqslant C(\ldots, R),$$

from which, in Ω',

$$|h_{ij}| \leqslant C(\ldots, \operatorname{dist}(\Omega', \partial\Omega)).$$

Furthermore the functions $h_{ij}(u(x))$ satisfy Hölder estimates of the form

$$[h_{ij}]^{B_\rho}_\mu \leqslant C(\ldots, \rho, R)$$

in each $B_\rho = \{|x| < \rho\}$. In order to translate this into estimates for $h_{ij}(u)$, note that the estimate (9) for the Dirichlet integral of u implies, by the Courant Lebesgue lemma, that there exists a $\rho = \rho(a, c, \mu, R)$, $0 < \rho < 1$, such that $x \in B_\rho$ if $u \in B_{R/2}(u_0)$. Since

$$x_k = \int_0^1 Dx_k(u_0 + \tau(u - u_0)) \cdot (u - u_0)d\tau,$$

the estimates (12,13) yield a dilation inequality of the form

$$|x| \leqslant C(\ldots, R)\,|u - u_0|$$

if $u \in B_{R/2}(u_0)$, and therefore

$$|h_{ij}(u) - h_{ij}(u_0)| \leqslant C(\ldots, R)\,|u - u_0|^{\mu},$$

which implies the Hölder estimates

$$[h_{ij}]_{\mu}^{\Omega'} \leqslant C(\ldots, \operatorname{dist}(\Omega', \partial\Omega)).$$

A priori estimates for the second derivatives of the radius vector X follow from the Gauss equations as required. ∎

5. Proof of Proposition 2

If $X \in C^3(B)$, then

$$D_i\left(\sqrt{g} K\, h^{ij} D_j X\right) = 2\sqrt{g} K\, \nu.$$

This formula is easily shown to hold in Fermi coordinates (see [**11**], pp.104, 106). By dotting with X and integrating over B_ρ, $0 < \rho < 1$, it follows that

$$\begin{aligned} 2\int_{\Sigma_\rho} K\nu \cdot X d\sigma &= -\int_{B_\rho} \sqrt{g} K\, g_{ij} h^{ij} du + \frac{1}{2}\int_{\partial B_\rho} \sqrt{g} K\, h^{ij} D_j\, |X|^2\, n_i ds \\ &= -2\int_{\Sigma_\rho} H\, d\sigma + \frac{1}{2}\int_{\partial B_\rho} \sqrt{g} K\, h^{ij} D_j\, |X|^2\, n_i ds. \end{aligned}$$

This relation holds true if $X \in C^2(B)$. By letting $\rho \to 1$ and incorporating the boundary condition (4), it follows that

$$\int_{\Sigma} K\nu \cdot X\, d\sigma = -\int_{\Sigma} H\, d\sigma.$$

Finally, $K > 0$ implies the required estimate

$$\begin{aligned} \int_{\Sigma} |H|\, d\sigma &\leqslant C(a, \kappa)\, |X| \\ &\leqslant C(a, \kappa). \end{aligned}$$

∎

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Department of Mathematics
The University of Iowa
Iowa City, IA 52242
United States of America
and
Centre for Mathematical Analysis
The Australian National University
GPO Box 4
Canberra ACT 2601
Australia