BULL. AUSTRAL. MATH. SOC. 55N22, 55Q05
VOL. 58 (1998) [373-382]

REMARKS ON THE TOPOLOGY OF SPATIAL POLYGON SPACES

YASUHIKO KAMIYAMA

Let M_n be the "polygon space" introduced by Kirwan and Klyachko. In this paper, we give new results on the topology of M_n for odd n. We determine $\pi_q(M_n)$ $(q \leqslant n-3)$. Then we describe M_n in the oriented cobordism ring Ω^{SO}_{2n-6}. We also give new and elementary proofs of the result on the ring structure of $H^*(M_n/\mathcal{S}_n; \mathbf{Q})$, where $\mathcal{S}_n$ denotes the symmetric group acting naturally on M_n.

1. INTRODUCTION

Let M_n be the variety of spatial polygons $P = (a_1, a_2, \dots, a_n)$ with the side vectors $a_i \in \mathbf{R}^3$ of length $|a_i| = 1$ $(1 \leqslant i \leqslant n)$. The polygons are considered up to motion in $\mathbf{R}^3$. The sum of the side vectors is zero:

$$a_1 + a_2 + \ldots + a_n = 0. \tag{1.1}$$

It is known that M_n admits a Kähler structure such that the complex dimension of M_n is $n-3$. For odd n, M_n is free from all singular points, while for even n, M_n has singular points.

For odd n, $H_*(M_n; \mathbf{Z})$ was determined by Kirwan and Klyachko [**6, 8**] (see Theorem 2.5). Then some results on the ring structure on $H^*(M_n/\mathcal{S}_n; \mathbf{Q})$ were proved by Brion and Kirwan [**2, 7**] (see Theorem 2.6), where $\mathcal{S}_n$ denotes the symmetric group acting naturally on M_n. We remark that the results in [**2, 6, 7, 8**] are proved by using theorems in symplectic geometry. Unfortunately, their methods cannot apply to M_n for even n, because of the singular points of M_n. Thus in [**5**], $H_*(M_n; \mathbf{Q})$ (n: even) is determined by another method.

Now let us assume n to be odd. The purpose of this paper is to prove new results on the topology on M_n. We study the following:

(a) We obtain new information on $\pi_*(M_n)$.
(b) We describe M_n in the oriented cobordism ring Ω^{SO}_{2n-6}.

Received 11th February, 1998
I am grateful to M. Tezuka for many useful comments.

First we give a detailed account of (a). Recall that $H_*(M_n;\mathbf{Z})$ was determined by Kirwan and Klyachko [**6, 8**]. But we have little information on $\pi_*(M_n)$, since theorems in symplectic geometry, which are used in [**6, 8**], are effective for homology but not effective for homotopy. Thus the purpose of (a) is to determine $\pi_q(M_n)$ $(q \leqslant n-3)$. In the course of the proof of $\pi_q(M_n)$ $(q \leqslant n-3)$, we can give new and elementary proofs of results in [**2, 6, 7, 8**] (see Theorems 2.5 and 2.6) without using theorems in symplectic geometry.

Next we give a detailed account of (b). It is clear that $M_3 = \{\text{1-point}\}$. Since we are assuming n to be odd, the first non-trivial example of M_n is the case $n = 5$. And in [**8**], Klyachko proved that as a projective surface, M_5 is the Del Pezzo surface of degree 5 (obtained from $\mathbf{C}P^2$ by blowing up four points in general position).

The purpose of (b) is to generalise this result from the viewpoint of cobordism. We give an orientation to M_n, which is defined from its Kähler structure. Then we describe M_n in the oriented cobordism ring Ω^{SO}_{2n-6}. (Note that Klyachko's result shows that $M_5 = -3\mathbf{C}P^2$ in Ω^{SO}_4.)

As a corollary, we prove that M_n is not a Spin-manifold for odd n $(n \geqslant 3)$.

Now we state our main results. For (a), we prove the following:

THEOREM A. *$\pi_q(M_n)$ $(q \leqslant n-3)$ is given as follows.*

(i) $\pi_q(M_n) \cong \pi_q\left(\left(S^2\right)^{n-1}\right)$ *for* $q = 1$ *or* $3 \leqslant q \leqslant n-3$.

(ii) $\pi_2(M_n) \cong \pi_2\left(\left(S^2\right)^{n-1}\right) \oplus \mathbf{Z} \cong \mathbf{Z}^n$.

Next for (b), we prove the following theorem. For odd n, we set $n = 2m+1$.

THEOREM B. *If we give an orientation to M_{2m+1}, which is defined from its Kähler structure, then M_{2m+1} is oriented cobordant to $(-1)^{m+1}\binom{2m-1}{m-1}\mathbf{C}P^{2m-2}$, where $\binom{2m-1}{m-1}$ denotes the binomial coefficient.*

From Theorem B, we prove the following:

COROLLARY C. *M_{2m+1} is not a Spin-manifold for $m \geqslant 2$.*

This paper is organised as follows. In Section 2, we prepare some notation. Then we state the results on the structure of $H_*(M_n;\mathbf{Z})$ which are proved in [**6, 8**], and the ring structure on $H^*(M_n/S_n;\mathbf{Q})$ which are proved in [**2, 7**]. In Section 3, we prove Theorem A. The essential part of the proof is to construct a Morse function explicitly, which seems to be interesting itself. In Section 4, we prove Theorem B and Corollary C. For the proof of Theorem B, we construct an oriented manifold with boundary which gives the required cobordism explicitly, which also seems to be interesting itself. In Section 5, we give new and elementary proofs of results in [**2, 6, 7, 8**] (see Theorems

2.5 and 2.6).

2. Preliminaries

Recall that M_n is defined from the space of spatial polygons by the action of the motion in $\mathbf{R}^3$. We set

$$(2.1) \qquad \mathcal{B}_n = \left\{ P = (a_1, a_2, \ldots, a_n) \in \left(S^2\right)^n ;\ a_1 + a_2 + \ldots + a_n = 0 \right\}.$$

Then by the definition of M_n, we have

$$(2.2) \qquad M_n = \mathcal{B}_n / SO(3).$$

Let $P = (a_1, a_2, \ldots, a_n) \in M_n$. By the $SO(3)$-action, we can always assume that $a_n = \mathbf{e}$, where we set $\mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbf{R}^3$. More precisely, we define $\mathcal{C}_n$ by

$$(2.3) \qquad \mathcal{C}_n = \left\{ P = (a_1, a_2, \ldots, a_{n-1}) \in \left(S^2\right)^{n-1} ; a_1 + a_2 + \ldots + a_{n-1} + \mathbf{e} = 0 \right\}.$$

Regard S^1 as the subgroup of $SO(3)$ consisting of elements which fix $\mathbf{e}$. Then S^1 naturally acts on $\mathcal{C}_n$. It is clear that

$$(2.4) \qquad M_n = \mathcal{C}_n / S^1.$$

We use (2.4) for the proofs of Theorems A and B. On the other hand, we use (2.2) in Section 5.

Finally we recall some results from [**2, 6, 7, 8**].

Theorem 2.5. [**6, 8**] *For odd n, $H_*(M_n; \mathbf{Z})$ is a free $\mathbf{Z}$-module and $P(M_n, t)$, the Poincaré polynomial of M_n, is given by*

$$P(M_n, t) = 1 + nt^2 + \ldots + \left\{ 1 + (n-1) + \binom{n-1}{2} + \ldots + \binom{n-1}{min\,(i, n-3-i)} \right\} t^{2i} + \ldots + t^{2n-6}.$$

Recall that the symmetric group $\mathcal{S}_n$ naturally acts on M_n, and we can define the orbit space $M_n/\mathcal{S}_n$. Then we have the following:

Theorem 2.6. [**2, 7**] *For $* \leqslant n-3$, we have the ring isomorphism*

$$H^*(M_n/\mathcal{S}_n; \mathbf{Q}) \cong \mathbf{Q}[\beta, p],$$

where $\deg \beta = 2$ *and* $\deg p = 4$.

https://doi.org/10.1017/S0004972700032342 Published online by Cambridge University Press

3. Proof of Theorem A

We adopt the definition $M_n = \mathcal{C}_n/S^1$ (see (2.4)). Note that Theorem A follows from Proposition 3.1 together with the homotopy long exact sequence of the principal bundle $S^1 \to \mathcal{C}_n \to M_n$. Let $i_n : \mathcal{C}_n \hookrightarrow (S^2)^{n-1}$ be the inclusion (see (2.3)).

Proposition 3.1. *$(i_n)_* : \pi_q(\mathcal{C}_n) \to \pi_q\Big(\big(S^2\big)^{n-1}\Big)$ is an isomorphism for $q \leqslant n-3$ and an epimorphism for $q = n-2$.*

In the rest of this section, we prove this proposition by constructing a Morse function explicitly. We define the function $f_n : \big(S^2\big)^{n-1} \to \mathbf{R}$ by

$$f_n(a_1, \dots, a_{n-1}) = |a_1 + \dots + a_{n-1} + \mathbf{e}|^2. \tag{3.2}$$

Note that $f_n^{-1}(0) = \mathcal{C}_n$. We need to know the critical points of f_n and the index at these points. To do this, we need to consider only the points $(a_1, \dots, a_{n-1}) \in \big(S^2\big)^{n-1}$ such that $f_n(a_1, \dots, a_{n-1}) > 0$, since $f_n^{-1}(0) = \mathcal{C}_n$. Now we can prove the following Propositions 3.3 and 3.5 in the same way as in [4]. Since the calculations are easy, we omit the details.

Proposition 3.3. *$(a_1, \dots, a_{n-1}) \in \big(S^2\big)^{n-1}$ is a critical point of f_n if and only if $a_i = \pm a_{n-1}$ $(1 \leqslant i \leqslant n-2)$.*

We set

$$S = \big\{(\varepsilon_1, \dots, \varepsilon_{n-2})\,;\, \varepsilon_i = \pm 1 \ (1 \leqslant i \leqslant n-2)\big\}. \tag{3.4}$$

For every $(\varepsilon_1, \dots, \varepsilon_{n-2}) \in S$, we can designate a critical submanifold of the form

$$\big\{(\varepsilon_1 a_{n-1}, \varepsilon_2 a_{n-1}, \dots, \varepsilon_{n-2} a_{n-1}, a_{n-1})\,;\, a_{n-1} \in S^2\big\},$$

which we denote by $N(\varepsilon_1, \dots, \varepsilon_{n-2})$. Let $\nu\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)$ be the normal bundle of $N(\varepsilon_1, \dots, \varepsilon_{n-2})$ in $\big(S^2\big)^{n-1}$.

For every $N(\varepsilon_1, \dots, \varepsilon_{n-2})$, we try to determine the index of

$$H(f_n) \mid \nu\Big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\Big),$$

the Hessian $H(f_n)$ restricted to the normal bundle $\nu\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)$. We say a critical submanifold $N(\varepsilon_1, \dots, \varepsilon_{n-2})$ of f_n is of type (k, l) if $+1$ appears k-times and -1 appears l-times in $(\varepsilon_1, \dots, \varepsilon_{n-2})$, such that $k + l = n - 2$. Then we have the following:

Proposition 3.5. *Let $N(\varepsilon_1, \dots, \varepsilon_{n-2})$ be a critical submanifold of type (k, l). Then the index of $H(f_n) \mid \nu\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)$ is given by*

$$\begin{cases} 2k & k > l \\ 2(l-1) & k < l. \end{cases}$$

Now we complete the proof of Proposition 3.1. Let $\nu^-\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)$ be the negative normal bundle, that is, the subbundle of $\nu\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)$ on which $H(f_n)$ is negative definite. Let $D\Big(\nu^-\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)\Big)$ be the disc bundle associated to $\nu^-\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)$. Then the Morse theory generalised by Bott [**1**] tells us that $\left(S^2\right)^{n-1}$ is homotopically equivalent to a CW complex which is obtained from $\mathcal{C}_n$ by attaching cells of the form $D\Big(\nu^-\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)\Big)$:

$$\left(S^2\right)^{n-1} \simeq \mathcal{C}_n \cup \bigcup_{(\varepsilon_1, \dots, \varepsilon_{n-2}) \in S} D\Big(\nu^-\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)\Big).$$

As a cell, $D\Big(\nu^-\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)\Big)$ has a dimension

$$\begin{cases} 2k+2 & k > l \\ 2l & k < l, \end{cases} \tag{3.6}$$

by Proposition 3.5. This implies that for every $(\varepsilon_1, \dots, \varepsilon_{n-2}) \in S$, $\dim D\Big(\nu^-\big(N(\varepsilon_1, \dots, \varepsilon_{n-2})\big)\Big) \geqslant n-1$. Hence we see that $\left(S^2\right)^{n-1}$ is homotopically equivalent to a CW complex obtained from $\mathcal{C}_n$ by attaching cells of dimensions greater than or equal to $n-1$. Hence Proposition 3.1 follows.

4. Proofs of Theorem B and Corollary C

Proof of Theorem B: Theorem B is proved by constructing a manifold with boundary which gives the required cobordism explicitly. We adopt the definition $M_n = \mathcal{C}_n / S^1$ (see (2.4)). For a real number $r \geqslant 1$, we set

$$\mathcal{C}_{n,r} = \Big\{P = (a_1, a_2, \dots, a_{n-1}) \in \left(S^2\right)^{n-1};\, a_1 + a_2 + \dots + a_{n-1} + r\mathbf{e} = 0\Big\}. \tag{4.1}$$

Thus $\mathcal{C}_{n,1} = \mathcal{C}_n$. Then set

$$\mathcal{D}_n = \bigcup_{r \geqslant 1} \mathcal{C}_{n,r}. \tag{4.2}$$

https://doi.org/10.1017/S0004972700032342 Published online by Cambridge University Press

It is clear that $\partial\mathcal{D}_n$, the boundary of $\mathcal{D}_n$, is exactly $\mathcal{C}_n$. S^1 acts naturally on $\mathcal{D}_n$ and this action is semifree, that is, the set of fixed points consists of $\{(a_1,\dots,a_{n-1}) \in \mathcal{D}_n;\ a_i = \pm\mathbf{e}\ (1 \leqslant i \leqslant n-1)\}$, and except the fixed points, S^1 acts freely.

We remove a small open disc around every fixed point in $\mathcal{D}_n$, and denote this space by $\overline{\mathcal{D}}_n$. Finally set

$$W_n = \overline{\mathcal{D}}_n/S^1. \tag{4.3}$$

We set $n = 2m+1$. If we forget the orientation, then ∂W_{2m+1} consists of one M_{2m+1} and $\left(\binom{2m}{0}+\binom{2m}{1}+\dots+\binom{2m}{m-1}\right)$-times $\mathbf{C}P^{2m-2}$. (Since the fixed point set of the S^1 action on $\mathcal{D}_n$ consists of $\left(\binom{2m}{0}+\binom{2m}{1}+\dots+\binom{2m}{m-1}\right)$-points, this number appears. About $\mathbf{C}P^{2m-2}$, see below.)

We need to be careful about how these $\mathbf{C}P^{2m-2}$ are oriented. For a fixed point $(a_1,\dots,a_{n-1}) \in \mathcal{D}_n$, we designate $(\varepsilon_1,\dots,\varepsilon_{n-1})$ $(\varepsilon_i = \pm 1)$ so that $a_i = \varepsilon_i\mathbf{e}$ $(1 \leqslant i \leqslant n-1)$. Thus every fixed point is labeled by $(\varepsilon_1,\dots,\varepsilon_{n-1})$ $(\varepsilon_i = \pm 1)$.

We give an orientation to S^2 so that $\begin{pmatrix} y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{1-y^2-z^2} \\ y \\ z \end{pmatrix}$ is a positive local coordinate. Then $\begin{pmatrix} y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -\sqrt{1-y^2-z^2} \\ y \\ z \end{pmatrix}$ is a negative local coordinate. For a fixed point $(a_1,\dots,a_{n-1}) \in \mathcal{D}_n$, which is labeled by $(\varepsilon_1,\dots,\varepsilon_{n-1})$, we define a local coordinate in $(S^2)^{n-1}$ around $(a_1,\dots,a_{n-1}) \in \mathcal{D}_n$ by

$$\left(\begin{pmatrix} \varepsilon_1\sqrt{1-|b_1|^2} \\ b_1 \end{pmatrix}, \begin{pmatrix} \varepsilon_2\sqrt{1-|b_2|^2} \\ b_2 \end{pmatrix}, \dots, \begin{pmatrix} \varepsilon_{n-1}\sqrt{1-|b_{n-1}|^2} \\ b_{n-1} \end{pmatrix}\right), \tag{4.4}$$

where $b_i \in \mathbf{R}^2$ such that $|b_i| < 1$ $(1 \leqslant i \leqslant n-1)$. This coordinate is positive if and only if -1 appears an even number of times in $(\varepsilon_1,\dots,\varepsilon_{n-1})$. In order to construct a local coordinate in $\mathcal{D}_n$ around $(a_1,\dots,a_{n-1}) \in \mathcal{D}_n$, we put the restriction

$$b_1 + \dots + b_{n-1} = 0 \tag{4.5}$$

on (4.4). (Recall that $a_1 + \dots + a_{n-1} + r\mathbf{e} = 0$.)

Thus if we forget the orientation, the boundary of an open disc around $(a_1,\dots,a_{n-1}) \in \mathcal{D}_n$ is given by $\left\{(b_1,\dots,b_{2m-1}) \in (\mathbf{R}^2)^{2m-1};\ |b_1|^2 + \dots + |b_{2m-1}|^2 = 1\right\}$, which is homeomorphic to S^{4m-3}. It is elementary to prove that S^1 acts on this S^{4m-3} in the usual way, that is, by complex multiplication. Thus ∂W_n consists of M_n and $\mathbf{C}P^{2m-2}$.

Now we take the orientation into account. Take a $\mathbf{C}P^{2m-2}$, which is labeled by $(\varepsilon_1, \dots, \varepsilon_{n-1})$. This $\mathbf{C}P^{2m-2}$ has a positive orientation if and only if -1 appears an even number of times in $(\varepsilon_1, \dots, \varepsilon_{n-1})$. Then in Ω^{SO}_{4m-4}, we have

$$\text{(4.6)} \qquad \begin{aligned} \delta M_{2m+1} &= \left(\binom{2m}{0} - \binom{2m}{1} + \dots + (-1)^{m-1}\binom{2m}{m-1}\right)\mathbf{C}P^{2m-2} \\ &= (-1)^{m+1}\binom{2m-1}{m-1}, \end{aligned}$$

where $\delta = \pm 1$, which is determined if we determine whether the orientation on M_{2m+1} induced from that of W_{2m+1} coincides with the orientation on M_{2m+1} induced from that of the Kähler structure on M_{2m+1}.

LEMMA 4.7. *$\delta = 1$ in* (4.6).

PROOF: We use the following Theorem. ∎

THEOREM 4.8. [**6, 8**] *The Hodge numbers of M_{2m+1} are given by*

$$h^{p,q}(M_{2m+1}) = \begin{cases} \binom{2m}{0} + \binom{2m}{1} + \dots + \binom{2m}{\min(p, 2m-2-p)} & p = q \\ 0 & \textit{otherwise.} \end{cases}$$

Now by Hodge's signature theorem (see for example [**3,** pp.126]), we have

$$\text{(4.9)} \qquad \begin{aligned} \tau(M_{2m+1}) &= \sum_{p,q} (-1)^q h^{p,q}(M_{2m+1}) \\ &= (-1)^{m+1}\binom{2m-1}{m-1}, \end{aligned}$$

where $\tau(M_{2m+1})$ denotes the signature. Hence we must have $\delta = 1$ in (4.6). This completes the proof of Lemma 4.7, and hence also that of Theorem B. ∎

PROOF OF COROLLARY C: Assume that M_{2m+1} is a Spin-manifold. Then $\widehat{A}(M_{2m+1})$, the $\widehat{A}$-genus of M_{2m+1}, is an integer. By Theorem B together with the well-known fact that $\widehat{A}(\mathbf{C}P^{2m-2}) = (-1)^{m+1}2^{-4(m-1)}\binom{2m-2}{m-1}$ (see for example [**9,** pp.163]), we have

$$\text{(4.10)} \qquad \widehat{A}(M_{2m+1}) = 2^{-4(m-1)}\binom{2m-1}{m-1}\binom{2m-2}{m-1}.$$

It is elementary to prove that this is less than 1 for $m \geqslant 2$. This is a contradiction. This completes the proof of Corollary C. ∎

REMARK 4.11. A theorem of Oshanin [**10**] tells us that for a Spin-manifold M^{8k+4} of dimension $8k+4$, $\tau(M^{8k+4})$ is divisible by 16. But we cannot deduce Corollary C from this theorem applied to (4.9) when m is even, that is, when $\dim_{\mathbf{R}} M_{2m+1} \equiv 4\ (8)$. In fact, when $m = 62$, $\tau(M_{125})$ is divisible by 16.

5. Proofs of Theorems 2.5 and 2.6

Proof of Theorem 2.5: We adopt the definition $M_n = \mathcal{C}_n/S^1$ (see (2.4)). Consider the Serre spectral sequence of the fibration $\mathcal{C}_n \to M_n \to \mathbf{C}P^\infty$. By Proposition 3.1, an argument based on dimension shows that $E_2^{s,t} \cong E_\infty^{s,t}$ $(s+t \leqslant n-3)$. Thus we can determine $H_q(M_n;\mathbf{Z})$ $(q \leqslant n-3)$. Then by the Poincaré duality with the universal coefficient theorem, we can determine $H_q(M_n;\mathbf{Z})$ $(q \geqslant n-2)$. Thus we have determined $H_*(M_n;\mathbf{Z})$. In particular, this is torsion-free. This completes the proof of Theorem 2.5. ∎

Next we go to the proof of Theorem 2.6. We assume the truth of the following Proposition 5.1 for the moment.

Proposition 5.1. *(i) For $* \leqslant n-3$, we have the ring isomorphism:*

$$H^*(M_n;\mathbf{Q}) \cong \mathbf{Q}[\alpha_1,\dots,\alpha_n,p]/\sim,$$

where $\deg\alpha_i = 2$ $(1 \leqslant i \leqslant n)$, $\deg p = 4$, and $\sim$ denotes the relations $\alpha_i^2 = rp$ $(1 \leqslant i \leqslant n)$ for some $r \in \mathbf{Q}$. (r does not depend on i.) Thus every element in $H^*(M_n;\mathbf{Q})$ $(* \leqslant n-3)$ are sums of elements of the form $\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}p^j$ with $i_1 < i_2 < \dots < i_k$ and $j \geqslant 0$.

(ii) Under the isomorphism in (i), the $\mathcal{S}_n$-action on $H^*(M_n;\mathbf{Q})$ corresponds to the following action on $\mathbf{Q}[\alpha_1,\dots,\alpha_n,p]/\sim$:

(a) *$\mathcal{S}_n$ acts on $\{\alpha_1,\dots,\alpha_n\}$ by permutation.*
(b) *$\mathcal{S}_n$ acts trivially on p.*

Proof of Theorem 2.6: Since $\mathcal{S}_n$ is a finite group, we have

$$H^*(M_n/\mathcal{S}_n;\mathbf{Q}) \cong H^*(M_n;\mathbf{Q})^{\mathcal{S}_n},$$

where the right hand side denotes the fixed point set under the $\mathcal{S}_n$-action on $H^*(M_n;\mathbf{Q})$. Set $\beta = \alpha_1 + \dots + \alpha_n$. Then Proposition 5.1 (i) tells us that β and p are algebraically independent in dimensions less than or equal to $n-3$. Hence we have the result from Proposition 5.1. This completes the proof of Theorem 2.6. ∎

In the rest of this section, we prove Proposition 5.1. To do so, we first prove the following Proposition 5.2. We adopt the definition $M_n = \mathcal{B}_n/SO(3)$ (see (2.2)). Let $j_n : \mathcal{B}_n \hookrightarrow (S^2)^n$ be the inclusion (see (2.1)).

Proposition 5.2. *$(j_n)_* : H_q(\mathcal{B}_n;\mathbf{Z}) \to H_q((S^2)^n;\mathbf{Z})$ is an isomorphism for $q \leqslant n-3$.*

Proof: Let A_n be the complement of $\mathcal{B}_n$ in $(S^2)^n$ (see (2.1)). Thus

$$A_n = \left\{P = (a_1,\dots,a_n) \in (S^2)^n \,;\, a_1 + \dots + a_n \neq 0\right\}. \tag{5.3}$$

https://doi.org/10.1017/S0004972700032342 Published online by Cambridge University Press

Instead of $f_n : (S^2)^{n-1} \to \mathbf{R}$ as in Section 3, we consider the function $g_n : A_n \to \mathbf{R}$ defined by

(5.4) $$g_n(a_1, \dots, a_n) = -|a_1 + \dots + a_n|^2.$$

Then by the same argument as in the proofs of Propositions 3.3 and 3.5, we see that A_n has the homotopy type of an $(n+1)$-dimensional CW complex.

Now by the Poincaré-Lefschetz duality $H_q((S^2)^n, \mathcal{B}_n; \mathbf{Z}) \cong H^{2n-q}(A_n; \mathbf{Z})$, we have $H_q((S^2)^n, \mathcal{B}_n; \mathbf{Z}) = 0$ $(q \leqslant n-2)$. Hence Proposition 5.2 follows. ∎

We construct $\alpha_1, \dots, \alpha_n \in H^2(M_n; \mathbf{Q})$. Recall that we have a principal bundle

(5.5) $$SO(3) \longrightarrow \mathcal{B}_n \xrightarrow{\pi_n} M_n,$$

where π_n denotes the projection. Since $H^*(SO(3); \mathbf{Q}) \cong H^*(S^3; \mathbf{Q})$, we have the following Gysin sequence:

(5.6)

$$\dots \longrightarrow H^q(M_n; \mathbf{Q}) \xrightarrow{\cup p} H^{q+4}(M_n; \mathbf{Q}) \xrightarrow{\pi_n^*} H^{q+4}(\mathcal{B}_n; \mathbf{Q}) \longrightarrow H^{q+1}(M_n; \mathbf{Q}) \longrightarrow \dots,$$

where $p \in H^4(M_n; \mathbf{Q})$ denotes the first Pontryagin class of (5.5):

(5.7) $$p = p_1(\mathcal{B}_n).$$

By Proposition 5.2, we have

(5.8) $$(j_n)^* : H^2(\mathcal{B}_n; \mathbf{Q}) \cong H^2\left((S^2)^n; \mathbf{Q}\right).$$

Let $\sigma \in H_2(S^2; \mathbf{Q})$ be the canonical generator, and set $\sigma_i = 1 \times \dots \times 1 \times \sigma \times 1 \times \dots \times 1 \in H^2\left((S^2)^n; \mathbf{Q}\right)$, where the i-th element is σ. We define $x_i \in H^2(\mathcal{B}_n; \mathbf{Q})$ to be the element which corresponds to σ_i under the isomorphism (5.8).

Since $(\pi_n)^* : H^2(M_n; \mathbf{Q}) \to H^2(\mathcal{B}_n; \mathbf{Q})$ is an isomorphism by the Gysin sequence, we set

(5.9) $$\alpha_i = \left((\pi_n)^*\right)^{-1}(x_i) \ (1 \leqslant i \leqslant n).$$

First we study the $\mathcal{S}_n$-action on α_i $(1 \leqslant i \leqslant n)$ and p. It is clear that $\mathcal{S}_n$ acts on $\{\alpha_1, \dots, \alpha_n\}$ by permutation. On the other hand, the $\mathcal{S}_n$-action on M_n lifts to the action on $\mathcal{B}_n$ (see (5.5)), that is, every $g \in \mathcal{S}_n$ defines a bundle map of (5.5). Hence $\mathcal{S}_n$ acts trivially on $p = p_1(\mathcal{B}_n)$.

Next we study the ring structure on $H^*(M_n; \mathbf{Q})$. We need to prove only the assertion on α_i^2 $(1 \leqslant i \leqslant n)$, since the other assertions are clear from Proposition 5.2 together with the Gysin sequence (5.6). As for α_i^2, since $\pi_n^* \alpha_1^2 = 0$ by Proposition 5.2, we can set $\alpha_1^2 = rp$ for some $r \in \mathbf{Q}$ by the Gysin sequence. Consider the $\mathcal{S}_n$-action on α_i $(1 \leqslant i \leqslant n)$ and p. By Proposition 5.1 (ii) we see that $\alpha_i^2 = rp$ $(1 \leqslant i \leqslant n)$. Hence the assertion on α_i^2 follows. This completes the proof of Proposition 5.1, and hence also that of Theorem 2.6. ∎

https://doi.org/10.1017/S0004972700032342 Published online by Cambridge University Press

References

[1] R. Bott, 'Nondegenerate critical manifolds', *Ann. of Math.* **60** (1954), 248–261.

[2] M. Brion, 'Cohomologie équivariante des points semi-stables', *J. Reine Angew.* **421** (1991), 125–140.

[3] P. Griffith and J. Harris, *Principles of algebraic geometry* (John Wiley and Sons, New York, 1978).

[4] Y. Kamiyama and M. Tezuka, 'Topology and geometry of equilateral polygon linkages in the Euclidean plane', (preprint).

[5] Y. Kamiyama, 'The homology of singular polygon spaces', (preprint).

[6] F. Kirwan, *The cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes **31** (Princeton University Press, 1984).

[7] F. Kirwan, 'Cohomology rings of moduli spaces of bundles over Riemannian surfaces', *J. Amer. Math. Soc.* **5** (1992), 853–906.

[8] A. Klyachko, 'Spatial polygons and stable configurations of points in the projective line', in *Algebraic geometry and its applications*, Aspects of Mathematics **25** (Vieweg, Braunschweig, 1994), pp. 67–84.

[9] H.B. Lawson and M. Michelsohn, *Spin geometry* (Princeton University Press, Princeton, N.J., 1989).

[10] S. Oshanin, 'The signature of *SU*-manifolds', *Mat. Zametki* **13** (1973), 97–102: *Math. Notes*, **13** (1973), 57–60.

Department of Mathematics
University of the Ryukyus
Nishihara-Cho
Okinawa 903-01
Japan
e-mail: kamiyama@ sci.u-ryukyu.ac.jp