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ON INTEGRALS AND SUMMABLE TRIGONOMETRIC SERIES

BY
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ABSTRACT. In considering a problem on certain summable (C, k) trigonometric series, R. D. James [13] used a symmetric p^{k+2}-integral defined earlier to recapture the coefficients of the series from the sum function. James' formulas for the coefficients are more complicated than the usual Euler-Fourier form since the p^{k+2}-integral is of order $k+2$. It is shown that a generalized integral of order one for each non-negative integer k can be suitably defined to reduce James' formulas to the usual form.

1. Introduction. One of the problems in the theory of trigonometric series is that of suitably defining a trigonometric integral which is general enough to integrate the sum function of any everywhere convergent series of the form

$$a_0/2+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx), \tag{1}$$

and to give back the coefficients a_n, b_n in terms of the sum function. This problem has been initially studied and solved by A. Denjoy [10], and has been also solved later by J. Marcinkiewicz and A. Zygmund [14], R. D. James [11] (see also [19], Ch. XI), and by J. C. Burkill [5] (see also [4], [2]). A natural extension of the problem is to consider summable series instead of convergent series. This has been done by James [13] for certain Cesàro summable series, and by S. J. Taylor [17] for certain Abel summable series (see also [6], where G. Cross has considered a combination of ideas from [13] and [17]). We will concern ourselves only with the following result stated in Theorem 6.2 by James [13], the notations of which are introduced here and will also be used later.

THEOREM (A, k). Suppose that the series (1) is summable (C, k) to a finite function $f(x)$ for all $x\in[0,2\pi]\sim E$, where E is at most countable, and let $f(x)=0$ for $x\in E$. If $A_n^{k-1}(x)=0(n^k)$ for $x\in E$ and $B_n^{k-1}(x)=0(n^k)$ for all $x\in[0,2\pi]$, then $f(x), f(x)\cos px$, $f(x)\sin px$ are each p^{k+2}-integrable and the

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coefficients of (1) are given by

$$a_p = \frac{\gamma_k}{2^{k+2}\pi^{k+2}} \int_{(\alpha_i)}^{0} f(x) \cos px \, d_{k+2}x, \tag{2}$$

$$b_p = \frac{\gamma_k}{2^{k+1}\pi^{k+2}} \int_{(\alpha_i)}^{0} f(x) \sin px \, d_{k+2}x, \tag{3}$$

where

$$\gamma_k = \begin{cases} (2m)!/(m!)^2 & \text{if} \quad k = 2m-2 \\ (2m+1)!/m!(m+1)! & \text{if} \quad k = 2m-1, \end{cases}$$

and (α_i) is also chosen in a suitable way according to k is even or odd.

Here the p^{k+2}-integral, also due to James first in [12] and then modified in [13] (see also the remarks in the last section), is of $(n+2)$th order so that the coefficient formulas (2) and (3) do not appear in the uaual Euler-Fourier form. The purpose of this note is to show that for each nonnegative integer k, a generalized integral of "first" order, to be denoted as G_{k+1}-integral, can be suitably defined to replace the p^{k+2}-integral in the theorem so that the formulas (2) and (3) will reduce to the usual form (see Theorem (2, k)).

To indicate what is involved, let us note some facts about the integrals in the various solutions of the original problem for convergent series. The "totalisation symétrique à deux degrés" of Denjoy [10], and the p^2-integral of James [11] are of second order involving essentially the concept of recapturing a function $F(x)$ (up to a linear term) from the second symmetric derivative $D^2F(x)$. The T-integral of Marcinkiewicz and Zygmund [14], and the SCP-integral of Burkill [5] are of first order involving essentially the concept of recapturing an "integrable" function $\phi(x)$ (only almost everywhere and up to a constant term) from its "first averaged derivative." This is the symmetric Borel derivative $B_s\phi(x)$ in the case of the T-integral, the symmetric Cesàro derivative SCD$\phi(x)$ in the case of the SCP-integral, and in either case it is just the second symmetric derivative $D^2F(x)$ of the indefinite "integral" $F(x)$ of the function $\phi(x)$. (At this instant, it seems worthwhile to remark that the T- and the SCP-integrals are equivalent should the original definitions be slightly modified [2]). Note that if $F(x)$ is the indefinite "integral" of $\phi(x)$ then the derivative $F'(x)$ exists and is equal to $\phi(x)$ almost everywhere. This leads us to consider the concept of "recapturing" the first derivative $F'(x)$ (only almost everywhere and up to a constant term) from the second symmetric derivative $D^2F(x)$. The G_{k+1}-integral to be defined by the Perron method in the next section involves essentially the naturally extended concept of "recapturing" the $(k+1)$th generalized derivative (i.e., the $(k+1)$th Peano derivative) $F_{(k+1)}(x)$ from the $(k+2)$th generalized symmetric derivative (which is also called the de la Vallée Poussin derivative of order $(k+2)$ $D^{k+2}F(x)$).

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The notations $F_{(k)}(x)$, $D^kF(x)$ and some of the other related notations and notions to be adopted later on are those used in [11], [15] and [7]. Hence it should be remarked that our notion of smoothness of order $(k+2)$ for functions is that of order $(k+1)$ discussed by E. M. Stein and A. Zygmund in the paper [16], where, among others, the "differentiability almost everywhere" (in the sense of Peano) of smooth functions is beautifully investigated.

2. The G_{k+1}-integral. Throughout this section, the letter k denotes a fixed but arbitrary non-negative integer.

Let $[a, b]$ be a non-degenerate compact interval and B a basis on $[a, b]$, i.e., B is a subset of $[a, b]$ with Lebesgue measure $|B| = b-a$ and $a, b \in B$. Let $\bar{\mathcal{M}}_k([a, b])$ denote the collection of all (real-valued) functions H defined on $[a, b]$ such that

($\bar{\mathcal{M}}_k$1) $H_{(k)}(x)$ exists and is finite (for $k=0$, this means by convention that the function $H \equiv H_{(0)}$ is continuous at x) for all $x \in [a, b]$;
($\bar{\mathcal{M}}_k$2) H is smooth of order $(k+2)$ at every point of $[a, b]$;
($\bar{\mathcal{M}}_k$3) $\underline{D}^{k+2}H(x) > -\infty$ for nearly every $x \in [a, b]$.

Given a (real-valued) function f defined at least almost everywhere on $[a, b]$, a (real-valued) function M defined on B is said to be a *G_{k+1}-major function of f on $[a, b]$ with basis B* if there exists a function H such that

(i) $H \in \bar{\mathcal{M}}_k([a, b])$;
(ii) $H_{(k+1)}(x) = M(x)$ for all $x \in B$;
(iii) $H_{(k+1)}(a) = M(a) = 0$;
(iv) $\underline{D}^{k+2}H(x) \geq f(x)$ for almost all $x \in [a, b]$.

And as usual, m is said to be a G_{k+1}-minor function of f on $[a, b]$ with basis B if $-m$ is a G_{k+1}-major function of $-f$.

The following result is fundamental for the theory of the G_{k+1}-integral.

LEMMA 1. *Suppose that M is a G_{k+1}-major function and m a G_{k+1}-minor function for the function f on $[a, b]$ with basis B. Then $M-m$ is monotone increasing and non-negative on B.*

Proof. Let H be a function satisfying the definition for the G_{k+1}-major function M, and h that for m, and denote $F = H-h$. Then F has the property $\mathcal{R}$ on $[a, b]$ since F is smooth of order $k+2$ on $[a, b]$ (see [18]). Then one shows from Theorem 3.1 and Theorem 4.1 in [15] that F is $(k+2)$-convex on $[a, b]$. Hence, by Theorem 7 in [1], $F_{(k+1)}$ is monotone increasing on where it exists, and hence on B in particular. Since $M(a)-m(a) = 0$, the conclusion follows.

Now, we come to the definition of G_{k+1}-integrability and the G_{k+1}-integral. Suppose that the function f has both G_{k+1}-major and minor functions on $[a, b]$

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with basis B, and suppose that

$$\inf M(b) = \sup m(b) \neq \pm\infty \tag{4}$$

where M runs over all the G_{k+1}-major functions and m over all the G_{k+1}-minor functions. Then f is said to be G_{k+1}-integrable on $[a, b]$ with basis B and that the common value in (4) is called the G_{k+1}-integral of f on $[a, b]$ with basis B and is denoted as

$$(G_{k+1}, B) - \int_a^b f(x)\, dx.$$

Many usual properties for integrals of Perron type of first order (see [2]) can be proved for the G_{k+1}-integral based on the definition and the fundamental Lemma 1. We will not do it here. Instead, for later use, we give the following result, which is a simple consequence of the definition and a deep result given by Marcinkiewicz and Zygmund in [14].

THEOREM (1, K). *Let F be a function such that $F_{(k)}(x)$ exists everywhere, and*

$$-\infty < \underline{D}^{k+2}F(x) \leq \bar{D}^{k+2}F(x) > +\infty \tag{5}$$

except possibly for x in a set of at most countably many points at each of which the function F is smooth of order $(k+2)$. Then there exists a set B whose complement has measure zero such that for any $a, b \in B$ with $a < b$, the function $F_{(k+2)}$ is G_{k+1}-integrable on $[a, b]$ with basis $B_1 = B \cap [a, b]$ and such that

$$(G_{k+1}, B_1) - \int_a^b F_{(k+2)}(x)\, dx = F_{(k+1)}(b) - F_{(k+1)}(a). \tag{6}$$

Proof. First note that by an argument similar to that on page 253 in [16], the condition (5) implies that the condition in Theorem 1, [14] is satisfied. Hence it follows from that theorem that $F_{(k+2)}(x)$ exists almost everywhere, and of course that $F_{(k+2)}(x) = D^{k+2}F(x)$ for all x at which $F_{(k+2)}(x)$ exists. Since $F_{(k+2)}(x)$ exists almost everywhere, so does $F_{(k+1)}(x)$. Let B be the set of all points x at which $F_{(k+1)}(x)$ exists. Then the complement of B has measure zero. Let $a, b \in B$ with $a < b$ and let $B_1 = B \cap [a, b]$. Then B_1 is a basis on $[a, b]$. To complete the proof, clearly it suffices to show that the function $M(x) = F_{(k+1)}(x) - F_{(k+1)}(a)$ for all $x \in B_1$ serves both as a G_{k+1}-major function and as a G_{k+1}-minor function for the function $F_{(k+2)}(x)$ on $[a, b]$ with basis B_1. From the assumption on the function F, we see that it suffices to show that F is smooth of order $(n+2)$ *everywhere.* This is immediate since at the points x where the inequalities (5) hold the function F is also smooth of order $(k+2)$. The proof is hence completed.

We end this section by remarking that, for generality the G_1-integral is sandwiched between the P^2-integral of James [11] and the SCP-integral of

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Burkill [5], and in general, the G_{k+1}-integral between the P^{k+2}-integral [13], and the $SC_{kk+1}P$-integral [3], which is a natural extension of the SCP-integral. Note that the $SC_{k+1}P$-integral is also an integral of Perron type of first order but we do not think that it is general enough to do the job mentioned in the introduction. (See [9], where using the $SC_{k+1}P$-integral, a result weaker than theorem (A, k) and theorem $(2, k)$ in the next section are given.) That the G_{k+1}-integral is powerful enough to do this will be given in the next section.

3. Summable series. Now we come to show that the G_{k+1}-integral will do the job discussed in the introduction. Notations and notions adopted here will be those used in [13]. We will be concerning trigonometric series of the form [1]. The following conditions and theorem as stated in [13] will be needed.

$$a_n = o(n^k),\ b_n = o(n^k). \tag{7}$$

$$A_n^{k-1}(x) = o(n^k). \tag{8}$$

$$a_0/2 + \sum_{n=1}^{\infty} a_n(x) = f(x) \qquad (C, k). \tag{9}$$

THEOREM (B, k) (Theorem 3.1, [13]). (α) *If condition* (7) *is satisfied, then the series obtained by integrating* (1) *formally term-by-term* $k+2$ *times converges uniformly to a continuous function* $F(x)$.

(β) *If conditions* (7) *and* (8) *are satisfied, then for the function* F *in* (α), $D^{k+2-2r}F(x)$ *exists for* $1 \le r \le (k+1)/2$, *and* F *is smooth of order* $k+2$ *at* x.

(γ) *If conditions* (7) *and* (9) *are satisfied, then for the function* F *in* (α), F *is smooth of order* $k+2$ *at* x *and*

$$\frac{a_0 x^{2r}}{2(2r)!} + (-1)^r \sum_{n=1}^{\infty} \frac{a_n(x)}{n^{2r}} = D^{k+2-2r}F(x) \quad (C, k-2r) \tag{10}$$

for $0 \le r \le (k+1)/2$.

We will also need the following result from [14].

THEOREM (C, k). *If* (1) *is summable* (C,k) $k \ge 0$, *for* $x \in E$, $|E| > 0$, *then for almost all* $x \in E$, *the series obtained by integrating* (1) *formally term-by-term once is summable* $(C, k-1)$.

We are now in a position to state and to prove the following result, a modification of Theorem (A, k) in the introduction, and using the G_{k+1}-integral instead of the P^{k+2}-integral.

THEOREM $(2, k)$. *Suppose that the series* (1) *is summable* (C, k) *to* $f(x)$ *for almost all* x, *and*

$$-\infty < s^k(x) \le S^k(x) < +\infty \tag{11}$$

for all x *except on a set* E *of at most countably many points. For* $k \ge 1$, *suppose*

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further that

(12) $$A_n^{k-1}(x) = o(n^k) \quad \text{for all} \quad x \in E,$$

(13) $$B_n^{k-1}(x) = o(n^k) \quad \text{for all} \quad x.$$

Then there exists a set B whose complement has measure zero and for each $u \in B$, the function $f(x)$, $f(x) \cos nx$, and $f(x) \sin nx$ are G_{k+1}-integrable on $[u, u+2\pi]$ with basis $B \cap [u, u+2\pi]$, and furthermore

(14) $$a_n = \frac{1}{\pi} (G_{k+1}, B \cap [u, u+2\pi]) - \int_u^{u+2\pi} f(x) \cos nx \, dx,$$

(15) $$b_n = \frac{1}{\pi} (G_{k+1}, B \cap [u, u+2\pi]) - \int_u^{u+2\pi} f(x) \sin nx \, dx,$$

where $n = 0, 1, 2, 3, \ldots$ in (14) and $n = 1, 2, 3, \ldots$ in (15).

Proof. For $k = 0$, the proof is similar to that given in [14] for the T-integral or that given in [4] for the SCP-integral, and hence is omitted here. We suppose that $k \geq 1$ and remark that the proof to be given based on that given in [13] for the P^{k+2}-integral with some modifications.

First, note that since the series (1) is summable (C, k) almost everywhere, condition (7) holds. Hence by Theorem $(1, k)$, (α), the series obtained by integrating (1) formally term-by-term $k+2$ times converges uniformly to a continuous function $F(x)$. We show that the function F satisfies all the assumptions in Theorem $(1, k)$. At $x \notin E$, since (11) holds, it follows from Lemma 5.2 in [13] that the condition (5) holds, and hence in particular, F is smooth of order $(k+2)$. At $x \in E$, since (12) holds, it follows from Theorem (B, k), (β) that F is also smooth of order $(k+2)$. Furthermore, for every x, since (13) holds, one can apply Theorem $(B, k-1)$, (β) to the series

(16) $$\frac{a_0 x}{2} + \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)/n$$

and concludes that F is smooth of order $(k+1)$, too. Thus, F being smooth of order both $(k+2)$ and $(k+1)$, one concludes that $F_{(k)}(x)$ exists for every x (cf. Theorem 3.2, [13]). Thus, we have showed that the assumptions of Theorem $(1, k)$ are satisfied for F. Furthermore, by Theorem (B, k), (γ), $f(x) = D^{k+2}F(x)$ for all points x at which the series (1) is summable (C, k), and hence $f(x) = F_{(k+2)}(x)$ almost everywhere. Thus, one concludes easily that the conclusion of Theorem $(1, k)$ holds with $F_{(k+2)}(x)$ replaced by $f(x)$.

Now, note further that it follows from Theorem (C, k) that the series (16) is summable $(C, k-1)$, say to $\phi(x)$, almost everywhere. Applying Theorem $(B, k-1)$, (γ) to (16), one sees that

$$\phi(x) = D^{k+1}F(x),$$

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which together with the property that F is smooth of order $(k+2)$ implies that $F_{(k+1)}(x)$ exists and is equal to $\phi(x)$ for all points x at which $\phi(x)$ is defined. Therefore, the set B in the conclusion of Theorem $(1, k)$ can be taken to be the set of all x at which the series (16) is summable $(C, k-1)$. Let this set be denoted as B_0 and let $u \in B_0$. Then it follows from (6) that we have

$$(G_{k+1}, B_0 \cap [u, u+2\pi]) - \int_u^{u+2\pi} f(x)\, dx = \phi(u+2\pi) - \phi(u) = \pi a_0.$$

To complete the proof of the theorem, note that using the same formal multiplication method as that used in the proof of Theorem 4.2 in [13] and following the line of arguments given above, we obtain that for each positive integer n, there exist two sets B_n, B_n^*, such that the complement of each of the sets B_n and B_n^* has measure zero, and for $u \in B_u$, $v \in B_n^*$, the following hold:

$$(G_{k+1}, B_n \cap [u, u+2\pi]) - \int_u^{u+2\pi} f(x) \cos nx\, dx = \pi a_n,$$

$$(G_{k+1}, B_n^* \cap [v, v+2\pi]) - \int_v^{v+2\pi} f(x) \sin nx\, dx = \pi b_n.$$

Let $B = B_0 \cap [\bigcap_{n=1}^{\infty} (B_n \cap B_n^*)]$. Then B is a set with all the properties we wanted, completing the proof.

REMARKS. As mentioned in the introduction, the P^{k+2}-integral is due to James, first in [12] and then modified in [13]. However, a defect in James' work has been noted by both S. N. Mukhopadhyay [15] and G. Cross [7], each of them has given a complete definition for the P^{k+2}-integral and showed that Theorem (A, k) is valid for the resulted P^{k+2}-integral. (See also [8].) It should also be noted that in [13], [7] and [8] there is still another flaw to be completed in connection with the scattered set in the definitions of major and minor functions there. This flaw seems to be first observed by Mukhopadhyay, and has been drawn to the author's attention by the referee and Professors P. S. Bullen and G. Cross. We only mention that the flaw will disappear provided that the major and minor functions there are required to satisfy, for example, the inequalities in (2.1.5) of [7] for *all* x instead of only for x in the scattered set involved.

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