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THE GROUP OF EIGENVALUES OF A RANK ONE TRANSFORMATION

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ABSTRACT. In this paper, several characterizations are given of the group of eigenvalues of a rank one transformation. One of these is intimately related to the corresponding expression for the maximal spectral type of a rank one transformation given in an earlier paper.

1. **Introduction.** The purpose of this paper is to compute the group $e(T)$ of L^∞ eigenvalues of a general rank one transformation T. These will be the L^2 eigenvalues when the underlying space is of finite measure. The possibility of such a calculation was suggested by J-F. Méla in connection with our earlier paper [1]. Our expression for the eigenvalue group is intimately related to the corresponding expression for the maximal spectral type of T calculated in [1]. This raises certain natural questions about the group of quasi-invariance of the maximal spectral type of T. We prove our results for measure preserving transformations, but they can be extended to non-singular transformations obtained by cutting and stacking.

Descriptions of eigenvalue groups of certain non-singular flows were given by M. Osikawa [4] and by Y. Ito, T. Kamae and I. Shiokawa [3]. These authors were motivated by certain questions in non-singular weak equivalence theory. From the point of view of spectral theory, however, it is advantageous to recast their work using the "cutting and stacking" description of a rank one transformation and some results on Fourier transforms (characteristic functions) of products of circle valued independent random variables, revealing thereby the close resemblance of an expression for $e(T)$ to the expression for the maximal type of T (up to a discrete measure) obtained in [1]. Thus the present paper complements the work in [1].

2. **Preliminary calculations.**

2.1. We recall the construction of a rank one transformation from [1]. Divide the unit interval Ω_0 into m_1 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the k-th stage we divide the stack obtained at the $(k-1)$-st stage into m_k equal columns add spacers and obtain a new stack of height h_k. If during the k-th stage of our construction the number of spacers put above the j-th column of the $(k-1)$-st stack is

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$a_j^{(k)}$, $0 \leq a_j^{(k)} < \infty$, $1 \leq j \leq m_k$, then we have

$$h_k = m_k h_{k-1} + \sum_{j=1}^{m_k} a_j^{(k)}.$$

Proceeding thus we get a rank one transformation T on a certain measure space $(X, \mathcal{B}, m)$ which may be finite or σ-finite depending on the number of spacers added. For each $k = 1, 2, 3, \ldots$ let Ω_k and Ω^k denote respectively the base and the top of the k-th stack; of course $\Omega_k \subseteq \Omega_0$. There is no loss of generality in assuming in addition that $\Omega^k \subseteq \Omega_0$, *i.e.*, no spacers are added on the last column at any stage in the construction. For given a rank one transformation T constructed by cutting and stacking as above, we can construct as follows an isomorphic transformation S with no spacers added on the last column at any stage: initially, cut Ω_0 into m_1 equal pieces, add $b_j^{(1)} = a_j^{(1)}$ spacers on the j-th column, $1 \leq j < m_1$, and stack. No spacers are added on the last column, *i.e.* $b_{m_1}^{(1)} = 0$. Cut Ω_1 into m_2 equal parts add

$$b_j^{(2)} = a_j^{(2)} + a_{m_1}^{(1)}$$

spacers on the j-th column $1 \leq j < m_2$ and stack; again $b_{m_2}^{(2)} = 0$. At the k-th stage of the construction cut Ω_{k-1} into m_k equal pieces add

$$b_j^{(k)} = a_j^{(k)} + \sum_{l=1}^{k-1} a_{m_l}^{(l)}$$

spacers on the j-th column, $1 \leq j < m_k$, and stack; again $b_{m_k}^{(k)} = 0$. It is easily verified that the two transformations S and T with spacers $a_j^{(k)}$ and $b_j^{(k)}$ respectively are isomorphic, but no spacers are added on the last column at any stage in the construction of S. From now on we assume that $\Omega^k \subset \Omega_0$ for all k.

We denote the m_k equal columns obtained by dividing the $(k-1)$-st stack by $C_1^k, \ldots, C_{m_k}^k$. For $1 \leq i \leq m_k$, write

$$Q_i^k = \text{ union of parts of } \Omega_0 \text{ in the column } C_i^k.$$

Then $\{Q_1^k, \ldots, Q_{m_k}^k\}$ gives a partition $\mathcal{P}_k$ of Ω_0, and the partitions

$$\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots$$

form an independent sequence of partitions of Ω_0; $\mathcal{P}_0$ being the trivial partition. They correspond to the partitions of the product space

$$\Omega = \prod_{k=1}^{\infty} \{0, 1, 2, \ldots, m_k - 1\}$$

given by the co-ordinate functions. Let τ denote the transformation on Ω_0 induced by T. We know that τ is isomorphic to the odometer action on Ω.

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2.2 The functions γ_k. We now define a sequence γ_k, $k = 0, 1, 2, 3, \ldots$ of independent integral valued random variables on Ω_0. First define

$$\lambda_0(\omega) = 0 \quad \text{for all } \omega \in \Omega_0.$$

$$\lambda_1(\omega) = \text{ first entry time under } T \text{ of } \omega \text{ into } \Omega^1, \text{ with } \lambda_1(\omega) = 0 \text{ if } \omega \in \Omega^1.$$

In general

$$\lambda_k(\omega) = \text{ first entry time under } T \text{ of } \omega \text{ into } \Omega^k, \text{ with } \lambda_k(\omega) = 0 \text{ if } \omega \in \Omega^k.$$

The sequence γ_k, $k = 0, 1, 2, 3, \ldots$, of independent integral valued random variables is defined as follows:

$$\gamma_0(\omega) = \lambda_0(\omega) = 0 \quad \text{for all } \omega \in \Omega_0,$$
$$\gamma_k(\omega) = \lambda_k(\omega) - \lambda_{k-1}(\omega), \quad k = 1, 2, 3, \ldots.$$

We have

(1) $$\gamma_k(\omega) = \text{ first entry time of } T^{\lambda_{k-1}(\omega)}(\omega) \text{ into } \Omega^k,$$
$$\lambda_k(\omega) = \gamma_0(\omega) + \cdots + \gamma_k(\omega).$$

Note that $T^{\lambda_{k-1}(\omega)}(\omega) \in \Omega^{k-1}$, whence (1) shows that $\gamma_k(\omega)$ is constant on each piece of the partition $\mathcal{P}_k$; thus $\gamma_0, \gamma_1, \gamma_2, \ldots$ form a sequence of independent random variables; γ_k assumes the value 0 on $Q^k_{m_k}$. Further let us write

$$\gamma_{k,i} = \text{ value of } \gamma_k \text{ on } Q^k_{m_k - i}, \quad 1 \leq i < m_k.$$

The values $0, \gamma_{k,1}, \ldots, \gamma_{k,m_k-1}$ assumed by γ_k are related in a natural and useful manner to the values $0, R_{1,k}, R_{2,k}, \ldots, R_{m_k-1,k}$, $k = 1, 2, 3, \ldots$ which occur in the expression for the maximal type of a rank one transformation described in our paper [1]. We have

$$\gamma_{k,i}(T) = R_{i,k}(T^{-1}), \quad \gamma_{k,i}(T^{-1}) = R_{i,k}(T).$$

To see this one notes that the inverse of the rank one transformation T is also a rank one transformation obtained by cutting and stacking and one has a construction of T^{-1} in which Ω^k, Ω_k are respectively the base and the top of the k-th stack for T^{-1}.

For $\omega \in \Omega_0$ let $l(\omega)$ be the last integer p for which $\omega \in \Omega^p$, *i.e.* $l(\omega) = p$, where p is given by

$$\lambda_0(\omega) = \lambda_1(\omega) = \cdots = \lambda_p(\omega) = 0, \quad \lambda_{p+1}(\omega) \neq 0.$$

Let $f(\omega)$ equal the first re-entry time of ω into Ω_0:

$$f(\omega) = (\text{number of spacers above } \omega) + 1.$$

Then

$$\gamma_k(\omega) = 0, \quad \text{for } 1 \leq k \leq l(\omega),$$
$$\gamma_k(\omega) = \lambda_k\big(\tau(\omega)\big) + f(\omega), \quad k = l(\omega) + 1,$$
$$\gamma_k(\omega) = \gamma_k\big(\tau(\omega)\big), \quad k > l(\omega) + 1.$$

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We therefore have in view of (1):

(2)
$$\sum_{p=1}^{\infty}\Big(\gamma_p(\omega)-\gamma_p\big(\tau(\omega)\big)\Big)=f(\omega)+\lambda_{l(\omega)+1}\big(\tau(\omega)\big)-\sum_{p=1}^{l(\omega)+1}\gamma_p\big(\tau(\omega)\big)$$
$$=f(\omega)=(\text{number of spacers above }\omega)+1.$$

Now let Σ_k denote the group of permutations on $\{0,1,2,\ldots,m_k-1\}$ and Σ the restricted direct product of the Σ_k acting on

$$\Omega=\prod_{k=1}^{\infty}\{0,1,\ldots,m_k-1\}$$

by changing finitely many co-ordinates. We may view Σ as acting on Ω_0. Then the orbits of Σ and τ agree except on a countable subset of Ω_0. Note that if $\sigma\in\Sigma$, $\sigma=(\sigma_1,\ldots,\sigma_k,e,e,\ldots)$, then for each $n>k$, σ leaves invariant each element of $\mathcal{P}_n$. [Here e denotes the identity permutation on $(0,1,\ldots,m_k-1)$ for all k.]. In particular, since each γ_n is $\mathcal{P}_n$ measurable, $\gamma_n\circ\sigma=\gamma_n$ for all $n>k$.

3. **The eigenvalue group: Osikawa's criterion.**

3.1. Let $e(T)$ denote the group of eigenvalues of T and let f be as in Section 2. The proposition and Theorem 1 below are essentially due to Osikawa [4].

PROPOSITION. *Let $s\in[0,1)$. Then $e^{2\pi is}\in e(T)$ if and only if there exists a measurable function $\phi\colon\Omega_0\longrightarrow[0,1)$ such that*

(3)
$$\phi\big(\tau(\omega)\big)=\phi(\omega)+sf(\omega)\pmod 1.$$

PROOF. If a function ϕ satisfying (3) exists then $e^{2\pi i\phi}$ can be extended from Ω_0 to all of X in a natural way so that the extended function is an eigenfunction with eigenvalue $e^{2\pi is}$: indeed if $x\in X$ is the p-th spacer above ω, so that $x=T^p(\omega)$, define $\phi(x)$ by

(4)
$$\phi(x)=\phi(\omega)+ps\pmod 1.$$

The function $e^{2\pi i\phi}$, where ϕ is the extended function, is then an eigenfunction with eigenvalue $e^{2\pi is}$.

On the other hand if $e^{2\pi is}$ is an eigenvalue with eigenfunction ψ of absolute value one, then $\psi=e^{2\pi i\phi_1}$ for some measurable function ϕ_1 defined on X with $0\le\phi_1<1$. Set $\phi=\phi_1\,|_{\Omega_0}$, then ϕ satisfies

$$\phi\big(\tau(\omega)\big)=\phi(\omega)+sf(\omega)\pmod 1,$$

which completes the proof of the proposition.

Let μ denote the Lebesgue measure on $\Omega_0=[0,1)$.

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THEOREM 1. *Let $s \in [0,1)$, then $e^{2\pi is} \in e(T)$ if and only if there exist real constants $c_n, n = 1, 2, \dots$ such that*

$$\sum_{k=1}^{\infty}\Big(s\gamma_k(\omega) - c_k\Big) \tag{5}$$

converges (mod 1) *for μ a.e. ω.*

PROOF. Suppose for an $s \in [0,1)$, the series (5) converges (mod 1)μ a.e. to a function ϕ. Then (mod 1), for μ a.e. ω,

$$\phi\Big(\tau(\omega)\Big) - \phi(\omega) = \sum_{k=1}^{\infty} s\Big(\gamma_k\Big(\tau(\omega)\Big) - \gamma_k(\omega)\Big)$$
$$= -sf(\omega) = (1-s)f(\omega),$$

by (2). By the proposition above we see that $e^{-2\pi is}$ is an eigenvalue of T. Since $e(T)$ is a group, $e^{2\pi is}$ is also an eigenvalue of T whenever (5) holds.

Conversely if $e^{-2\pi is} \in e(T)$ then by the proposition and (2) there exists $\phi\colon \Omega_0 \longrightarrow [0,1)$ such that (mod 1),

$$\phi\Big(\tau^{\nu}(\omega)\Big) - \phi(\omega) = \sum_{k=1}^{\infty}(1-s)\Big(\gamma_k(\tau^{\nu}\omega) - \gamma_k(\omega)\Big),$$

for all $\nu \in \mathbf{Z}$. If $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n, e, e, \dots) \in \Sigma$, then $\sigma(\omega) = \tau^{\nu(\omega)}(\omega)$ for some measurable function ν. Hence we have :

$$\phi\Big(\sigma(\omega)\Big) - \phi(\omega) = \sum_{k=1}^{\infty}(1-s)\Big(\gamma_k(\sigma\omega) - \gamma_k(\omega)\Big)$$
$$= \sum_{k=1}^{n}(1-s)\Big(\gamma_k(\sigma\omega) - \gamma_k(\omega)\Big) \cdot (\text{mod } 1),$$

since $\gamma_k\Big(\sigma(\omega)\Big) = \gamma(\omega)$ for $k > n$. (Recall that γ_k is $\mathcal{P}_k$ measurable.) Define

$$\phi_n(\omega) = \sum_{k=1}^{n}(1-s)\gamma_k(\omega),$$

and note that ϕ_n is $\mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \vee \mathcal{P}_n$ measurable. The function $\psi_n = \phi - \phi_n$ satisfies

$$(\phi - \phi_n)(\omega) = \phi(\omega) - \sum_{k=1}^{n}(1-s)\gamma_k(\omega) \pmod 1$$

which is invariant under all $\sigma = (\sigma_1, \dots, \sigma_n, e, e, \dots)$ and therefore measurable $\bigvee_{k=n+1}^{\infty}\mathcal{P}_k$.

Now $\phi = \phi_n + \psi_n$ and

$$e^{2\pi i\phi_n}\mathrm{E}(e^{2\pi i\psi_n}) = \mathrm{E}(e^{2\pi i\phi} \mid \mathcal{P}_1 \vee \cdots \vee \mathcal{P}_n) \longrightarrow e^{2\pi i\phi} \text{ a.e.}$$

as $n \longrightarrow \infty$. [Here E denotes the expectation or the conditional expectation.] Clearly there exist real constants A_n such that $\phi_n - A_n \longrightarrow \phi$ (mod 1), indeed we can take $A_n =$

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$\operatorname{Arg} \mathrm{E}(e^{2\pi i \psi_n})$. If we set $A_0 = 0$ and $c_k = A_k - A_{k-1}$, $k = 1, 2, \ldots$, then it follows that (mod 1)

$$\phi_n(\omega) - A_n = \sum_{k=1}^{n} \Big((1-s)\gamma_k(\omega) - c_k\Big) \longrightarrow \phi \text{ a.e. } [\mu].$$

This proves the theorem.

3.2 Restatement of Theorem 1. For any real number a let $[a]$ denote the largest integer $\leq a$, $\{a\} = a - [a]$ and

$$\langle a \rangle = \{a\} \quad \text{if } 0 \leq \{a\} \leq 1/2, \; \langle a \rangle = \{a\} - 1 \quad \text{if } 1/2 < \{a\} < 1.$$

We note that $|\langle a \rangle| \leq 1/2$ so that $\sum_{k=1}^{\infty} a_n$ converges (mod 1) if and only if $\sum_{k=1}^{\infty} \langle a_n \rangle$ converges.

Using these remarks we can restate Theorem 1 in the following form.

THEOREM 2. *For $s \in [0, 1)$, $e^{2\pi i s} \in e(T)$ if and only if there exist real constants $c_k, k = 1, 2, \ldots$ such that any one of the following series converges* (mod 1) *a.e.* $[\mu]$,

(a) $$\sum_{k=1}^{\infty} (\{s\gamma_k\} - c_k),$$

(b) $$\sum_{k=1}^{\infty} (\langle s\gamma_k(\omega) \rangle - c_k),$$

(c) $$\sum_{k=1}^{\infty} (\langle s\gamma_k(\omega) - c_k \rangle).$$

We can replace s by $-s$ or $1 - s$ in any of (a), (b), (c) above since eigenvalues form a group.

4. **The eigenvalue group: structural criterion.**

4.1. We now give a criterion for $e^{2\pi i s}$ to be an eigenvalue of T in terms of the quantities $\gamma_{k,j}$, $0 \leq j \leq m_k - 1$, $k = 1, 2, 3, \ldots$ which determine the rank one transformation T. We need Theorem 3 below which is an analog for the circle group of a similar theorem for the real line. (See Doob [2], p. 115, Theorem 2.7.) Recall that an infinite product $\prod_{k=1}^{\infty} a_k$ of complex numbers is said to be *convergent* if there is an M such that $\prod_{k=M}^{N} a_k$ converges to a non-zero complex number as N tends to infinity, which in turn holds true if and only if $\prod_{k=M}^{N} a_k$ tends to one as M, N tend to infinity. In case $0 \leq a_k \leq 1$, the non-convergence of the infinite product $\prod_{k=1}^{\infty} a_k$ is equivalent to the convergence to zero as N tends to infinity of the product $\prod_{k=M}^{N} a_k$ for every M.

Let Y be a random variable taking values in the circle group S^1. We will assume that our random variables are defined on a probability space $(W, \mathcal{C}, P)$. Let ν denote the distribution of Y and $\hat{\nu}$ its Fourier transform. Let $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ denote respectively the expectation and variance of Y. We note that

$$\mathrm{E}(Y^n) = \int_{S^1} z^n \, d\nu = \hat{\nu}(n), \quad n \in \mathbf{Z},$$

$$\mathrm{Var}(Y) = \int_{S^1} |z - \mathrm{E}(Y)|^2 \, d\nu = 1 - |\mathrm{E}(Y)|^2 = 1 - |\hat{\nu}(1)|^2.$$

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THEOREM 3. *Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of independent S^1 valued random variables with distributions $\nu_1, \nu_2, \nu_3, \ldots$ respectively. Then the following are equivalent:*

(a) There exist real constants c_k, $k = 1, 2, 3, \ldots$ such that if $Z_n = \prod_{k=1}^n Y_k e^{ic_k}$ then Z_n, $n = 1, 2, 3, \ldots$ converges a.e. over a subsequence,

(b) for all integers $p \in \mathbf{Z}$, the infinite product

$$\prod_{k=1}^{\infty} |\hat{\nu}_k(p)|^2$$

converges,

(c) $\sum_{k=1}^{\infty} \mathrm{Var}(Y_k)$ converges,

(d) for some $p \neq 0$, the infinite product

$$\prod_{k=1}^{\infty} |\hat{\nu}_k(p)|^2$$

converges.

PROOF. (a) implies (b). If $Z_{n_j}, j = 1, 2, 3, \ldots$ converges a.e. then

$$Z_{n_l}(Z_{n_j})^{-1} = \prod_{k=n_j+1}^{n_l} Y_k e^{ic_k} \longrightarrow 1$$

a.e. as $j, l \longrightarrow \infty$, whence for all p, $\prod_{k=n_j+1}^{n_l} \hat{\nu}_k(p) e^{ipc_k} \longrightarrow 1$ as $j, l \longrightarrow \infty$. Therefore since $|\hat{\nu}_k(p)| \leq 1$, $\prod_{k=1}^{\infty} |\hat{\nu}_k(p)|^2$ is a convergent infinite product for all p.

Since $\mathrm{Var}(Y_k) = 1 - |\hat{\nu}_k(1)|^2$, it is easy to see that (b) implies (c) and that (c) implies (d).

We prove that (d) implies (a). Suppose that for some $p \neq 0$, $\prod_{k=1}^{\infty} |\hat{\nu}_k(p)|^2$ is a convergent infinite product. Then

$$\prod_{k=j}^{l} |\hat{\nu}_k(p)|^2 \longrightarrow 1$$

as $j, l \longrightarrow \infty$. Since $|\hat{\nu}_k(q)| \leq 1$ the limit as $n \longrightarrow \infty$ of $\prod_{k=l}^{n} |\hat{\nu}_k(q)|^2$ exists for each q and the resulting limit as a function of q is the Fourier transform of a probability measure, say ρ_ℓ. The functions $\hat{\rho}_\ell$ are non-decreasing and their limit as $\ell \longrightarrow \infty$ is the Fourier transform of a probability measure, say ρ. Since $\hat{\rho}(p) = 1$ and $p \neq 0$ the measure ρ is the point mass at 1.

Let X_k be the random variable $X_k(x, y) = Y_k(x) \cdot \overline{Y_k}(y)$. (The bar denotes the complex conjugate.) Its distribution has Fourier transform $|\hat{\nu}_k(\cdot)|^2$. The finite products $\prod_{k=j}^{l} X_k$ converge in distribution to the point mass at 1 as $j, l \longrightarrow \infty$. Hence they also converge in measure to the constant function 1. It follows that $\prod_{k=1}^{n} X_k$, $n = 1, 2, 3, \ldots$ converges a.e. over an increasing subsequence $n_1, n_2, n_3, \ldots$ of natural numbers. By Fubini's theorem we see that for some y the products $\prod_{k=1}^{n_j} Y_k(x) \cdot \overline{Y_k}(y)$, $j = 1, 2, 3, \ldots$ converge for a.e. x as $j \longrightarrow \infty$. If we write $Y_k(y) = e^{ic_k}$, (a) follows, completing the proof of the theorem.

4.2. We apply this theorem to the random variables $Y_k = e^{2\pi i s \gamma_k}$, $k = 1, 2, 3, \ldots$ of Theorem 1. Note that, in this case, if the products $\prod_{k=1}^{n} Y_k \cdot e^{ic_k}$, $k = 1, 2, 3, \ldots$ converge a.e.

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over a subsequence then the argument used in the proof of Theorem 1 shows that the resulting limit extends to an eigenfunction of T with eigenvalue $e^{2\pi is}$. Hence by Theorem 1 the same product converges a.e. over the full sequence of natural numbers, possibly for some different constants c_k. Also note that

$$\mathrm{E}(Y_k) = \frac{1}{m_k}\sum_{j=0}^{m_k-1} e^{2\pi is\gamma_{k,j}},$$

$$\mathrm{Var}(Y_k) = 1 - \frac{1}{m_k^2}\Big|\sum_{j=0}^{m_k-1} e^{2\pi is\gamma_{k,j}}\Big|^2.$$

In view of Theorem 1 above we have at once the following characterization of the group $e(T)$. Write

$$\tilde{P}_k(z) = \sum_{j=0}^{m_k-1} z^{-\gamma_{k,j}}.$$

THEOREM 4. *For $s \in [0,1)$, the following are equivalent:*

(a)

$$e^{2\pi is} \in e(T);$$

(b) the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{m_k^2}|\tilde{P}_k(e^{2\pi is})|^2$$

is convergent;

(c)

$$\sum_{k=1}^{\infty} \mathrm{Var}(e^{2\pi is\gamma_k}) = \sum_{k=1}^{\infty}\Big(1 - \frac{1}{m_k^2}|\tilde{P}_k(e^{2\pi is})|^2\Big)$$

is finite.

COROLLARY. *If either of the series*

$$\sum_{k=1}^{\infty}\Big(\frac{1}{m_k}\sum_{j=0}^{m_k-1}|1 - e^{2\pi is\gamma_{k,j}}|\Big)$$

or

$$\sum_{k=1}^{\infty}\Big(\frac{1}{m_k}\sum_{j=0}^{m_k-1}|1 - e^{2\pi is\gamma_{k,j}}|^2\Big)$$

is finite then $e^{2\pi is} \in e(T)$.

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PROOF. If the first series converges, then so does the second. We have

$$\begin{aligned}1-\frac{1}{m_k^2}\Big|\sum_{j=0}^{m_k-1} e^{2\pi i s\gamma_{k,j}}\Big|^2 &= \frac{1}{m_k^2}\sum_{j=0}^{m_k-1}\sum_{\ell=0}^{m_k-1}(1-e^{2\pi i s\gamma_{k,j}}e^{-2\pi i s\gamma_{k,\ell}})\\ &= \frac{1}{m_k^2}\sum_{j<\ell}|e^{2\pi i s\gamma_{k,j}}-e^{2\pi i s\gamma_{k,\ell}}|^2\\ &= \frac{1}{m_k^2}\sum_{j<\ell}|(1-e^{2\pi i s\gamma_{k,j}})-(1-e^{2\pi i s\gamma_{k,\ell}})|^2\\ &\le \frac{2}{m_k^2}\sum_{j<\ell}(|1-e^{2\pi i s\gamma_{k,j}}|^2+|1-e^{2\pi i s\gamma_{k,\ell}}|^2)\\ &= \frac{2(m_k-1)}{m_k^2}\sum_{j=0}^{m_k-1}|1-e^{2\pi i s\gamma_{k,j}}|^2.\end{aligned}$$

Thus convergence of the second series implies condition (c) of Theorem 4, which proves the corollary.

4.3 Comments on Theorem 4. We note the close resemblance (already mentioned in the introduction) between the criterion for $e(T)$ obtained above and the expression for the maximal spectral type (up to discrete measures) obtained in our paper [1]. Since T and T^{-1} are spectrally equivalent, and as remarked in 2.2., $R_{i,k}(T) = \gamma_{k,i}(T^{-1})$ and $R_{i,k}(T^{-1}) = \gamma_{k,i}(T)$, it follows that both the sequences of polynomials $P_k(z) = \sum_{j=0}^{m_k-1} z^{-R_{i,k}}$ and $\tilde{P_k}(z)$ give the eigenvalue group $e(T) = e(T^{-1})$. Thus $z \in e(T)$ if and only if $\prod_{k=1}^{\infty}\frac{1}{m_k^2}|P_k(z)|^2$ converges or equivalently if $\prod_{k=1}^{\infty}\frac{1}{m_k^2}|\tilde{P_k}(z)|^2$ converges. The maximal spectral type σ (denoted by σ_0 in [1]) of T or T^{-1} is given, up to a discrete measure, by either of the generalized Riesz products $\prod_{k=1}^{\infty}\frac{1}{m_k}|P_k(z)|^2$ or $\prod_{k=1}^{\infty}\frac{1}{m_k}|\tilde{P_k}(z)|^2$. (The generalized Riesz product $\prod_{k=1}^{\infty}\frac{1}{m_k}|P_k(z)|^2$ is understood as the weak limit of the probability measures $\prod_{k=1}^{n}\frac{1}{m_k}|P_k(z)|^2\,dz$ as $n \longrightarrow \infty$.)

4.4.

THEOREM 5. *(a) If for $s \in [0,1)$, $e^{2\pi i s} \in e(T)$, then the series $\sum_{k=1}^{\infty}\mathrm{Var}(|2\pi\langle s\gamma_k\rangle|)$ is convergent.*

(b) If the series $\sum_{k=1}^{\infty}\mathrm{Var}(2\pi\langle s\gamma_k\rangle)$ is convergent then $e^{2\pi i s} \in e(T)$.

PROOF. (a) Suppose $e^{2\pi i s} \in e(T)$, $0 \le s < 1$, then

$$1-\frac{1}{m_k^2}\Big|\sum_{j=0}^{m_k-1} e^{2\pi i s\gamma_{k,j}}\Big|^2 \longrightarrow 0$$

as $k \longrightarrow \infty$. Without loss of generality we assume that $|\frac{1}{m_k}\sum_{j=0}^{m_k-1} e^{2\pi i s\gamma_{k,j}}| > 1/2$. For $z \neq 0$ write $z = |z|e^{i\theta}$, $-\pi \le \theta < \pi$. The map $\psi: z \longrightarrow |\theta|$ is Lipschitz on any compact subset of the complex plane not containing the origin. Hence it is Lipschitz on $1/2 \le |z| \le 1$. Let C be the Lipschitz constant on this domain. Then

$$\Big|\psi(e^{2\pi i s\gamma_k})-\psi\Big(\frac{1}{m_k}\sum_{j=0}^{m_k-1} e^{2\pi i s\gamma_{k,j}}\Big)\Big|^2 \le C^2\Big|e^{2\pi i s\gamma_k}-\frac{1}{m_k}\sum_{j=0}^{m_k-1} e^{2\pi i s\gamma_{k,j}}\Big|^2.$$

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Since the variance of a random variable is smaller than the second moment around any other point,

$$\operatorname{Var}\big(\psi(e^{2\pi i s\gamma_k})\big) = \operatorname{Var}(2\pi|\langle s\gamma_k\rangle|) \le C^2 \operatorname{Var}(e^{2\pi i s\gamma_k}).$$

Thus (a) follows by Theorem 4.

(b) The map $\phi(z) = e^{iz}$ is Lipschitz on any compact subset of the complex plane. Let C be Lipschitz constant for the domain $|z| \le 1$. We have

$$|e^{2\pi i s\gamma_k} - e^{(i\mathrm{E}(2\pi\langle s\gamma_k\rangle))}| \le C|2\pi\langle s\gamma_k\rangle - \mathrm{E}(2\pi\langle s\gamma_k\rangle)|.$$

Hence, by a similar argument as in (a), if the series $\sum_{k=1}^{\infty} \operatorname{Var}(2\pi\langle s\gamma_k\rangle)$ is finite then the series $\sum_{k=1}^{\infty} \operatorname{Var}(e^{2\pi i s\gamma_k})$ is finite and by Theorem 4, $e^{2\pi i s} \in e(T)$. This proves (b).

REMARK. In case the m_k are bounded then it follows from a theorem of Y. Ito, T. Kamae and I. Shiokawa [3] that the converse of (b) holds, *i.e.* if $e^{2\pi i s} \in e(T)$ then $\sum_{k=1}^{\infty} \operatorname{Var}(2\pi\langle s\gamma_k\rangle)$ is finite.

4.5 An example. In the case of Chacon's transformation, the height h_{k-1} of the $(k-1)$-st stack is $h_{k-1} = \frac{3^k-1}{2}$ (see [1]), and γ_k assumes three values $0, 3^k, \frac{3^k+1}{2}$, with equal probability. The series

$$\sum_{k=1}^{\infty}\Big(1 - \frac{1}{3^2}|1 + e^{2\pi i s 3^k} + e^{2\pi i s\frac{3^k+1}{2}}|^2\Big)$$

can be shown to be divergent for all $s \neq 0$ so that Chacon's transformation has no non-trivial eigenvalues. This proves the well known fact that Chacon's transformation is weakly mixing.

5. An expression for $\frac{d\sigma_\alpha}{d\sigma}$, $\alpha \in e(T)$.

5.1. We first describe a very concrete necessary and sufficient condition for $e^{2\pi i s}$, $s \in [0, 1)$ to be an eigenvalue of T. For each $k = 1, 2, 3, \ldots$, we define a function ψ_k on Ω_0 as follows: Let

$$q_k(\omega) = \text{ least integer } \ge 0 \text{ such that } T^{-q_k(\omega)}(\omega) \in \Omega_k$$
$$= h_k - \lambda_k(\omega) - 1.$$

If $\omega \notin \Omega^k$, $q_k(\tau\omega) = q_k(\omega) + f(\omega)$. Define

$$\psi_k(\omega) = e^{2\pi i s q_k(\omega)} = e^{2\pi i s(-\lambda_k(\omega)+h_k-1)}.$$

If $\lim_{n\to\infty} \psi_{k_n}(\omega)$ exists a.e. along some subsequence $k_n \longrightarrow \infty$, then the limit function ψ satisfies $\psi(\tau\omega) = e^{2\pi i s f(\omega)}\psi(\omega)$, so that, by the proposition, $e^{2\pi i s} \in e(T)$. Conversely if $e^{2\pi i s} \in e(T)$ for some $s \in [0, 1)$, then there exist real constants c_k such that $\sum_{k=1}^{\infty}\big(s\gamma_k(\omega) - c_k\big)$ converges a.e. (mod 1). Equivalently

$$\sum_{k=1}^{n}\big(s\gamma_k(\omega) - c_k\big) = s\lambda_n(\omega) - \sum_{k=1}^{n} c_k = s\lambda_n(\omega) - A_n$$

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converges a.e. (mod 1), where $A_n = \sum_{k=1}^{n} c_k$. Since the A_n are constants, $s\lambda_k$ converges a.e. (mod 1) along a subsequence. For the same reason, since s, h_k are constants,

$$sq_k(\omega) = sh_k - s\lambda_k(\omega) - s$$

converges a.e. (mod 1) along a further subsequence, say k_n, to a function ϕ, so that $e^{2\pi i s q_{k_n}}$ converges a.e. to $e^{2\pi i\phi}$. We thus have:

THEOREM 6. *For $s \in [0, 1)$, $e^{2\pi is} \in e(T)$ if and only if the sequence $\psi_k = e^{2\pi isq_k}$, $k = 1, 2, 3, \ldots$ converges along a subsequence to a function ψ. This function ψ then extends in a natural way to an eigenfunction of T with eigenvalue $e^{2\pi is}$.*

Note that our argument in fact shows that $e^{2\pi is} \in e(T)$ if and only if given any increasing sequence k_n, $n = 1, 2, 3, \ldots$ of natural numbers there is a subsequence of it over which the functions ψ_k, $k = 1, 2, 3, \ldots$ converge a.e. to a function ψ which then extends to an eigenfunction of T with eigenvalue $e^{2\pi is}$. Any two such limits differ by a multiplicative constant of absolute value one. Note also that $e^{2\pi is} \in e(T)$ if and only if the ψ_k converge over a subsequence in the L^2 norm.

We note that the functions ψ_k vanish outside Ω_0. Since Ω_0 has finite measure the ψ_k are in $L^2(X, \mathcal{B}, m)$ with bounded L^2 norms. Any weak limit ψ of the collection $\{\psi_k : k = 1, 2, 3, \ldots\}$ satisfies the relation

$$\psi(\tau\omega) = e^{2\pi isf(\omega)}\psi(\omega).$$

If such a ψ is non-zero then it extends to an eigenfunction of T, and ψ is then an a.e. limit of the ψ_k over a subsequence. Thus we see that either the ψ_k converge weakly to zero or the ψ_k converge a.e. over a subsequence to a function which extends to an eigenfunction with eigenvalue $e^{2\pi is}$.

5.2. The maximal spectral type σ of U_T is given (up to a discrete measure) by the weak limit as $n \longrightarrow \infty$ of the measures $\prod_{k=1}^{n} \frac{1}{m_k}|P_k(z)|^2\, dz$. We will assume in the rest of this section that the weak limit is indeed precisely equal to the maximal spectral type of U_T. Such is the case, for example, when the measure m is infinite or when none of the P_k vanish on S^1. If $\alpha \in S^1$, then the translate σ_α of σ by α is given by the weak limit of the measures $\prod_{k=1}^{n} \frac{1}{m_k}|P_k(\alpha z)|^2$. It is known that if $\alpha \in e(T)$ then σ_α and σ are mutually absolutely continuous.

Fix $s \in [0, 1)$, write $\alpha = e^{2\pi is}$ and let ψ_k be the functions as in Theorem 6 for this s. The correspondence $U_T^n 1_{\Omega_0} \leftrightarrow z^n$, $n \in \mathbf{Z}$ extends by linearity to an invertible isometry S from the closed linear span $\mathcal{H}$ of $\{U_T^n 1_{\Omega_0} : n \in \mathbf{Z}\}$ to $L^2(S^1, \sigma)$. We know from [1] that

$$1_{\Omega_0} = \Big(\prod_{j=1}^{k} P_j(U_T)\Big) 1_{\Omega_k},$$

and one sees similarly that

$$\psi_k = \Big(\prod_{j=1}^{k} P_j(\bar{\alpha} U_T)\Big) 1_{\Omega_k},$$

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$$S1_{\Omega_0} = \Big(\prod_{j=1}^{k} P_j(\bar z)\Big) S1_{\Omega_k},$$

$$S\psi_k = \Big(\prod_{j=1}^{k} P_j(\bar\alpha \bar z)\Big) S1_{\Omega_k}.$$

Since $S1_{\Omega_0} = 1$, we see that

$$S\psi_k = \prod_{j=1}^{k} \frac{P_j(\bar\alpha\bar z)}{P_j(\bar z)}.$$

By Theorem 6, $\alpha \in e(T)$ if and only if the ψ_k converge over a subsequence to a function ψ in the L^2 norm. Hence $\alpha \in e(T)$ if and only if $S\psi_k$ converge over a subsequence in the L^2 norm. If ψ_k converge over a subsequence in the L^2 norm to a function ψ, then $(S\psi_k)$ will converge in the L^2 norm over the same subsequence to $S\psi$. Any two subsequential limits of the ψ_k differ by a constant of absolute value one, hence any two subsequential limits of the $S\psi_k$ will also differ by a constant of absolute value one. In view of the remark after Theorem 6, we see that if $\alpha \in e(T)$ then

$$\prod_{j=1}^{k} \left| \frac{P_j(\bar\alpha\bar z)}{P_j(\bar z)} \right|$$

converges in L^2 norm as $k \longrightarrow \infty$ to the function $|S\psi|$, the convergence being over the full sequence of natural numbers. Hence, if $\alpha \in e(T)$ then

$$\prod_{j=1}^{k} \left| \frac{P_j(\bar\alpha\bar z)}{P_j(\bar z)} \right|^2$$

converges in $L^1(S^1, \sigma)$ to $|S\psi|^2$.

When $\alpha \in e(T)$, a subsequential limit ψ of the ψ_k is the restriction to Ω_0 of an eigenfunction ψ' with eigenvalue α. We have for such a subsequential limit ψ and $n \in \mathbf{Z}$;

$$\begin{aligned}
(U_T^n\psi, \psi) &= (U_T^n\psi' 1_{\Omega_0}, \psi' 1_{\Omega_0}) \\
&= (\alpha^n \psi' U_T^n 1_{\Omega_0}, \psi' 1_{\Omega_0}) \\
&= \alpha^n (U_T^n 1_{\Omega_0}, 1_{\Omega_0}) \\
&= \int_{S^1} (\alpha z)^n \, d\sigma \\
&= \int_{S^1} z^n \, d\sigma_\alpha, \quad (\text{where } \sigma_\alpha(A) = \sigma(\alpha^{-1}A)) \\
&= \int_{S^1} z^n \frac{d\sigma_\alpha}{d\sigma} \, d\sigma.
\end{aligned}$$

But

$$(U_T^n\psi, \psi) = \int_{S^1} z^n |S\psi|^2 \, d\sigma, \quad n \in \mathbf{Z}.$$

Thus

$$\frac{d\sigma_\alpha}{d\sigma} = |S\psi|^2,$$

and we have proved:

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THEOREM 7. *If $\alpha \in e(T)$ then*

$$\frac{d\sigma_\alpha}{d\sigma} = \lim_{k\to\infty} \prod_{j=1}^{k} \left| \frac{P_j(\bar\alpha \bar z)}{P_j(\bar z)} \right|^2,$$

convergence being in the L^1 norm.

We conclude with the query whether, when $\alpha \notin e(T)$, the measures σ and σ_α are mutually singular and further if

$$\lim_{k\to\infty} \prod_{j=1}^{k} \left| \frac{P_j(\bar\alpha \bar z)}{P_j(\bar z)} \right|^2 = 0 \text{ a.e. } [\sigma]$$

in that case?

REFERENCES

1. J. R. Choksi and M. G. Nadkarni, *The maximal spectral type of a rank one transformation*, Canad. Math. Bull. **37**(1994), 29–36.
2. J. L. Doob, *Stochastic Processes*, Wiley Interscience, New York, 1953.
3. Y. Ito, T. Kamae and I. Shiokawa, *Point spectrum and Hausdorff dimension, Number Theory and Combinatorics*, (eds. J. Akiyama et al), World Scientific Publishing Co. Tokyo, 1985, 209–227.
4. M. Osikawa, *Point spectrum of non-singular flows*, Publ. Res. Inst. Math. Sci. **13**(1977), 167–172.

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