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EISENSTEIN SERIES FOR REDUCTIVE GROUPS OVER GLOBAL FUNCTION FIELDS II

The General Case

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Introduction. This paper is a continuation of [**5**]. As stated there, the problem is to explicitly decompose the space $L^2 = L^2(G(F)\backslash G(\mathbf{A}))$ into simpler invariant subspaces, and to deal with the associated continuous spectrum in case G is a connected reductive algebraic group defined over a global function field. In that paper the solution was begun by studying Eisenstein series associated to cusp forms on Levi components of parabolic subgroups; these Eisenstein series and the associated intertwining operators were shown to be rational functions satisfying functional equations. To go further it is necessary to consider more general Eisenstein series and intertwining operators, and to show that they have similar properties. Such Eisenstein series arise from the cuspidal ones by a residue taking process, which is detailed in a disguised form suitable for induction in the first part of this paper: the main result is a preliminary form of the spectral decomposition. In the second part these results are used to show that the residual Eisenstein series and intertwining operators do possess all the requisite properties, including functional equations; these properties are then exploited to transform the preliminary spectral decomposition into a more explicit form. Elsewhere we sketch some complements to the theory described above. Firstly a variant of the theory is sketched which takes care of metaplectic coverings: this is valid over global fields of arbitrary characteristic. Secondly we prove a function field analogue of an induction theorem for automorphic representations due, in the number field case, to Langlands [**4**]. The present paper can also serve as an introduction to the number field version of the same problem (carried out by Langlands ca. 1964, and published in [**3**]): the analytic problems inherent in the latter case evanesce for function fields, leaving the essential induction argument untrammeled by technical analytical difficulties.

The ideas behind this paper are simple enough and are the same as those employed in [**3**] and sketched intuitively in [**1**]; we recapitulate them briefly.

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In [5] it was seen that $\mathscr{L}(\xi)$, which can be viewed as L^2 for argument's sake, is equal to

$$\bigoplus_{\{P\}} \mathscr{L}(\{P\}, \xi)$$

where $\{P\}$ runs through the classes of associate standard parabolic subgroups of G, including $\{G\}$; the space $\mathscr{L}(\{P\}, \xi)$ is closed and $G(\mathbf{A})$-invariant. For simplicity suppose P is the minimal standard parabolic so that $\{P\}$ consists of one element, and write $P = NM$, where M is the Levi component associated to some fixed chosen maximal split torus T_0. There is then an associated complex manifold $D_M(\xi)$ of quasicharacters on $Z_M(\mathbf{A})$, which we can think of as an infinite "tube" whose axis in the infinite direction consists of "real" quasicharacters, and whose axis in the tubular direction is the imaginary axis: namely if $\zeta \in D_M(\xi)$ then Re $\zeta = |\zeta|$ with $|\zeta|(z) = |\zeta(z)|$, and $D_M{}^0(\xi) = \{\zeta| \quad |\zeta| \text{ trivial}\}$ is the imaginary axis.

The space $\mathscr{L}(\{P\}, \xi)$ can be thought of as a space of Poincare θ series, or as the closure of a space of holomorphic cross sections of a vector bundle $\mathscr{C}$ on an open subset of $D_M(\xi)$. Viewed in the latter way, it is complete with respect to the inner product

$$(0.1) \quad (\Phi, \Psi) = \int_{\mathrm{Re}\zeta=\zeta_0} \sum_{w\in W} \langle M(w, \zeta)\Phi(\zeta), \Psi(-w\bar{\zeta})\rangle d\zeta$$

where W is the relative Weyl group, $M(w, \zeta)$ is the intertwining operator associated to the Eisenstein series in [5] and $\zeta_0 - \delta_P \in C_P$ (the "small" Weyl chamber), if δ_P is the modular function for P. The symbol $\langle\ ,\ \rangle$ refers to a sesquilinear pairing that need not concern us here.

The space $D_M{}^0(\xi)$ lends itself to a natural Lebesgue measure; let $\mathscr{L}$ denote the space of cross sections of $\mathscr{C}$ on $D_M{}^0(\xi)$ which are square integrable with respect to this measure, and consider the operator on $\mathscr{L}$ given by

$$Q: \Phi \to \frac{1}{\#(W)} \sum_{w\in W} M(w^{-1}, w\zeta)\Phi(w\zeta).$$

The functional equations established in [5] for $M(w, \zeta)$ viz

$$M(st, \zeta) = M(s, t\zeta)M(t, \zeta)$$

imply that Q is a projection with range consisting of those Φ for which

$$\Phi(s, \zeta) = M(s, \zeta)\Phi(\zeta).$$

Since $M(w, \zeta)^* = M(w^{-1}, -w\bar{\zeta})$, the operator Q is self adjoint and the inner product $(Q\Phi, \Psi)$ is given by (0.1) with ζ_0 replaced by 0.

Suppose in (0.1) that ζ_0 were 0 (at present it satisfies $\zeta_0 - \delta_P \in C_P$). The above would imply that $\mathscr{L}(\{P\}, \xi)$ was isomorphic to the range of Q. But ζ_0 is not 0, so we must perform a contour integration, picking

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up residues due to poles of the $M(w, \zeta)$. To avoid this, one considers those Φ in $\mathscr{L}(\{P\}, \xi)$ whose zeroes cancel the poles of the $M(w, \zeta)$ in the region over which the contour manipulations take place. Consequently the subspace of $\mathscr{L}(\{P\}, \xi)$ generated by such Φ will be isomorphic to the range of Q. The inner product of the projection of $\mathscr{L}(\{\mathrm{P}\}, \xi)$ onto the orthogonal complement of this subspace is given by the residues associated to the $M(w, \zeta)$ picked up as one moves the contour of integration. The result is a sum of integrals of the same type as before, but taken over "hyperplanes". Langland's method is to repeat the argument and proceed by an intricate induction until nothing is left (c.f. Sections 5 and 4.14 of the present paper).

There are some unpleasant difficulties encountered in carrying out this argument in detail, which are not merely notational. In the first place, for number fields one must eventually integrate over regions where one has little control of $\|M(w, \zeta)\|$; $D_M(\xi)$ is not compact in the imaginary direction in this case, and one eventually ends up in regions where $M(w, \zeta)$ is not well understood. Langlands circumvents this by an ingenious use of spectral theory (for unbounded self-adjoint operators). This accounts for the analytic difficulties alluded to earlier; in the function field version $D_M(\xi)$ is tubular and these problems do not arise.

In either case one must eventually compute residues of poles which are not simple (see [**3**] appendix III for an example), and an appropriate notation for handling this must be developed; this explains some of the preliminaries of Section 2 of this paper. Another problem that arises is that Res $M(w, \zeta)$ is not necessarily analytic on the final axis of integration (loc. cit.): one must live with a weaker statement (c.f. 4.14 (iii) of this paper). Here we should point out that the intertwining operators that one constructs for the final, explicit spectral decomposition are not merely residues of the original intertwining operators. Their method of construction is given in part II of this paper, and they themselves are holomorphic on the imaginary axis.

Now for an aperçu of this paper. We begin part I with a review of standard parabolic subgroups, and the various complex manifolds associated to them; this section should be read carefully as it is the basis for induction arguments. In Section 2 the necessary language for residue-taking is set up, as well as some other preliminaries. Then in Section 3 we define the notion of an Eisenstein system; this is defined by conditions tailor made for induction in Section 5. The first example is that of the Eisenstein series and associated constant terms defined in [**5**]; the results of [**5**] show that these satisfy the conditions for an Eisenstein system. We then show how to construct the only other Eisenstein systems of any interest to us: those obtained by taking residues of a known one. Section 4 is largely concerned with properties of Eisenstein systems which play a role in Section 5; in particular we show how to associate closed invariant subspaces of automorphic forms to appropriate collections of Eisenstein systems. This sets the stage for Section 5 which is the heart of the paper: here the spectral decomposition is initially obtained in terms of subspaces associated to various Eisenstein systems obtained by inductively taking residues. To transform this decomposition into a more canonical and explicit one using only the recipe of inducing from parabolic subgroups leads to part II.

Thus in Section 6 we return to Eisenstein series defined in a naive way and show how part I implies that these have all the properties that they should. The corresponding intertwining operators are constructed in Section 7 and shown to possess all requisite properties, including functional equations. These results are then used in Section 8 to put the spectral decomposition into the desired form and describe it more explicitly.

Part I is the means by which one proceeds to the final theorems of part II, but perhaps the best way to proceed is to read enough of part I to understand the statements of the results, and then move on to part II.

Two related results will appear elsewhere. Firstly, if F is a global field of any characteristic, G a reductive group defined over F and $\tilde{G}$ is a finite central covering of $G(\mathbf{A})$ with a "splitting" over $G(F)$, then we show how to adapt the usual theory to decompose $L^2(G(F)\backslash\tilde{G})$; this is designed for metaplectic coverings. The second result says that automorphic representations arise as constituents of representations induced from cuspidal representations of parabolic subgroups; it is a function field version of [**4**].

The paper is more or less self-contained with two exceptions. The first and more serious is that no proof of Langlands' ubiquitous Lemma 7.4 [**3**] is given. Langlands gives a proof in [**3**], and a more detailed proof is presented in [**7**]. The function field account has nothing to add (or subtract), so has been omitted. The second exception is minor: it is concerned with square integrable automorphic forms, and does not properly belong in the paper; in any event it will appear elsewhere.

Finally, I have made systematic use of the manifolds $D_M(\xi)$ as suggested by Langlands in appendix II of [**3**]; this approach is necessary for function fields and deserves to be better known in general.

Anyone who compares them will see that this paper owes its existence to [**3**]; I only wish to add that I have found the forthcoming monograph [**7**] by Osborne and Warner to be very helpful in many places, especially in dealing with the induction step for the auxiliary spectral decomposition. I take this opportunity to thank them for providing me with a preliminary version so expeditiously.

Conventions.

F : global field of char $p > 0$

$e(x)$: exp $(x \log q)$, $q = \#$ field of constants of F
δ_P : modular function associated to a parabolic subgroup P
$\mathbf{A}$: the ring of adeles associated to F.

1. Parabolic subgroups and related complex manifolds.

1.1. Let G be a connected reductive group defined over F, T' a maximal torus, not necessarily defined over F. Let $\bar{F}$ be an algebraic closure for F and suppose that a set of roots R' and simple roots Δ' has been given for the pair $(G_{\bar{F}}, T_{\bar{F}}')$. Let T_0 be a maximal F splt torus with $T_{0\bar{F}} \subseteq T_{\bar{F}}'$ and R the set of roots G with respect to T_0. As remarked in [**6**] I.2.1 we can choose a set of simple roots Δ for G with respect to T_0 in such a way that if $\alpha \in \Delta'$ then either $\alpha|_{T_0}$ is trivial or an element of Δ; if this is so then there is a correspondence between subsets of Δ and Gal $(\bar{F}/F)$-stable subsets of Δ_ϕ', where Δ_ϕ' consists of those elements of Δ' not trivial on T_0. The set Δ corresponds to a minimal parabolic subgroup P_0 defined over F and there is a canonical Levi decomposition $P_0 = N_0M_0$ where N_0 is the unipotent radical of P_0, and M_0 is $Z_G(T_0)$.

As explained in [**5**] 1.2.2 and elsewhere, there is a correspondence between parabolic subgroups of G which contain P_0 and subsets of Δ; if $\Theta \subseteq \Delta$ we write P_Θ for the corresponding parabolic subgroup. Parabolics obtained in this way are said to be *standard parabolic subgroups*; they form a set of representatives for the F-conjugacy classes of parabolic subgroups of G. In particular $P_\phi = P_0$, $P_\Delta = G$; the standard maximal parabolics correspond to sets of the form $\Delta - \{\alpha\}$, and we sometimes denote them by P^α rather than $P_{\Delta-\{\alpha\}}$.

In this paper, unless otherwise stated, we shall only work with standard parabolic subgroups, and "parabolic" will be taken to mean "standard parabolic subgroup".

1.2. Let P be as above, then it has a canonical Levi decomposition $P = NM$ with N the unipotent radical, and $M = Z_G(T)$; here $T \subseteq T_0$ is defined by

$$T = \bigcap_{\alpha \in \Theta} (\ker \alpha)$$

if $P = P_\Theta$. By abuse of notation we shall usually refer to M as "the" Levi component of P, meaning it has been chosen as above.

Let X_M be the group of rational characters of $Z_G\backslash M$. If k is a field we shall write $X_M(k)$ for $X_M \otimes_Z k$. In particular we have the real (resp. complex) finite dimensional vector space $X_M(\mathbf{R})$ (resp. $X_M(\mathbf{C})$). Its elements may be interpreted as homomorphisms

$$M(\mathbf{A}) \to \mathbf{R}_+^\times \quad (\text{resp. } \mathbf{C}^\times)$$

which are trivial on $Z_G(\mathbf{A})M(F)$. It should be noted that $X_M(\mathbf{R})$ can equally well be viewed as homomorphisms

$$Z_G(\mathbf{A})Z_M(F)\backslash Z_M(\mathbf{A}) \quad \to \mathbf{R}_+^\times.$$

Set $X_M^*(\mathbf{R}) = \operatorname{Mor}(X_M(\mathbf{R}), \mathbf{R})$. There is a homomorphism $H_M : M(\mathbf{A}) \to X_M^*(\mathbf{R})$ which can be described as follows. If $m \in M(\mathbf{A})$, then for $\chi \in X_M(\mathbf{R})$,

$$\chi(m) = e\langle H_M(m), \chi\rangle.$$

The image of H_M is a lattice L_M^* in $X_M^*(\mathbf{R})$. If we restrict H_M to $Z_M(\mathbf{A})$, and denote it by H_{Z_M}, then we obtain a lattice $L_{Z_M}^* \subseteq L_M^*$ and these two lattices are, in general, distinct. The kernel of H_M (resp. H_{Z_M}) is denoted by M^0 (resp. Z_M^0); it does not correspond to a reductive algebraic group in general, but for many purposes it behaves like one: it contains $M(F)$, unipotent radicals, and any compact subgroup of $M(\mathbf{A})$. In particular one can define the notion of cusp form on M^0.

1.3. Given $P \supseteq P_0$ one then has $M \supseteq M_0$ which gives rise to a natural projection

$$X_0^*(\mathbf{R}) \to X_M^*(\mathbf{R})$$

and injection

$$X_M(\mathbf{R}) \to X_0(\mathbf{R})$$

where for brevity we set $X_0^*(\mathbf{R}) = X_{M_0}^*(\mathbf{R})$, etc. Let T be the split component of Z_M, and Δ_P (resp. Σ_P) the set of simple roots (resp. roots) for the pair (P, T). These last are elements of $X_M(\mathbf{Q})$ hence elements of $X_M(\mathbf{R})$; any element α of Δ_P is the restriction of a unique root $\alpha_0 \in \Delta$. We shall write α^v for the projection of the coroot α_0^v to $X_M^*(\mathbf{R})$, and call it a *simple coroot*.

More generally, suppose $P_2 \supseteq P_1$ with Levi decompositions $P_i = N_iM_i$, then there are maps $X_1^*(\mathbf{R}) \twoheadrightarrow X_2^*(\mathbf{R})$, $X_2(\mathbf{R}) \hookrightarrow X_1(\mathbf{R})$ coming from $M_1 \subset M_2$. The group $P_1 \cap M_2$ is a parabolic subgroup of M_2 with unipotent radical $N_1^2 = N_1 \cap M_2$; if Δ_1^2 is the set of simple roots for the pair $(P_1 \cap M_2, T_1)$ then $\Delta_1^2 \subseteq \Delta_1$ and in this way subsets of Δ_1 correspond to parabolic subgroups containing P_1. The map $P_1 \to P_1 \cap M_2$ is a bijection between parabolics contained in P_2 and parabolic subgroups of M_2 and $X_2^*(\mathbf{R})$ can be identified with the vector subspace of $X_1^*(\mathbf{R})$

$$\{\chi | \quad \alpha(\chi) = 0, \alpha \in \Delta_1^2\}$$

as follows easily from the definition of T_2 and the alternative interpretation of $X_2(\mathbf{R})$ given in 1.2.

If $(X_1^2)^*$ is the subspace of $X_1^*(\mathbf{R})$ annihilated by $X_2(\mathbf{R})$ then

$$X_1^*(\mathbf{R}) = X_2^*(\mathbf{R}) \oplus (X_1^2)^*$$

and the subspace of $X_1(\mathbf{R})$ spanned by Δ_1^2 is in duality with $(X_1^2)^*$. We denote it by X_1^2 and then

$$X_1(\mathbf{R}) = X_2(\mathbf{R}) \oplus X_1^2.$$

Similarly

$$(X_0^2)^* = (X_0^1)^* \oplus (X_1^2)^*$$

and the set $\{\alpha^v : \alpha \in \Delta_1^2\}$ is a basis for $(X_1^2)^*$.

Finally, we note that if one identifies $X_0^*(\mathbf{R})$ with $X_0(\mathbf{R})$ by means of a bilinear form invariant under the Weyl group, then the decompositions elaborated above can be viewed as orthogonal sums. This point of view will often be taken, without comment.

In particular we can expand our definition of coroot: if $(\ , \ )$ is such a bilinear form then for each $\alpha \in \Sigma_P$ we define (by restriction of the associated inner product) $\alpha^v \in X_M^*(\mathbf{R}) \simeq X_M(\mathbf{R})$ by

$$\alpha^v = 2\alpha/(\alpha, \alpha).$$

The set of these coroots (resp. simple coroots) is denoted by Σ_P^v (resp. Δ_P^v).

1.4. In [**5**] 2.1 we defined a complex analytic manifold $D_M(\xi)$; as it plays a central role in all that is to follow, we may as well recall its definition. Let ξ be a character of $Z(F)\backslash Z(\mathbf{A})$, then as a set, $D_M(\xi)$ consists of those quasicharacters of $Z_M(F)\backslash Z_M(\mathbf{A})$ which prolong ξ. The group $X_M(\mathbf{C})$ acts on $D_M(\xi)$ via

$$\chi \rightarrow \chi \cdot e\langle H_{Z_M}(\cdot), \omega\rangle$$

if $\chi \in D_M(\xi)$, $\omega \in X_M(\mathbf{C})$, and the stabilizer of any χ is simply ιL_{Z_M}, where

$$L_{Z_M} = \{\omega \in X_M(\mathbf{R}) | \quad \langle H_M(z), \omega\rangle \in 2\pi\mathbf{Z}/\log q, z \in Z_M(\mathbf{A})\}.$$

In this way $D_M(\xi)$ acquires the structure of a complex analytic manifold, characterized by the fact that each connected component can be (non canonically) identified with the complex Lie group $\iota L_{Z_M}\backslash X_M(\mathbf{C})$.

For each component of $D_M(\xi)$, choose a character ω as a basepoint for that component; we write $\Omega(\omega)$ to indicate the relation.

If $\zeta \in D_M(\xi)$, define $\operatorname{Re}\zeta$ by $\operatorname{Re}\zeta(z) = |\zeta(z)|$; this is an element of $X_M(\mathbf{R})$. The set of characters $D_M^0(\xi)$ is the set of ζ for which $\operatorname{Re}\zeta$ is trivial: it is a disjoint sum of compact connected real manifolds, each of which is diffeomorphic to $L_{Z_M}\backslash X_M(\mathbf{R})$.

1.5. Let H be an affine subspace of $X_M(\mathbf{C})$; in view of the map

$$X_M(\mathbf{C}) \rightarrow \iota L_{Z_M}\backslash X_M(\mathbf{C}) \rightarrow \amalg_\omega \Omega(\omega) = D_M(\xi)$$

we see that H gives rise to a submanifold of $D_M(\xi)$, which we shall call an *affine subspace* of $D_M(\xi)$; in particular an affine hyperplane will arise from an affine subspace of codimension 1. The only affine subspaces of interest to us in this paper will be defined by equations of the form

$$\alpha^v(\quad) = \lambda_\alpha, \ \alpha \in \Xi$$

where $\Xi \subseteq \Sigma_P$. By a *standard affine subspace* we shall mean one for which $\Xi \subseteq \Delta_P$.

In this paper the words "affine subspace", "affine hyperplane" will always refer to one defined by equations as above with $\Xi \subseteq \Sigma_P$.

1.6. Unfortunately, for the induction that lies at the core of this paper we shall have to operate at a degree of generality a little higher than that expounded in 1.3-1.5. For this, let Z_0 be a closed subgroup of $Z_M(\mathbf{A})$: in practice Z_0 will be $Z_{*M}(\mathbf{A})$ where $*P \supseteq P$ is some larger parabolic, possibly G itself. In any case we assume that $Z_0 Z_M(F)$ is closed in $Z_M(\mathbf{A})$. Let ξ be a quasicharacter on Z_0, assumed trivial on $Z_0 \cap Z_M(F)$: note that ξ is not necessarily assumed to be a character. We now write $X_M(\mathbf{R})$ for the (finite dimensional) real vector space consisting of quasicharacters

$$\chi : Z_0 Z_M(F) \backslash Z_M(\mathbf{A}) \rightarrow \mathbf{R}_+^{\times}$$

and $X_M(\mathbf{C})$ for $X_M(\mathbf{R}) \otimes \mathbf{C}$.

Let $D_M(\xi)$ denote the set of quasicharacters on $Z_M(F) \backslash Z_M(\mathbf{A})$ which prolong ξ. The complex group $X_M(\mathbf{C})$ acts on $D_M(\xi)$ as before, and each element of $D_M(\xi)$ has the same stabilizer, namely ιL_{Z_M} where L_{Z_M} is a lattice which can be defined analogously to that defined earlier. Thus we see that $D_M(\xi)$ can be given a natural structure of complex analytic manifold, in which each connected component is holomorphically equivalent to $\iota L_{Z_M} \backslash X_M(\mathbf{C})$. Such a holomorphic equivalence depends on a choice of base point in each connected component: we can always suppose that these are chosen to be of the form $\omega \nu_0$ where ν_0 is a fixed real quasi character which prolongs $|\xi| = \operatorname{Re} \xi$, and ω is some character.

In particular suppose that $Z_0 = Z_{*M}(\mathbf{A})$ where $*P \supseteq P$ is some larger parabolic, possibly G itself. Then one may certainly speak of roots, coroots, weights etc. corresponding to $X_M(\mathbf{R})$: in fact in this situation $X_M(\mathbf{R})$ is precisely the $X_1{}^2$ of 1.3 where $*P = P_2 \supseteq P = P_1$.

Henceforth when the symbol $D_M(\xi)$ appears, it will be with respect to some (implicitly) defined $Z_0 \subseteq Z_M(\mathbf{A})$. Two cases are of special interest: $\xi = \chi\delta_{*P}$ where $*P \supseteq P$, and χ is a character on $Z_{*M}(\mathbf{A})$; secondly, ξ a character on $Z_G(\mathbf{A}) \subseteq Z_M(\mathbf{A})$. In general the meaning will be clear from the context; in particular when roots etc. are mentioned, it is assumed that $Z_0 = Z_{*M}(\mathbf{A})$ or $Z_G(\mathbf{A})$.

In general, if $Z_0 = Z_{M_Q}(\mathbf{A})$ we shall preface the object of interest with a "Q" thus ${}^Q D_M(\xi)$, ${}^Q X_M(\mathbf{C})$ etc. Finally, we shall abuse notation and write $Z_Q(\mathbf{A})$ for $Z_{M_Q}(\mathbf{A})$ to avoid a proliferation of subscripts.

1.7. We shall also need to define various chambers which enter into the picture later on. Firstly we shall frequently write ${}^Q\mathfrak{a}_M$ (resp. ${}^Q\mathfrak{a}_M(\mathbf{C})$) for ${}^Q X_M(\mathbf{R})$ (resp. $X_M(\mathbf{C})$).

If $x \in \mathfrak{a}_M$, we shall write $x > 0$ if

$$\alpha^v(x) > 0 \text{ for each } \alpha^v \in \Delta_P{}^v.$$

We then set

$${}^Q\mathfrak{a}_M{}^+ = \{x \in {}^Q\mathfrak{a}_M | x > 0\},$$

and define the (small) Weyl chamber ${}^QC_P$ by

$${}^QC_P = \mathfrak{a}_M{}^+ = \{x | x > {}^Q\delta_P\}.$$

We define the dual (large) chamber by

$${}^QC_P{}^* = {}^{+Q}\mathfrak{a}_M$$

where

$${}^{+Q}\mathfrak{a}_M = \{\textstyle\sum c_i\alpha_i, \alpha_i \in {}^Q\Delta_P, c_i > 0\}.$$

It is always the case that ${}^{+Q}\mathfrak{a}_M \supseteq {}^Q\mathfrak{a}_M{}^+$.

More generally let $\mathfrak{c} \subseteq {}^Q\mathfrak{a} = {}^Q\mathfrak{a}_M$ be defined by equations

$$\alpha^v(\chi) = 0, \quad \alpha^v \in \Xi \subseteq {}^Q\Sigma_P{}^v,$$

and let $\mathfrak{b}$ be the orthogonal complement of $\mathfrak{c}$ in $\mathfrak{a}$, then we can define

$${}^+\mathfrak{b} = \{\textstyle\sum c_i\alpha_i | \quad \alpha_i{}^v \in \Xi, C_i > 0\}.$$

Similarly $\mathfrak{b}^+$ can be defined and then ${}^+\mathfrak{b} \supseteq \mathfrak{b}^+$.

If $\mathfrak{r}$ is an affine subspace of ${}^Q\mathfrak{a} = {}^Q\mathfrak{a}_M$, then let ${}^Q\mathfrak{a}(\mathfrak{r})$ denote the orthogonal complement of $\mathfrak{r}^v \cap \mathfrak{a}$ in $\mathfrak{a}$. From the above we can then define ${}^{+Q}\mathfrak{a}(\mathfrak{r})$. Similarly if $\mathfrak{b}$ is the orthogonal complement of the largest standard subspace of $\mathfrak{r}^v \cap {}^Q\mathfrak{a}$, we can define ${}^+\mathfrak{b}$. (See 2.4 for $\mathfrak{r}^v$.)

1.8. Let $L_{Z_M}{}^*$ be the image of $H_M : Z_M(\mathbf{A}) \to {}^Q\mathfrak{a}_M{}^*$. By a *trigonometric polynomial* on $\iota L_{Z_M} \backslash {}^Q\mathfrak{a}_M(\mathbf{C})$ we shall mean an element of $\mathbf{C}[L_{Z_M}{}^*]$. Note that this last always contains ${}^Q\mathfrak{a}_M{}^*(\mathbf{C})$.

Finally, if $R > 1$, the space $\mathfrak{a}_M(R)$ is defined as in [**5**] III.1.1, as is $\mathscr{H}(R)$.

2. Preliminaries for Eisenstein systems.

2.1. Suppose P, Q are parabolics with $Q \supseteq P$, and let χ be a character of $Z_Q(\mathbf{A})$ prolonging ξ. For much of part I we shall be interested, at least implicitly, in the space ${}^Q\mathscr{C}_0\ (P, K', \chi\delta_Q)$ of functions

$$\phi : N(\mathbf{A})P(F)\backslash G(\mathbf{A}) \to \mathbf{C}$$

satisfying the following conditions

(i) ϕ is right K'-invariant.

(ii) $\phi(zg) = \chi\delta_Q(z)\phi(g)$, $z \in Z_Q(\mathbf{A})$, $g \in G(\mathbf{A})$.

(iii) Let ${}^\psi M^0$ denote the kernel of the homomorphism

$${}^\psi H_M : M(\mathbf{A}) \to \text{Mor}\ ({}^QX_M(\mathbf{R}), \mathbf{R}).$$

Then for each g, the function $m \to \phi(mg)$, $m \in M_Q(\mathbf{A})$ is to have compact support mod ${}^{\psi}M^0$.

(iv) For each g, the function on ${}^{\psi}M^0$ given by $m \to \phi(mg)$ is to lie in the space of cusp forms on ${}^{\psi}M^0$ transforming according to $\chi\delta_Q$.

A word is necessary concerning (iv), since we have not defined what is meant by "cusp forms transforming according to $\chi\delta_Q$" (the point is that $\chi\delta_Q$ is not a character). The space of cusp forms on ${}^{\psi}M^0$ transforming according to $\chi\delta_Q$ is denoted by $\mathscr{L}(\{{}^{\psi}M^0\}, \chi\delta_Q)$ and is defined to be the space of functions

$$\phi : {}^{\psi}M(F)\backslash{}^{\psi}M^0 \to \mathbf{C}$$

satisfying the following conditions

(i)′ For each $z \in Z_Q(\mathbf{A})$, $\phi(zm) = \chi\delta_Q(z)\phi(m)$.

(ii)′ $$\int_{Z_Q(\mathbf{A})M(F)\backslash{}^{\psi}M^0} |\phi(g)|^2\delta_Q^{-2}(g)dg < \infty.$$

(iii)′ If $R = N_RM_R$ is a parabolic subgroup of ${}^{\psi}M = M$, then

$$\int_{N_R(F)\backslash N_R(\mathbf{A})} \phi(nm)dn = 0$$

for each $m \in {}^{\psi}M^0$ (note that $N_R(\mathbf{A}) \subseteq {}^{\psi}M^0$).

Following [**5**] 1.5.7 one can show that the space of cusp forms transforming according to $\chi\delta_Q$, right invariant by $K' \cap {}^{\psi}M^0$ is finite dimensional; we denote this space by $\mathscr{L}(\{{}^{\psi}M^0\}, K', \chi\delta_Q)$.

Thus the definition of

$${}^{Q}\mathscr{C}_0(P, \chi\delta_Q) = \bigcup_{K'} {}^{Q}\mathscr{C}_0(P, K', \chi\delta_Q)$$

is a version tailor made for induction of that of that of [**5**] 2.1.

2.2. If $\phi \in {}^{Q}\mathscr{C}_0(P, \chi\delta_Q)$ then as in [**5**] 2. 2.3-4 the series

$$\phi^\wedge(g) = \sum_{P(F)\backslash Q(F)} \phi(\gamma g) = e\langle\delta_Q, H_Q(g)\rangle \sum \delta_Q^{-1}(\gamma g)\phi(\gamma g)$$

is a finite sum and gives rise to a function on $N_Q(\mathbf{A})Q(F)\backslash G(\mathbf{A})$ which has compact support modulo $Z_Q(\mathbf{A})$.

Let ${}^{Q}\mathscr{L}(\chi\delta_Q)$ be the space of functions

$$N_Q(\mathbf{A})Q(F)\backslash G(\mathbf{A}) \to \mathbf{C}$$

transforming by $\chi\delta_Q$ (i.e., $\phi(zg) = \chi\delta_Q(z)\phi(g)$) and satisfying

$$\int_{N_Q(\mathbf{A})\,Z_Q(\mathbf{A})\,Q(F)\backslash G(\mathbf{A})} |\phi(g)|^2\delta_Q^{-2}(g)dg < \infty.$$

We shall denote the closure in ${}^{Q}\mathscr{L}(\chi\delta_Q)$ of the space of such $\phi^\wedge$ by

${}^{Q}\mathscr{L}(P, \chi\delta_Q)$. Let ${}^{Q}\{P\}$ consist of those elements of $\{P\}$ for which $Q \supseteq P$, and set

$$ {}^{Q}\mathscr{L}(\{P\}, \chi\delta_Q) = \bigoplus_{{}^{Q}\{P\}} {}^{Q}\mathscr{L}(P, \chi\delta_Q). $$

This space is invariant under right translations by $G(\mathbf{A})$: indeed if $\phi \in {}^{Q}\mathscr{C}_0(P, K', \chi\delta_Q)$ then the function $g \to \phi(gg')$ for fixed g' is right invariant by $g'K'g'^{-1}$, hence by $g'K'g'^{-1} \cap K'$, and an easy approximation argument implies right invariance for the closure ${}^{Q}\mathscr{L}(P, \chi\delta_Q)$.

The aim of part I is to describe the decomposition of ${}^{Q}\mathscr{L}(\{P\}, \chi\delta_Q)$ under the action of $G(\mathbf{A})$ in terms of residues of Eisenstein systems. In part II we shall see that this leads to an explicit description of the spectral decomposition for the space $\mathscr{L}(\xi)$.

2.3. The next thing we must do is define the notion of residue. Suppose as in 1.5 that $\mathfrak{r}$ is an affine subspace of ${}^{Q}X_M(\mathbf{C})$, giving rise to an affine subspace of $\iota^{Q}L_{Z_M}\backslash{}^{Q}X_M(\mathbf{C})$; we denote the latter affine space also by $\mathfrak{r}$. Let $\mathfrak{t}$ be an affine hyperplane contained in R. Finally suppose that f is a meromorphic function defined on $\mathfrak{r}$, all of whose singularities lie along hyperplanes of the type described in 1.5. We are going to define a new meromorphic function, $\operatorname{Res}_{\mathfrak{t}} f$, on $\mathfrak{t}$; to do this choose a real unit normal to $\mathfrak{t}$, call it H_0, and let $\Lambda \in \mathfrak{t}$. Choose $\delta > 0$ small enough so that (i) $f(\Lambda + zH_0)$ has no singularities for $0 < |z| < 2\delta$ (say) and (ii)

$$ \Lambda \cdot L_{Z_M} \cap \{\Lambda + zH_0| \quad |z| < 2\delta\} = \{\Lambda\}. $$

We define $\operatorname{Res}_{\mathfrak{t}} f$ by

$$ \operatorname{Res}_{\mathfrak{t}} f(\Lambda) = \frac{1}{\iota} \int_{|z|=\delta} f(\Lambda + zH_0)dz $$
$$ = \delta \,.\, 2\pi \int_0^1 f(\Lambda + \delta e^{2\pi\iota\theta} H_0)e^{2\pi\iota\theta} d\theta. $$

Note that $\mathfrak{t}$ is not supposed to be a singular hyperplane of f, but of course in practice that is the case of interest. The singularities of $\operatorname{Res}_{\mathfrak{t}} f$ then lie on the intersections with $\mathfrak{t}$ of the singular hyperplanes of f. This definition can be carried over to ${}^{Q}D_M(\xi)$.

2.4. Eventually we shall be concerned with residues arising from non-simple poles, and for this one will need to take derivatives; thus to use this definition of residue we must make several more.

First, suppose $\mathfrak{r}$ is an affine subspace in ${}^{Q}X_M(\mathbf{C})$; then $\mathfrak{r}$ can be written in the form $\mathfrak{r} = X(\mathfrak{r}) + \mathfrak{r}^v$, where $\mathfrak{r}^v$ is a complex subspace of ${}^{Q}X_M(\mathbf{C})$ defined by real linear equations of the form $\{H|\alpha^v(H) = 0\}$, and $X(\mathfrak{r})$ is a vector orthogonal to $\mathfrak{r}^v$. The (co) tangent space of $\iota L_{Z_M}\backslash{}^{Q}X_M(\mathbf{C})$ at 0 is just the (co) tangent space of ${}^{Q}X_M(\mathbf{C})$ at 0. Therefore if $S(\mathfrak{r})$ is the

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symmetric algebra on the orthogonal complement of $\mathfrak{r}$ in ${}^QX_M(\mathbf{C})$, then there is a canonical isomorphism $Z \rightarrow D(Z)$ of $S(\mathfrak{r})$ with a subalgebra of the algebra of holomorphic differential operators on ${}^QX_M(\mathbf{C})$ such that

$$D(Y)f(\Lambda) = \left(\frac{df}{dt}\right)(\Lambda + tY)|_{t=0}$$

if Y lies in the orthogonal complement of $\mathfrak{r}^v$, and f is defined and analytic in a neighborhood of $\Lambda \in \iota L_{Z_M}\backslash {}^QX_M(\mathbf{C})$.

2.5. Now suppose that $\Lambda \in {}^QX_M(\mathbf{C})$, and let V, V' be finite dimensional complex vector spaces endowed with a non degenerate sesquilinear pairing

$$\langle\ ,\ \rangle : V \times V' \rightarrow \mathbf{C}.$$

Let Φ be a function analytic in a neighborhood (in ${}^QX_M(\mathbf{C})$) of Λ, with values in V. Let Hom $(S(\mathfrak{r}), V)$ denote the space of linear transformations from $S(\mathfrak{r})$ to V, and define $d\Phi(\Lambda) \in \text{Hom}\ (S(\mathfrak{r}), V)$ by

$$d\Phi(\Lambda)(Y) = D(Y)\Phi(\Lambda).$$

In this way we obtain a map

$$d : \vartheta_{V,\Lambda} \rightarrow \text{Hom}\ (S(\mathfrak{r}), V)$$

where $\vartheta_{V,\Lambda}$ is the space of germs of V-valued analytic functions at Λ.

Since Hom $(S(\mathfrak{r}), V)$ may be viewed as the space of formal power series over the orthogonal complement of $\mathfrak{r}^v$, we can obtain $d\Phi(\Lambda)$ by expanding the function Φ in a Taylor series about Λ.

In particular, if W is another finite dimensional complex vector space, and $T \in \vartheta_{\text{Hom}(V,W),\Lambda}$, we obtain the element $dT(\Lambda)$. If also $F \in$ Hom $(S(\mathfrak{r}), \text{V})$, we define

$$dT(\Lambda)F \in \text{Hom}\ (S(\mathfrak{r}), W)$$

by the composition

$$S(\mathfrak{r}) \xrightarrow{\Delta} S(\mathfrak{r}) \otimes S(\mathfrak{r}) \xrightarrow{dT(\Lambda) \otimes F} \text{Hom}(V, W) \otimes V \xrightarrow{E} W$$

where Δ is the diagonal map and E is evaluation.

2.6. Suppose $\mathfrak{t} \subseteq \mathfrak{r}$ is also an affine subspace, so that $S(\mathfrak{r}) \subseteq S(\mathfrak{t})$; if $S_0(\mathfrak{t})$ denotes the symmetric algebra on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{r}$, then there is a natural isomorphism

$$S_0(\mathfrak{t}) \otimes \text{S}(\mathfrak{r}) \simeq S(\mathfrak{t}).$$

Suppose $T \in \text{Hom}\ (S(\mathfrak{t}), V)$, $X_0 \in S_0(\mathfrak{t})$, and define $X_0 \vee T$ to be that element of Hom $(S(\mathfrak{r}), V)$ given by

$$(X_0 \vee T)(X) = T(X_0 \otimes X).$$

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Moreover suppose that

$$F : U \to \mathrm{Hom}\ (S(\mathfrak{r}), V), \quad U \subseteq X_M(\mathbf{C})$$

is defined in a neighborhood U of Λ, such that for each $Z \in S(\mathfrak{r})$, the function $F(\cdot)(Z)$ is analytic on U (we could describe this by saying that F is *weakly analytic*). Then $dF(\Lambda) \in \mathrm{Hom}\ (S(\mathfrak{t}), V)$ is defined by

$$dF(\Lambda)(X_0 \otimes X) = D(X_0)(F(\Lambda)(X)).$$

In particular suppose that $f = d\Phi(\Lambda)$. Then

$$\begin{aligned} d(d\Phi(\Lambda))(X_0 \otimes X) &= D(X_0)(d\Phi(\Lambda)(X)) \\ &= D(X_0)D(X)\Phi(\Lambda) \\ &= d\Phi(\Lambda)(X_0 \otimes X) \end{aligned}$$

where now $d\Phi(\Lambda)$ is taken as an element of Hom $(S(\mathfrak{t}), V)$. Bearing this in mind, we see that

$$d(d\Phi)(\Lambda) = d\Phi(\Lambda).$$

2.7. *Adjoints*. Let Y be an element of the orthogonal complement of $\mathfrak{r}^v$; The map $Y \to Y^* = -\bar{Y}$ then gives rise to a conjugate linear isomorphism $* : S(\mathfrak{r}) \to S(\mathfrak{r})$. Define a pairing

$$(\ ,\ )_S : S(\mathfrak{r}) \otimes V' \times \mathrm{Hom}\ (S(\mathfrak{r}), V) \to \mathbf{C}$$

by

$$(Y \otimes u', F)_S = (u', F(Y^*))_{V \times V'}$$

where $(\ ,\ )_{V\times V'}$ denotes the pairing given a priori on $V \times V'$. The pairing $(\ ,\ )_S$ is then sesquilinear.

When the context is clear we shall omit the subscript "S" (or "$V \times V'$").

Now suppose f is a functional on $S(\mathfrak{r}) \otimes V'$, then one may always find $F \in \mathrm{Hom}\ (S(\mathfrak{r}), V)$ so that $f(w) = (w, F)_S$ for all $w \in S(\mathfrak{r}) \otimes V'$: this is a simple matter of applying the definitions. Suppose however that we are given a functional f on Hom $(S(\mathfrak{r}), V)$, then for there to be a $T \in S(\mathfrak{r}) \otimes V'$ such that

$$f(F) = \overline{(T, F)_S}$$

(all F), the functional f must be restricted.

Indeed if $F \in \mathrm{Hom}\ (S(\mathfrak{r}), V)$ define the *order* of F, $O(F)$, to be the lowest degree of the non zero terms occurring in the power series expansion of F. Now if

$$T : \mathrm{Hom}\ (S(\mathfrak{r}), V) \to W$$

is a linear map to the finite dimensional space W we say that T has finite order n if T vanishes on all F with $O(F) > n$, but $T(F) \neq 0$ for some F of order n.

Returning to the functional f above, and taking $W = \mathbf{C}$ we see that there is a unique $T \in S(\mathfrak{r}) \otimes V'$ with $f(F) = \overline{(T, F)_S}$ if, and only if, f has finite order n, for some n.

We shall apply this to the following situation. Suppose that

$$A : \text{Hom}\,(S(\mathfrak{r}), V) \to W$$

is a linear transformation of finite order and suppose that

$$(\ ,\ )_W : W \times W' \to C$$

is also a sesquilinear pairing. Fixing $w' \in W'$ for the moment, we find that

$$F \to (AF, w')_W$$

is a functional on Hom $(S(\mathfrak{r}), V)$ of finite order, hence there is a unique $T_{w'} \in S(\mathfrak{r}) \otimes V'$ such that

$$(AF, w')_W = \overline{(T_{w'}, F)} \text{ for all } F.$$

The map $w' \to T_{w'}$ is a linear transformation

$$A^* : W' \to S(\mathfrak{r}) \otimes V'.$$

In particular, suppose that $\mathfrak{t}$ is an affine subspace of $X_{M_2}(\mathbf{C})$ where P_2 is another parabolic, that $W = S(\mathfrak{t}) \otimes V_2'$, $W' = \text{Hom}\,(S(\mathfrak{t}), V_2)$ where V_2, V_2' are analogous to V, V' then

$$A^* : \text{Hom}\,(S(\mathfrak{t}), V_2) \to S(\mathfrak{r}) \otimes V'$$

is a linear transformation, called the *adjoint* of A.

Since for each $F \in \text{Hom}\,(S(\mathfrak{r}), V)$ the linear functional

$$F' \to (A^*F', F)_V = (AF, F')_W$$

is of finite order, we see that A^* is itself of finite order.

3. Eisenstein systems.

3.1. Fix a class of associated parabolic subgroups $\{P\}$, whose elements we denote by $P^1, \ldots, P^r$. Suppose R is some parabolic subgroup such that $R \supset P^i$, $R \supset P^j$ say. We write ${}^RW({}^G\mathfrak{a}^i, {}^G\mathfrak{a}^j)$ for the set of distinct linear transformations from ${}^G\mathfrak{a}^i$ to ${}^G\mathfrak{a}^j$ fixing $\mathfrak{a}_R$ pointwise; alternatively one can view it as the set of linear transformations ${}^R\mathfrak{a}^i$ to ${}^R\mathfrak{a}^j$, or again as the set of distinct linear transformations $M^i \to M^j$ arising from conjugation by $M_R(F)$. In any event it is a subset of $W({}^G\mathfrak{a}^i, {}^G\mathfrak{a}^j)$. We shall also write $W({}^R\mathfrak{a}^i, {}^R\mathfrak{a}^j)$ for ${}^RW({}^G\mathfrak{a}^i, {}^G\mathfrak{a}^j)$.

Suppose $\mathfrak{r}$ is an affine subspace of ${}^G\mathfrak{a}^i$, then we shall let ${}^RW^{(j)}(\mathfrak{r})$ denote the set of distinct linear transformations from $\mathfrak{r}$ to ${}^G\mathfrak{a}^j(\mathbf{C})$ obtained by restricting the elements of ${}^RW(\mathfrak{a}^i, \mathfrak{a}^j)$ to $\mathfrak{r} \subseteq \mathfrak{a}^i(\mathbf{C})$.

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Given $s \in {}^R W^j(\mathfrak{r})$, we define

$$\mathfrak{r}_s = \{-s\bar{\Lambda} | \Lambda \in \mathfrak{r}\}.$$

Thus $\mathfrak{r}_s$ is an affine subspace of $\mathfrak{a}^j(\mathbf{C})$.

3.2. In the previous section we defined ${}^Q\mathscr{C}_0(\{P\}, \chi\delta_Q)$; in this and subsequent sections we shall also be considering $\mathscr{C}_0(\{P\}, \omega\delta_P)$.

We shall always suppose that if $Q \supseteq P$ then $\omega\delta_P$ restricts to $\chi\delta_Q$ on $Z_Q(\mathbf{A})$. Moreover, if $P = P^i$ we frequently write V^i for $\mathscr{C}_0(P, \omega\delta_P)$ or $\mathscr{C}_0(P, K', \omega\delta_P)$.

3.3. With these observations in hand we now proceed to make the central definition of this paper.

Fix $\mathfrak{r} \subseteq {}^G\mathfrak{a}_M(\mathbf{C}) = {}^G\mathfrak{a}^i(\mathbf{C})$ as before. We suppose that for each $Q \supseteq P$ with $\mathfrak{r}^v \supseteq \mathfrak{a}_Q(\mathbf{C})$ there is given a function $E(g, F, \Lambda)$ transforming by $\chi\delta_Q$ on $Z_Q(\mathbf{A})$ i.e.,

$$E(zg, F, \Lambda) = \chi\delta_Q(z)E(g, F, \Lambda),$$
$$E(.\ ,\ .\ ,\ .) : N_Q(\mathbf{A})Q(F)\backslash G(\mathbf{A}) \times \mathrm{Hom}\ (S(\mathfrak{r}), V^i) \times {}^Q\mathfrak{r} \rightarrow \mathbf{C}.$$

The space $\mathfrak{r}$ together with the collection of functions $E(g, F, \Lambda)$ is called an *Eisenstein system belonging to* $\mathfrak{r}$ if for some $Q \supseteq P$, some g, some F the function $E(g, F, \Lambda) \not\equiv 0$, and if the conditions (i)-(iii) below are satisfied.

(i) Given $Q \supseteq P$, $g \in G(\mathbf{A})$, F, the function $E(g, F, .)$ is rational, and is a function on the image of ${}^Q\mathfrak{r}$ in $\iota L_{Z_M}\backslash{}^Q\mathfrak{a}^i$: here L_{Z_M} denotes what should properly be written ${}^Q L_{Z_M}$. As a function of g it is right invariant by some open compact $K' \subseteq K$, and it is linear in the argument F. Moreover $E(.\ ,\ .\ ,\ .)$ has finite order in the sense that for each g, Λ, the functional $E(g, ., \Lambda)$ is of order n, n an integer independent of g, Λ. Finally we can find a trigonometric polynomial $p(\Lambda) \in \mathbf{C}[{}^{*Q}L_{Z_M}]$ such that for all g, F, the function $p(\Lambda)E(g, F, \Lambda)$ is analytic in Λ.

(ii) Suppose that $P' = P^j$ with $Q \supseteq P'$ as well. Then for each $s \in {}^Q W^j(\mathfrak{r})$ there is a function $N(s, \Lambda)$ on $\mathfrak{r}$ with values in

$$\mathrm{Hom}\ (\mathrm{Hom}\ (S(\mathfrak{r}), V^i), S(\mathfrak{r}) \otimes V^j)$$
$$(V^j = \mathscr{C}_0(P', K', \omega^{s^{-1}}\delta_{P'}) \text{ or } \mathscr{C}_0(P', \omega^{s^{-1}}\delta_{P'}))$$

such that for each F in $\mathrm{Hom}\ (S(\mathfrak{r}), V^i)$ and each F' in $S(\mathfrak{r}_s) \otimes V^j$, the function

$$(N(s, \Lambda)F, F') \qquad \text{(c.f. 2.7)}$$

is rational in Λ on ${}^Q\mathfrak{r}$. Furthermore there is a positive integer n so that

$$(N(s, \Lambda)F, F') \equiv 0$$

if $O(F)$ or $O(F') > n$.

Define as usual

$$E^{P'}(g, F, \Lambda) = \int_{\psi_{N'(F)}\backslash \psi_{N'(\mathbf{A})}} E(ng, F, \Lambda)dn$$

where ${}^{\psi}P' = P' \cap M_Q$. Then we require that

$$E^{P'}(g, F, \Lambda) = \sum_{s \in {}^{Q}W^{j}(\mathfrak{r})} e(s\Lambda, \overline{H_{M'}(g)})N(s, \Lambda)F(g)$$

where $\overline{H_{M'}(g)}$ is defined as in [**5**] 2.1.3, and $N(s, \Lambda)F(g)$ is interpreted as a function on $G(\mathbf{A})$, c.f. 2.7.

On the other hand if $P' \notin \{P\}$, but $P' \subseteq Q$ then the cuspidal component of $E^{P'}(g, F, \Lambda)$ is zero i.e., $E^{P'} \backsim 0$ c.f. [**5**] 2.4.1.

(iii) Given $Q \supseteq R \supseteq P$ we have the pairs $Q \supseteq P$ and $R \supseteq P$ with corresponding functions $E(g, F, \Lambda)$ and $E_1(g, F, \Lambda)$ respectively, with $g \in N_Q(\mathbf{A})Q(F)\backslash G(\mathbf{A})$ in the first case, and $g \in N_R(\mathbf{A})R(F)\backslash G(\mathbf{A})$ in the second case.

According to [**5**] 4.2 there is a non empty connected open set $\mathscr{D}$ such that the series

$$(3.3.1) \qquad \sum_{\gamma \in R(F)\backslash Q(F)} E_1(\gamma g, F, \Lambda)$$

converges uniformly on compact subsets $\omega \subset \mathscr{D}$, with $\Lambda \in \omega$ (a fortiori on compact subsets of $G(\mathbf{A})$ since E_1 is, say, K'-invariant).

Then we require that $E(g, F, \Lambda)$ be equal to (3.3.1) on $\mathscr{D}$, and that moreover if $Q \supseteq R \supseteq P' \in \{P\}$, $P' = P^j$, and $s \in {}^{R}W^{j}(\mathfrak{r})$ then if $\Lambda \in \mathscr{D}$, $\Lambda' = \Lambda + \mu$ with μ fixed by ${}^{R}W^{j}(\mathfrak{r})$, then $N(s, \Lambda') = N_1(s, \Lambda)$.

3.5. *Remarks.* (a) Axiom (iii) is a transitivity condition which is the basis for induction arguments c.f. [**5**] Sections 2, 3, particularly the argument for analytic continuation.

(b) We point out that in digesting this set of conditions it might be helpful to remember that ${}^{Q}D_M(\chi\delta_Q)$ can be viewed equally well as a manifold of quasicharacters

$${}^{Q}D_{\psi_M}(\chi\delta_Q), {}^{\psi}P = P \cap M_Q.$$

Indeed the reader should bear in mind the fibration diagram

$$\begin{array}{ccc} {}^{Q}D_M(\lambda) & \rightarrow & D_M(\xi) \\ \downarrow & & \downarrow \\ \{\lambda\} & \rightarrow & D_Q(\xi) \end{array}$$

3.6. The definition above is a "trivialized" form of a vector bundle definition which we shall outline in a moment. The first definition is useful for many concrete arguments; the second should be borne in mind.

Firstly it is more conceptual, and secondly it is inescapable especially in Sections 6–7, since one ultimately relies upon Fourier transform arguments. Moreover for function fields, there is no Langlands decomposition $R(\mathbf{A}) = N_R(\mathbf{A})A_RM^0$ as in the number field case, and one must resort to fibration diagrams as above.

To give the alternate definition, we first recast some of our earlier definitions of associated function spaces in terms of vector bundles on manifolds, and consider affine spaces in these manifolds.

3.7. First, the spaces $\mathscr{C}_0(P, \omega\delta_P)$, ${}^Q\mathscr{C}_0(P, \chi\delta_Q)$. If $\zeta \in D_M(\xi) = {}^GD_M(\xi)$, then the space $\mathscr{C}_0(P, \zeta)$ has been defined earlier in [**5**] II.1. It was shown there that the collection $\mathscr{C}_0(P, \zeta)$ fits together to form an analytic vector bundle on $D_M(\xi)$. This bundle trivializes along each connected component, and then a holomorphic cross section can be regarded as a holomorphic function

$$\Phi_\zeta : \iota L_{Z_M}\backslash \mathfrak{a}_M(\mathbf{C}) \rightarrow \mathscr{C}_0(P, \omega\delta_P).$$

The space ${}^Q\mathscr{C}_0(P, \chi\delta_Q)$ can also be recast in this vein as well: replace $\chi\delta_Q$ by a quasicharacter $\lambda \in D_Q(\xi)$, then ${}^Q\mathscr{C}_0(P, \lambda)$ is defined as before, the only difference is that in 2.1, one replaces "$\chi\delta_Q$" by λ, and in 2.1 (ii)′, one replaces "δ_Q^{-2}" by "$|\lambda|^{-2}$". Similarly one has a definition of ${}^Q\mathscr{L}(\lambda)$.

If we fix $\lambda \in D_Q(\xi)$ for the moment, we then have ${}^QD_M(\lambda)$, and if $\zeta \in {}^QD_M(\lambda)$ we have $\mathscr{C}_0(P, \zeta)$; these spaces give rise to a vector bundle on ${}^QD_M(\lambda)$.

3.8. So far we have considered affine spaces $\mathfrak{r}$ in $\mathfrak{a}_M(\mathbf{C})$; we now consider them in $D_M(\xi)$, ${}^QD_M(\lambda)$ etc. For this, fix component representatives in ${}^QD_M(\lambda)$ of the form $\omega\nu_0$, where ω is a character, and ν_0 is a fixed real quasicharacter prolonging $|\lambda|$. Each component of ${}^QD_M(\lambda)$ is then holomorphically equivalent to $\iota^QL_{Z_M}\backslash{}^Q\mathfrak{a}_M(\mathbf{C})$ and in this way we can view ${}^Q\mathfrak{r}$ in ${}^QD_M(\lambda)$ via

$$(3.8.1)\quad {}^Q\mathfrak{a}_M(\mathbf{C}) \rightarrow \iota^QL_{Z_M}\backslash{}^Q\mathfrak{a}_M(\mathbf{C})$$

and

$$\coprod_\omega \iota^QL_{Z_M}\backslash{}^Q\mathfrak{a}_M(\mathbf{C}) \rightarrow {}^QD_M(\lambda).$$

Similarly we obtain an $\mathfrak{r}$ in $D_M(\xi)$.

Observe in passing that each component then has a universal covering corresponding to (3.8.1). This will figure surreptitiously in analytic continuation arguments later.

3.9. Now form the collection $\{\text{Hom}\,(S(\mathfrak{r}), V_\zeta{}^i)\}$ as ζ runs through ${}^QD_M(\lambda)$. Here $S(\mathfrak{r})$ is just as before, and $V_\zeta{}^i = \mathscr{C}_0(P, \zeta)$ or $\mathscr{C}_0(P, K', \zeta)$ (as the case may be), if $P = P^i$. This is a bundle in the sense that

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Hom $(S(\mathfrak{r}), V^i)$ $(V^i = \mathscr{C}_0(P, \omega\nu_0))$ translates to Hom $(S(\mathfrak{r}), V_\zeta{}^i)$ by the translation map

$$\Phi \to T_\zeta \Phi = e < \zeta, \bar{H}_M(\cdot) > \Phi.$$

Fix $\mathfrak{r} \subseteq {}^G\mathfrak{a}_M(\mathbf{C}) = {}^G\mathfrak{a}(\mathbf{C})$. We suppose that to each $Q \supseteq P$ with $\mathfrak{r} \supseteq \mathfrak{a}_Q(\mathbf{C})$ there is given a function $E(g, F, \zeta)$ transforming by λ on $Z_Q(\mathbf{A})$

$$E(. , . , \zeta) : N_Q(\mathbf{A})Q(F)\backslash G(\mathbf{A}) \times \text{Hom } (S(\mathfrak{r}), V_\zeta) \to \mathbf{C}.$$

The space $\mathfrak{r}$ and the collection $\{E(g, F_\zeta, \zeta)\}$ is called an Eisenstein system belonging to $\mathfrak{r}$ if for some $Q \supseteq P$, some F_ζ the function $E(. , F_\zeta) \not\equiv 0$, and if the analogues of (i)–(iii), which we denote by (i)′–(iii)′, are satisfied. Moreover, suppose $\omega\nu_0$ is chosen, then we ask that

$$E(g, T_\zeta F, \omega\nu_0\zeta) = E(g, F, \zeta), \quad \zeta \in {}^Q\mathfrak{r}.$$

On the left we have the new Eisenstein system, on the right we have the old one.

The conditions (i)′–(iii)′ are by and large the same as before up to a trivial rephrasing. Of course in (i)′ the definition of rationality is taken on each component, as is the notion of finite order. We do ask that the trigonometric polynomial $p(\Lambda)$ suffice for each component.

In (ii)′ we replace $N(s, \Lambda)$ by $M(s, \zeta)$ subject to

$$E^{P'}(g, F_\zeta, \zeta) = \sum_{s \in {}^Q W^j(\mathfrak{r})} M(s, \zeta) F_\zeta(g).$$

The relation between $N(s, \Lambda)$ and $M(s, \zeta)$ is simply that with $\omega\nu_0$ chosen,

$$N(s, \omega\nu_0\zeta) = T_{-\zeta'} M(s, \zeta) T_\zeta, \quad \zeta \in {}^Q\mathfrak{r}.$$

The transitivity condition (iii) is more or less as before: if $\zeta | Z_R(\mathbf{A}) = \mu$, then in one case we have a function on cross sections of a bundle on ${}^R D_M(\mu)$, and in the other a function on cross sections on ${}^Q D_M(\lambda) \subseteq {}^R D_M(\mu)$. The transitivity condition asks that the former function "restricts along the fibres" appropriately to the latter function.

3.10. Observe that the notions of Sections 2.3–2.7 carry over to the vector bundle framework: they are local concepts.

3.11. *Construction of Eisenstein systems.* All Eisenstein systems of interest to us will be constructed by an inductive process, by repeatedly taking residues. We shall first consider the Eisenstein system at the basis of the induction, and then show how taking residues of a given Eisenstein system gives rise to another Eisenstein system.

Example 3.11.1. (The standard example.) Suppose that $\mathfrak{r} = {}^G\mathfrak{a}(\mathbf{C}) = {}^G\mathfrak{a}(\mathbf{C})$. Then $S(\mathfrak{r})$ is none other than $\mathbf{C}$, and $F \rightarrow F(1)$ defines an isomorphism

$$\mathrm{Hom}\,(S(\mathfrak{r}), \mathscr{C}_0(P, K'\omega\delta_P)) \rightarrow \mathscr{C}_0(P, K', \omega\delta_P).$$

Consider the function $\Phi \in \mathscr{C}_0(P, K', \omega\delta_P)$. On $M_Q(\mathbf{A})$ it enjoys the following properties, if we set ${}^\psi P = P \cap M_Q$,

(i) It is right invariant by K'.

(ii) $\Phi(zg) = \omega\delta_P(z)\Phi(g)$ for $z \in Z_{\psi_M}(\mathbf{A}) = Z_M(\mathbf{A})$.

(iii) Let ${}^\psi M^0 = \ker\,(M(\mathbf{A}) \rightarrow {}^Q\mathfrak{a}^*)$. Then the space of functions on ${}^\psi M^0$, $m \rightarrow \Phi(mg)$ is a finite dimensional subspace of $\mathscr{L}(\{{}^\psi M_0\}, \chi\delta_Q)$ where $\chi = \omega|Z_Q(\mathbf{A})$. Indeed, let $b_1, \ldots, b_r$ be a set of coset representatives for K/K', then this space of functions is a subspace of

$$\mathscr{L}(\{{}^\psi M^0\}, K'', \chi\delta_Q), K'' = \bigcap_{1 \leqq i \leqq r} b_i K' b_i^{-1}.$$

It follows that $\mathscr{C}_0(P, K', \omega\delta_P)$ can be viewed as a similar space of functions on $M_Q(\mathbf{A})$ with respect to the parabolic subgroup ${}^\psi P = P \cap M_Q$. Let

$$\zeta \in \iota^Q L_{Z_M} \backslash {}^Q X_M(\mathbf{C}),$$

then the series

$$E(g, \Phi, \zeta) = \sum_{P(F)\backslash Q(F)} e\langle \bar{H}_M(\gamma g), \zeta\rangle \Phi(\gamma g)$$

which may be written as

$$e\langle H_Q(g), \delta_Q\rangle \sum_{{}^\psi P(F)\backslash M_Q(F)} e\langle \bar{H}_M(\gamma g), \zeta\rangle \Phi'(\gamma g)$$

with

$$\Phi'(g) = e\langle H_Q(g), -\delta_Q\rangle \Phi(g)$$

will converge absolutely for $\mathrm{Re}\,\zeta \in {}^Q C_P + {}^Q\delta_P$, as follows immediately from applying [**5**] 2.2.2. The results of [**5**] imply that this function may be analytically continued to a rational function on $\iota^Q L_{Z_M} \backslash {}^Q X_M(\mathbf{C})$, as can the associated intertwining operators $N(w, \zeta)$. Consequently if we put $\Phi = F(1)$ and define

$$E(g, F, \zeta) = E(g, \Phi, \zeta)$$

we see from the results of [**5**] 4 that this collection of functions satisfies the conditions for an Eisenstein system.

Example 3.11.2 (residual Eisenstein systems). Suppose that $E(.\,,.\,,.)$ is an Eisenstein system belonging to $\mathfrak{r}$, with $\mathfrak{r}$ an affine subspace of ${}^G\mathfrak{a}_M(\mathbf{C})$. Let $\mathfrak{t}$ be an affine hyperplane of $\mathfrak{r}$; we shall define an Eisenstein system belonging to $\mathfrak{t} \subseteq {}^G\mathfrak{a}(\mathbf{C})$, $\mathfrak{a} = \mathfrak{a}_M$.

First let $\Phi \in \vartheta_{\Lambda,V}$ with $V = \mathscr{C}_0(P, K', \omega\delta_P)$. From Section 2.3, $\mathrm{Res}_{Q\mathfrak{t}}\, E(g, d\Phi(\cdot), .)$ is defined in a neighbourhood of Λ in ${}^Q\mathfrak{t}$.

Let

$$\Phi(\Lambda + z\Lambda_0) = \sum_{l=0}^{\infty} \frac{z^l}{l!} d(D(\Lambda_0^{\,l}))\Phi(\Lambda)$$

where Λ_0 is a real unit normal to $\mathfrak{t}$ (thus $\Lambda_0^{\,l} \in S(\mathfrak{t})$, c.f. 2.3).

Suppose that if $F \in \mathrm{Hom}\,(S(\mathfrak{r}), V)$, then

$$E(g, F, \Lambda + z\Lambda_0) = \sum_{m=-\infty}^{\infty} z^m E_m(g, F, \Lambda)$$

where only a finite number of non zero terms in negative m actually occur.

Then

$$\mathrm{Res}_{Q\mathfrak{t}}\, E(g, d\Phi(\Lambda), \Lambda) = \sum_{l+m=-1} \frac{1}{l!} E_m(g, d(D(\Lambda_0^{\,l})\Phi(\Lambda), \Lambda)$$

provided that in the first definition of $\mathrm{Res}_{Q\mathfrak{t}}$ we choose the real unit normal Λ_0 to $\mathfrak{t}$ employed above.

From this we define a meromorphic function on

$$N_Q(\mathbf{A})Q(F)\backslash G(\mathbf{A}) \times \mathrm{Hom}\,(S(\mathfrak{t}), V) \times {}^Q\mathfrak{t}$$

which we denote by $\mathrm{Res}_{Q\mathfrak{t}}\, \mathrm{E}(. , . , .)$, as follows

$$\mathrm{Res}_{Q\mathfrak{t}}\, E(g, F, \Lambda) = \sum_{l+m=-1} \frac{1}{l!} E_m(g, \Lambda_0^{\,l} \vee F, \Lambda).$$

Let us show that this collection of functions gives rise to an Eisenstein system belonging to $\mathfrak{t}$, when $\mathfrak{t}$ is a singular hyperplane for some $E(. , . , .)$ with respect to a given $Q \supseteq P$ say (to ensure non triviality).

Condition (i) is automatic as it is inherited from $E(. , . , .)$. Condition (iii) can also be verified directly, once we have defined the functions $N(s, \Lambda)$, and verified condition (ii).

Suppose then that $P^{(j)} = P' \in \{P\}$ with $Q \supseteq P'$. From the definition of $\mathrm{Res}_{Q\mathfrak{t}}$, we have

$$\int_{{}^{\psi}N'(F)\backslash{}^{\psi}N'(\mathbf{A})} \mathrm{Res}_{Q\mathfrak{t}}\, E(ng, F, \Lambda) = \sum_{l+m=-1} \frac{1}{l!} \int E_m(g, \Lambda_0^{\,l} \vee F, \Lambda)$$

where the second integral is also taken over ${}^{\psi}N'(F)\backslash{}^{\psi}N'(\mathbf{A})$. Moreover

$$\int_{{}^{\psi}N'(F){}^{\psi}\backslash N'(\mathbf{A})} E(ng, \Lambda_0^l v F, \Lambda + z\Lambda_0) = \sum_{m=-\infty}^{\infty} z^m \int E_m(ng, \Lambda_0^{\,l} \vee F, \Lambda).$$

The expression on the left is, by definition, equal to

$$\sum_{s \in W^{(j)}(\mathfrak{r})} e\langle s\Lambda + zs\Lambda_0, \bar{H}_{M'}(g)\rangle N(s, \Lambda + z\Lambda_0)(\Lambda_0^{\,l} \vee F)(g).$$

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Set

$$N(s, \Lambda + z\Lambda_0) = \sum_{k=-\infty}^{\infty} z^k N_k(s, \Lambda)$$

and write

$$e\langle s\Lambda + zs\Lambda_0, \bar{H}_{M'}(g)\rangle = e\langle s\Lambda, \bar{H}_{M'}(g)\rangle \left(\sum_{h=0}^{\infty} \frac{1}{h!} (z \log q\langle s\Lambda_0, \overline{H_{M'}(g)}\rangle)^h\right).$$

On substituting and equating coefficients of z^{-1}, one finds that

$$\int_{\psi_{N'(F)}\backslash\psi_{N'(\mathbf{A})}} E_m(ng, \Lambda_0{}^l vF, \Lambda)dn$$

is equal to the sum over $s \in {}^QW^{(j)}(\mathfrak{r})$ of $e\langle s\Lambda, \bar{H}_{M'}(g)\rangle$ times

$$\sum_{h+k=m} \frac{1}{h!} (\log q)^h \langle s\Lambda_0, \bar{H}_{M'}(g)\rangle^h N_k(s, \Lambda)(\Lambda_0{}^l \vee F)(g).$$

Let $t \in {}^QW^{(j)}(\mathfrak{t})$, then we define $\mathrm{Res}_{Q_\mathfrak{t}} N(t, \Lambda)$ to be the linear transformation which takes

$$F \in \mathrm{Hom}\,(S(\mathfrak{t}), V)$$

to

$$\mathrm{Res}_{Q_\mathfrak{t}} N(t, \Lambda)F \in S(\mathfrak{t}_l) \otimes V$$

where the latter element is defined to be the sum over those $s \in {}^QW^{(j)}(\mathfrak{r})$ whose restriction to $\mathfrak{t}$ is equal to t, of

$$\sum_{h+l+k=-1} \frac{1}{h!l!} (s\Lambda_0)^h \otimes N_k(s, \Lambda)(\Lambda_0{}^l \vee F).$$

Applying the definitions we find immediately that

$$\int_{\psi_{N'(F)}\backslash\psi_{N'(\mathbf{A})}} \mathrm{Res}_{Q\mathfrak{t}}\, E(ng, F, \Lambda)dn$$

is equal to

$$\sum_{t\in {}^QW^{(j)}(\mathfrak{t})} e\langle t\Lambda, \bar{H}_{M'}(g)\rangle(\mathrm{Res}_{Q_\mathfrak{t}} N(t, \Lambda)F)(g)$$

if $P' = P^{(j)} \in {}^Q\{P\}$.

On the other hand if $P' \notin {}^Q\{P\}$, we find from condition (ii) satisfied by the original Eisenstein system $\{E(\cdot, \cdot, \cdot)\}$ that

$$\int_{\psi_{N'(F)}\backslash\psi_{N'(\mathbf{A})}} \mathrm{Res}_{Q\mathfrak{t}}\, E(ng, F, \Lambda)dn \equiv 0.$$

4. Properties of Eisenstein systems. In order to use Eisenstein systems effectively, we shall have to develop relevant properties of those Eisenstein systems which will ultimately be of interest to us. Naturally, one can expect that some of these results will be of a technical nature.

The first technical lemma is related to a square integrability criterion for automorphic forms; the latter result does not belong in this paper and is proved elsewhere. The next lemma tells us what the adjoint (as defined in 2.7) of the operator $N(s, \Lambda)$ is, at least in those cases of interest to us. The main result proved in this section is the construction of a closed invariant subspace of ${}^{Q}\mathscr{L}(P, \chi\delta_Q)$ associated to $\mathfrak{r}$ and each $\{E(.\,, .\,, .)\}$ belonging to $\mathfrak{r}$; this plays a basic role in Section 5.

4.1. Suppose that $\mathfrak{r}$ is affine in $\mathfrak{a}_{\mathbf{C}}{}^{(i)}$, $\mathfrak{t}$ is affine in $\mathfrak{a}_{\mathbf{C}}{}^{(j)}$, that $P^{(i)}$, $P^{(j)} \in {}^{Q}\{P\}$ and ${}^{Q}\mathfrak{a}_{\mathbf{C}}$ lies in $\mathfrak{r}^{v}$, and $\mathfrak{t}^{v}$. Define ${}^{Q}W(\mathfrak{r}, \mathfrak{t}) \subseteq {}^{Q}W^{(j)}(\mathfrak{r})$ by

$${}^{Q}W(\mathfrak{r}, \mathfrak{t}) = \{s \in {}^{Q}W^{(j)}(\mathfrak{r}) | \mathfrak{r}_s = \mathfrak{t}\}.$$

Note that if

$$\mathfrak{r}' = \text{span } \langle X(\mathfrak{r}), \mathfrak{r}^{v} \rangle \quad \text{and}$$
$$\bar{\mathfrak{r}}' = \{ \Lambda | \bar{\Lambda} \in \mathfrak{r}' \}$$

then the above set may also be viewed as a set of linear transformations from $\mathfrak{r}'$ to $\bar{\mathfrak{t}}'$ or from $\bar{\mathfrak{r}}'$ to $\mathfrak{t}'$.

Suppose that for each $1 \leqq i \leqq r$ there is given a finite collection $S^{(i)}$ of distinct affine subspaces of a given dimension m. We put

$$S = \bigcup_i S^{(i)}$$

and assume that to each $\mathfrak{r} \in S$ there is given an Eisenstein system belonging to $\mathfrak{r}$. Since each $\mathfrak{r}$ is defined by equations of the form $\alpha^{v}(\Lambda) = C_\alpha$ one can show that for each such $\mathfrak{r}$ there is an element (unique) $s_0 \in {}^{Q}W(\mathfrak{r}, \mathfrak{r})$ fixing $\mathfrak{r}^{v}$ pointwise. If Q is a parabolic we define

$${}^{Q}S^{(i)} = \{\mathfrak{r} \in S^{(i)} | \mathfrak{a}_Q(\mathbf{C}) \subseteq \mathfrak{r}^{v}\}$$
$${}^{Q}S = \bigcup_{1 \leqq i \leqq r} {}^{Q}S^{(i)}.$$

If ${}^{Q}W(\mathfrak{r}, \mathfrak{t}) \neq \emptyset$, then we say that $\mathfrak{r}$ and $\mathfrak{t}$ are equivalent: the remark above says ${}^{Q}W(\mathfrak{r}, \mathfrak{r}) \neq \emptyset$. This is necessary for the proof of 4.2 below.

4.2. With S, ${}^{Q}S$ as above, let $\mathfrak{r} \in S^{(i)}$, and let $\mathfrak{b}$ be the orthogonal complement of the standard subspace of maximal dimension which is contained in $\mathfrak{r}^{v} \cap \mathfrak{a}^{(i)}$. We shall say that $\mathfrak{r}$ is *geometric* if $\text{Re } X(\mathfrak{r}) \in {}^{+}\mathfrak{b}$ (i.e., is a sum of positive multiples of positive roots). If $\{(E.\,, .\,, .)\}$ belongs to $\mathfrak{r}$ we shall say that $\{E(.\,, .\,, .)\}$ is *complete* if $\mathfrak{r}$ is geometric and if $N(s, \Lambda) \equiv 0$ unless $s \in {}^{Q}W(\mathfrak{r}, \mathfrak{t})$ for some $\mathfrak{t}$ in ${}^{Q}S^{(j)}$ (note that this condition is not automatically satisfied by the definition of Eisenstein system).

All Eisenstein systems of interest to us will be shown later to be complete.

PROPOSITION. *Suppose that* $\mathfrak{r}(\{E(.,.,.)\})$ *is geometric (complete), for each* $\mathfrak{r} \in S$. *Then*

(i) *each* $X(\mathfrak{r})$ *is real, and*

(ii) *for any* $\mathfrak{a}_Q(\mathbf{C})$ *associated to* Q, *each equivalence class in* ${}^QS$ *contains an element* $\mathfrak{r}$ *such that* $\mathfrak{r}^v$ *is standard.*

To begin, observe that by definition we may write if $P' \in \{P\}$ with $Q \supseteq P, P'$, and ${}^\psi P' = P' \cap M_Q$,

$$(4.2.1) \qquad E^{P'}(g, F, \Lambda) = \int_{{}^\psi N'(F) \backslash {}^\psi N'(\mathbf{A})} E(ng, F, \Lambda) dn$$
$$= \sum_i e\langle \Lambda_i, H_{M'}(g)\rangle \Psi_i(g)$$

where the Λ_i are distinct, and

$$0 \not\equiv \Psi_i \in \mathbf{C}[{}^QL^{*\prime}] \otimes \mathscr{C}_0(P', \omega'\delta_{P'}).$$

Here $\mathbf{C}[{}^QL^{*\prime}]$ denotes the group algebra of ${}^QL^*_{Z_{M'}}$.

The proof of 4.2 relies upon the following two results which follow from more general assertions concerning automorphic forms.

4.3. Suppose that $E(g, F, \Lambda) \in {}^Q\mathscr{L}(\{P\}, \chi\delta_Q)$. Then for all P' as in (4.2.1) above, the Λ_i appearing in (4.2.1) are all real.

4.4 Suppose that each Λ_i appearing in (4.2.1) satisfies

$$-\text{Re } \Lambda_i \in {}^QC^*_{P'}$$

(for each P' as above). Then $E(g, F, \Lambda)$ as a function of g is an element of ${}^Q\mathscr{L}(\{P\}, \chi\delta_Q)$.

Remark. A special case of 4.4 is proved in [**5**] 2.4.

4.5. *Proof of* 4.2. Suppose C is an equivalence class in ${}^QS$, and let $\mathfrak{r} \in C$ be such that $\mathfrak{r}^v$ contains a standard subspace of maximal dimension. This standard subspace corresponds to a parabolic $Q \supseteq R \supseteq P$. If we work with R instead of Q then we may as well suppose that $\mathfrak{a}_Q$ is the subspace of maximal dimension: this may change C (by making it smaller), but that does not matter. As usual ${}^Q\mathfrak{r}$ denotes the projection of $\mathfrak{r}$ on ${}^Q\mathfrak{a}_{\mathbf{C}}$. We pause for a definition.

4.5.1. *Definition.* $\Lambda \in {}^Q\mathfrak{r}$ is *general* if (i) Λ lies on no singular hyperplane of those $E(.,..)$ defined on ${}^Q\mathfrak{r}$, and

(ii) if $s, t \in {}^QW^{(j)}(\mathfrak{r})$ with $s\Lambda = t\Lambda$ then $s = t$.

Returning to the proof of 4.2, we observe that there is at least one Q corresponding to $\mathfrak{a}_Q$ such that for some $F \in \text{Hom}(S(\mathfrak{r}), V)$ the function $E(g, F, \Lambda) \not\equiv 0$; indeed this follows from the definition of Eisenstein

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system and the fact that $\mathfrak{r}^v \supseteq \mathfrak{a}_Q(\mathbf{C})$. Let $\Lambda = X(\mathfrak{r}) + \iota H$ be general, with H real, and $P' = P^{(j)} \in {}^Q\{P\}$, so that

$$E^{P'}(g, F, \Lambda) = \sum_{s \in {}^Q W^{(j)}(\mathfrak{r})} e\langle s\Lambda, \bar{H}_{M'}(g)\rangle N(s, \Lambda)F(g)$$

where $N(s, \Lambda) \equiv 0$ unless $s \in {}^Q W^{(j)}(\mathfrak{r}, \mathfrak{t})$, for some $\mathfrak{t} \in {}^Q S^{(j)}$. Thus $N(s, \Lambda) \not\equiv 0$ implies that $\mathfrak{t} \in C$ and ${}^Q S^{(j)}$, hence the largest standard subspace in $\mathfrak{t}^v$ must be $\mathfrak{a}_Q$ again. Now by choice if $N(s, \Lambda)F \not\equiv 0$, then $-\text{Re}\,(s\Lambda) = \text{Re}\,X(\mathfrak{t})$, and hence $-\text{Re}\,(s\Lambda) \in {}^{+Q}C_{P'}$. Therefore, using 4.4 we conclude that

$$E(g, F, \Lambda) \in {}^Q\mathcal{L}(\{P\}, \chi\delta_Q) \quad \text{where } \chi = \omega|Z_Q(\mathbf{A}).$$

But then 4.3 implies that each $s\Lambda$ must be real. Suppose that $\mathfrak{r}^v$ were not the complexification of $\mathfrak{a}_Q(\mathbf{C})$: the definition of Eisenstein system ($\{E(.\,,\,.\,,\,.)\} \not\equiv \{0\}$) and the principle of analytic continuation would imply then that we could choose a non real Λ so that $E(g, F, \Lambda) \not\equiv 0$. This however is a contradiction: if $\Lambda = X(\mathfrak{r}) + \iota H$ is general as above with H real, then $s\Lambda = sX(\mathfrak{r}) + \iota sH$ and it follows immediately that $s\Lambda$ must be non real as well (note that $sX(\mathfrak{r})$ is orthogonal to sH).

Therefore $\mathfrak{r}^v$ is standard, equal to $\mathfrak{a}_Q(\mathbf{C})$, and thus ${}^Q\mathfrak{r} = \{X(\mathfrak{r})\}$. Applying 4.4, then 4.3 again, we conclude that $X(\mathfrak{r})$ must itself be real, and hence all $X(\mathfrak{t})$ are real for $\mathfrak{t} \in C$. This finishes the proof.

4.6. The next thing to do is to find analytical conditions on the Eisenstein systems associated to the collection $S = \bigcup_i S^{(i)}$ which imply that they are complete. In the induction argument of Section 5 the analytical conditions will be verified to each step, for the Eisenstein systems of interest to us. Before summarizing these conditions as a definition, let us make some conventions.

First choose a set of base points $\{\omega\delta_P\}$ for the connected components of ${}^Q D_M(\chi\delta_Q)$ where ω runs through a countable set of characters. Then, as remarked earlier, we may regard ${}^Q\mathfrak{r} \subseteq {}^Q D_M(\chi\delta_Q)$.

We define the axis $U({}^Q\mathfrak{r})$ component wise on ${}^Q D_M(\chi\delta_Q)$ by defining it on ${}^Q\mathfrak{a}_M(\mathbf{C})$ as

$$\{X(\mathfrak{r}) + \iota H | H \in {}^Q\mathfrak{r}^v \cap {}^Q\mathfrak{a}_M(\mathbf{R})\}.$$

We shall often be interested in the space just defined, in its own right, and shall again denote it by $U({}^Q\mathfrak{r})$. This should cause no confusion since the meaning will usually be clear from the content (it will usually occur in "trivialized" definitions).

More generally, if one replaces $X(\mathfrak{r})$ by some point Λ, the resulting axis is written

$$U({}^Q\mathfrak{r}, \Lambda).$$

Finally, observe that if we take the Fourier transform Φ_ζ of a function $\phi \in \mathscr{C}_0(P, K', \xi)$, then Φ_ζ can be regarded as a function

$$\Phi : U \to \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K'\omega\delta_P) \quad \text{(algebraic direct sum)}$$

with U as in 2.5 say. This is proved in [**5**] II.1.7, and forms the basis for setting

$$V(K') = V^i(K') = \bigoplus_{\{\omega\}} \mathscr{C}_0(P, K', \omega\delta_P)$$
$$V = V^{(i)} = \bigcup_{K'} V^i(K').$$

We shall then write $V_\omega{}^i$ etc. for $\mathscr{C}_0(P^i, \omega\delta_P)$ etc., agreeing with an earlier convention.

4.7. *Definition*. Let $S = \bigcup_{i=1}^r S^{(i)}$ be a finite collection of distinct affine subspaces, such that $\mathfrak{r} \in S^{(i)}$ is geometric. Suppose that for each $\mathfrak{r}$ there is an Eisenstein system belonging to $\mathfrak{r}$. We say that $\{E(g, F, \Lambda)\}$ (belonging to $\mathfrak{r}$) is *spectral* if it satisfies the following conditions:

(i) $X(\mathfrak{r}) \in X_i(R) \quad (R > 1)$, c.f. 1.8.

(ii) For each Q with $\mathfrak{a}_Q(\mathbf{C}) \subseteq \mathfrak{r}^v$, if $\{P^{(i)}, \ldots, P^{(r')}\}$ is the set ${}^Q\{P\}$, then there is an orthogonal projection $\mathscr{Q}$ from ${}^Q\mathscr{L}(\{P\}, \chi\delta_Q)$ onto a closed subspace.

(iii) For each $1 \leqq i \leqq r'$ there is a trigonometric polynomial r_i defined on $\iota^Q L_{Z_M}\backslash {}^Q\mathfrak{a}_{\mathbf{C}}{}^{(i)}$ which is not identically zero on ${}^Q\mathfrak{r}$ if $\mathfrak{r} \in {}^Q S^{(i)}$.

(iv) If $P^{(i)}, P^{(j)} \in Q\{P\}$, and $\Phi' \in \mathscr{H}^{(i)}(R) \otimes V$ set $\Phi(\Lambda) = r_i(\Lambda)\Phi'(\Lambda)$. Then if $\Psi \in \mathscr{H}^{(j)}(R) \otimes V'$, we require that $(\mathscr{Q}\phi\hat{}, \psi\hat{})$ be equal to the sum over $\mathfrak{r} \in {}^Q S^{(i)}$ of

$$\sum_{s \in {}^Q W^{(j)}(\mathfrak{r})} \int_{U({}^Q\mathfrak{r})} (N(s, \Lambda) d\Phi(\Lambda), d\Psi(-s\bar{\Lambda}))_S d\Lambda$$

where $(\, , \,)_S$ is as in 2.7.

4.8. Proposition. *Suppose that each Eisenstein system belonging to each* $\mathfrak{r}$ *in* $S = \bigcup S^{(i)}$ *is spectral. Then each such Eisenstein system is complete, and furthermore the adjoint of* $N(s, \Lambda)$ *with respect to pairing* $(\, , \,)_S$ *is given by*

$$N^*(s, \Lambda) = N(s^{-1}, -s\bar{\Lambda}).$$

The proof of this proposition is a triviality once we have pinpointed some technicalities which had best be proved as well as stated if only to make explicit some arguments which we later take for granted.

4.9. Lemma. *Let* Λ *be a general point in* $\mathfrak{r}$. *Suppose that*

$$\oplus u(s) \in \bigoplus_{s \in {}^Q W^j(\mathfrak{r})} \operatorname{Hom}(S(\mathfrak{r}_s), V^j_{s\omega})$$

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is such that

$$0 = (\bigoplus_s u(s), \bigoplus_s d\Psi_i(-s\bar{\Lambda}))_S$$

for all $\bigoplus_{s \in {}^Q W^j(\mathfrak{r})} \Psi_i(-s\bar{\Lambda})$ *with* $\Psi_i \in \mathscr{H}_i(R) \otimes V_\omega{}^i$. *Then* $\bigoplus_s u(s) \equiv 0$.

Proof. We can without loss suppose $V_\omega{}^i$ is one dimensional (take coordinates), and then it must be shown that if

$$0 = \bigoplus_s D(u(s))\overline{\Psi_i(-s\bar{\Lambda})} \text{ for all } \Psi_i$$

then each $u(s) = 0$.

This is trivial because if Λ is general then all $s\Lambda$ are distinct, and then we can choose Ψ so that the sum above does not vanish if some $u(s) \neq 0$.

We shall use this lemma to prove something more directly related to Proposition 4.8.

4.10. LEMMA. *Let* $M_\omega(s, \Lambda)$ *be a function on* ${}^Q W^j(\mathfrak{r}) \times {}^Q\mathfrak{r}$ *with values in* Hom (Hom $(S(\mathfrak{r}), V_\omega{}^i)$, $S(\mathfrak{r}) \otimes V_\omega{}^j$). *Suppose that* $(M_\omega(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\Lambda))_S$, *with* $\Phi = r_i\Phi'$ *as in* 4.8, *and* $\Psi \in \mathscr{H}^{(j)}(R) \otimes V^{(j)}$, *is holomorphic in a neighbourhood of* $U({}^Q\mathfrak{r})$, *and that*

$$(M_\omega(s, \Lambda)F, F')_S \equiv 0$$

if $O(F)$ *or* $O(F')$ *is large enough. Put*

$$M(s, \Lambda) = \bigoplus_\omega M_\omega(s, \Lambda).$$

Suppose that

$$\sum_{s \in {}^Q W^{(j)}(\mathfrak{r})} \int_{U({}^Q\mathfrak{r})} (M(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\Lambda))_S d\Lambda$$

is zero. Then $M(s, \Lambda) \equiv 0$, *for all* s, *and all* Λ.

Proof. Argue by contradiction: suppose that $M(s, \Lambda) \not\equiv 0$. Then for some integer $n \geqq 0$ there is a F_ω, s, Λ, so that $M(s, \Lambda)F_\omega \neq 0$ but if $O(F') > n$, then $M(s, \Lambda)F' = 0$ for all s, Λ.

We can pick an element h in

$$\mathbf{C}[{}^Q L_{Z_M}{}^*] \otimes V_\omega$$

so that

$$O(F_\omega - dh) > n.$$

For the moment, let $f(\Lambda)$ be any scalar valued function holomorphic in a neighbourhood of $U({}^Q\mathfrak{r})$, and set

$$\Phi(\Lambda) = r_i(\Lambda)f(\Lambda)h(\Lambda).$$

Then

$$O(d\Phi - r_i(\Lambda)f(\Lambda)F_\omega) > n,$$

so that $(M(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\bar{\Lambda}))_S$ is simply equal to

$$r_i(\Lambda)f(\Lambda)(M(s, \Lambda)F_\omega, d\Psi(-s\bar{\Lambda}))_S.$$

Now put

$$g(\Lambda) = \sum_{s \in {}^QW^{(j)}(\mathfrak{r})} r_i(\Lambda)(M(s, \Lambda)F_\omega, d\Psi(-s\bar{\Lambda}))_S.$$

Then

$$\int_{U({}^Q\mathfrak{r})} f(\Lambda)g(\Lambda)d\Lambda \equiv 0.$$

In particular the Fourier coefficients of $g(\Lambda)$ are zero, hence $g(\Lambda)$ is zero. The preceding lemma implies that each $M_\omega(s, \Lambda)F_\omega = 0$, which contradicts our assumption.

4.11. *Proof of Proposition* 4.8. By assumption $(\mathcal{Q}\phi^\wedge, \Psi^\wedge)$ is

$$\sum_{s \in {}^QW^{(j)}(\mathfrak{r})} \int_{U({}^Q\mathfrak{r})} (N(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\bar{\Lambda}))_S d\Lambda$$

where $\Phi(\Lambda) = r_i(\Lambda)\Phi'(\Lambda)$ with $\Phi'(\Lambda) \in \mathscr{H}_i(R) \otimes V^i$. Take $\Psi(\Lambda) = r_j(\Lambda)\Psi'(\Lambda)$ in this expression, replace i by j, interchange Φ and Ψ and take complex conjugates. Subtracting the result from the original expression and recalling that $\mathcal{Q}$ is orthogonal hence self adjoint, we find that the sum over $\mathfrak{r} \in {}^QS^{(i)}$ of the sum over $s \in {}^QW^{(j)}(\mathfrak{r})$ of

$$\int_{U({}^Q\mathfrak{r})} (M(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\bar{\Lambda}))d\Lambda$$

is zero, where

$$(4.11.1) \quad M(s, \Lambda) = \begin{cases} d^*r_j(-s\bar{\Lambda})N(s, \Lambda) & \text{if } \mathfrak{r}_s \notin {}^QS^{(j)} \\ d^*r_j(-s\bar{\Lambda})\{N(s, \Lambda) - N^*(s^{-1}, -s\bar{\Lambda})\} & \text{otherwise.} \end{cases}$$

We are not quite in a position to apply the preceding lemma because of the presence of the sum over all $\mathfrak{r} \in {}^QS^{(i)}$; to fix this up we proceed as follows. Choose a polynomial on $\iota L_{Z_M} \backslash \mathfrak{a}_{\mathbf{C}}{}^{(i)}$, denoted by $q(\Lambda)$ with the following properties.

(i) $q(\Lambda)$ is non zero on ${}^Q\mathfrak{r}_0$ say, vanishing to a high enough order on each ${}^Q\mathfrak{t}$, $\mathfrak{r}_0 \neq \mathfrak{t} \in {}^QS^{(i)}$ so that if $s \in {}^QW^{(j)}(\mathfrak{t})$ then

$$N(s, \Lambda) \circ dq(\Lambda) \equiv 0 \quad \text{for } \Lambda \in {}^Q\mathfrak{t},$$

(ii) If $\mathfrak{t} \in {}^QS^{(j)}$, then $q(\Lambda)$ vanishes to such a high order on each ${}^Q\mathfrak{t}_s$, $s \in {}^QW^{(i)}(\mathfrak{t})$, that

$$d^*q(-s\bar{\Lambda}) \circ N(s, \Lambda) \equiv 0.$$

Now in the preceding argument replace $\Phi(\Lambda)$ by $q(\Lambda)\Phi(\Lambda)$, then the result is that only the term involving $\mathfrak{r} = \mathfrak{r}_0$ remains, and the expression

(4.11.1) is replaced by

$$M(s, \Lambda) = \begin{cases} d^*r_j(-s\bar{\Lambda})N(s, \Lambda) \circ dq(\Lambda) & \text{if } \mathfrak{r}_s \notin {}^QS^{(j)} \\ d^*r_j(-s\bar{\Lambda})\{N(s, \Lambda) - N^*(s^{-1}, -s\bar{\Lambda})\}dq(\Lambda) & \\ & \text{if } \mathfrak{r}_s \in {}^QS^{(j)}. \end{cases}$$

Now we can apply Lemma 4.10 to conclude that

$$d^*r_j(-s\bar{\Lambda})N(s, \Lambda) \circ dq(\Lambda) \equiv 0 \quad \text{if } \mathfrak{r}_s \notin {}^QS^{(j)}$$
$$d^*r_j(-a\bar{\Lambda})\{Ns, \Lambda) - N^*(s^{-1}, -s\bar{\Lambda})\} \circ dq(\Lambda) \equiv 0, \text{ if not}$$

which implies

$$N(s, \Lambda) \equiv 0 \quad \text{if } \mathfrak{r}_s \notin {}^QS^{(j)}$$

and

$$N(s, \Lambda) = N^*(s^{-1}, -s\bar{\Lambda})$$

as desired.

4.12. *Remark.* A similar kind of analytic argument implies that under the above circumstances, ${}^QW^{(j)}(\mathfrak{r}) \neq \emptyset$.

4.13. Collections of spectral Eisenstein systems will provide us with a spectral decomposition of ${}^Q\mathscr{L}(\{P\}, \chi\delta_Q)$. The next definition tells us how a collection of spectral Eisenstein systems as in 4.8 furnishes a closed subspace of each ${}^Q\mathscr{L}(\{P\}, \chi\delta_Q)$, and the next proposition describes this space in more detail.

Definition. Let $S^{(i)}$ be a collection of distinct affine subspaces of $\mathfrak{a}_{\mathbf{C}}{}^{(i)}$, and set

$$S = \bigcup_{i=1}^{r} S^{(i)}.$$

Suppose that to each $\mathfrak{r} \in S$ there is associated a spectral Eisenstein system. If Q is a parabolic subgroup such that ${}^Q\{P\}$ is non empty, we define ${}^Q\mathscr{L}_S(\{P\}, \chi\delta_Q)$ to be the closed subspace of ${}^Q\mathscr{L}(\{P\}, \chi\delta_Q)$ generated by functions of the form $\mathscr{Q}_S\phi\hat{}$ where $\phi\hat{}$ corresponds to $r_i(\Lambda)\Phi(\Lambda)$ for some i, with

$$\Phi(\Lambda) \in \mathscr{H}^{(i)}(R) \otimes V^{(i)}.$$

4.14. Fix S as in the definition, and let $C_1, \ldots, C_u$ denote the equivalence classes in ${}^QS$ and for each $1 \leqq k \leqq u$, choose $\mathfrak{r}_k$ in C_k so that $\mathfrak{r}_k$ is standard (Proposition 4.2). Define ${}^QW(\mathfrak{r}_k, C_k)$ to be the union over $\mathfrak{t} \in C_k$ of the ${}^QW(\mathfrak{r}_k, \mathfrak{t})$. For each $\mathfrak{r}_k$ we let e_k denote the element in ${}^QW(\mathfrak{r}_k)$ fixing each element of $\mathfrak{r}_k$ (c.f. 4.2). Set

$${}^QW^{(i)}(\mathfrak{r}_k, C_k) = \{s \in {}^QW(\mathfrak{r}_k, C_k) | j_s = i\}$$

if j_s is defined to be such that $(\mathfrak{r}_k)_s \subseteq \mathfrak{a}_{\mathbf{C}}{}^{(j_s)}$.

PROPOSITION. *Let* $P = P^{(i)} \in {}^Q\{P\}$, $\Phi \in \mathscr{H}^{(i)}(R) \otimes V^{(i)}$, *and for each* $\mathfrak{r}_k$, *let* ω_k *denote the order of* ${}^QW(\mathfrak{r}_k, \mathfrak{r}_k)$.

(i) $$\sum_{s \in {}^QW^{(i)}(\mathfrak{r}_k, C_k)} E(g, d\Phi(se_k\Lambda), se_k\Lambda)$$
is analytic on $U({}^Q\mathfrak{r}_k)$.

(ii) *The projection of* $\phi\hat{}(g)$ (*corresponding to* $\Phi(\Lambda)$) *onto* ${}^Q\mathscr{L}_S(\{P\}, \chi\delta_Q)$ *is equal to*

$$\sum_{k=1}^{u} \frac{1}{\omega_k} \int_{U({}^Q\mathfrak{r}_k)} \sum_{s \in {}^QW^{(i)}(\mathfrak{r}_k, C_k)} E(g, d\Phi(se_k\Lambda), se_k\Lambda)d\Lambda.$$

(iii) *If* $P^{(j)} \in {}^Q\{P\}$, $\Psi \in \mathscr{H}^{(j)}(R) \otimes V^{(j)}$, *then*

$$[\Phi, \Psi]^{(k)} = \sum_{t \in {}^QW^{(j)}(\mathfrak{r}_k, C_k)} \times \sum_{s \in {}^QW^{(i)}(\mathfrak{r}_k, C_k)} (N(te_ks^{-1}, se_k\Lambda)d\Phi(se_k\Lambda), d\Psi(-t\bar{\Lambda}))_S$$

is analytic on $U({}^Q\mathfrak{r}_k)$.

(iv) *The inner product of the projection of* $\phi\hat{}(g)$ *on* ${}^Q\mathscr{L}_S(\{P\}, \chi\delta_Q)$ (*as in* (ii)) *with* $\psi\hat{}(g)$ (*corresponding to* $\Psi(\Lambda)$ *in* (iii)) *is equal to*

$$\sum_{k=1}^{u} \frac{1}{\omega_k} \int_{U({}^Q\mathfrak{r})} [\Phi, \Psi]^{(k)} d\Lambda.$$

Proof. To begin with, consider the inner product $(\mathscr{Q}_S\phi\hat{}, \psi\hat{})$; it is equal to the sum over $\mathfrak{r} \in {}^QS^{(i)}$ of the sum over $s \in {}^QW^{(j)}(\mathfrak{r})$ of

$$\int_{U({}^Q\mathfrak{r})} N(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\bar{\Lambda}))d\Lambda.$$

In turn we may rearrange this so that it is equal to the sum from $k = 1$ to $k = u$ of

$$(4.14.1) \quad \sum_{t \in {}^QW^{(j)}(\mathfrak{r}_k, C_k)} \sum_{s \in {}^QW^{(i)}(\mathfrak{r}_k, C_k)} \frac{1}{\omega_k} \int_{U({}^Q\mathfrak{r}_k)} [\Phi, \Psi, s, t]_\Lambda^{(k)} d\Lambda$$

where

$$[\Phi, \Psi, s, t]_\Lambda^{(h)} = (N(te_ks^{-1}, se_k\Lambda)d\Phi(se_k\Lambda), d\Psi(-t\bar{\Lambda})).$$

Next, define the space $\mathrm{Hom}_k(V^{(i)})$ by

$$\mathrm{Hom}_k(V^{(i)}) = \bigoplus_{s \in {}^QW^i(\mathfrak{r}_k, C_k)} \bigoplus_{\omega} \mathrm{Hom}(S((\mathfrak{r}_k)_s), V_\omega^{(i)}).$$

If $F = \oplus F_s \in \mathrm{Hom}_k(V^{(i)})$, $F' = \oplus F_s' \in \mathrm{Hom}_k(V^{(j)})$ we define $[F, F']_\Lambda^{(k)}$ to be equal to

$$\sum_t \sum_s (N(te_ks^{-1}, se_k\Lambda)F_s, F_t').$$

It follows, via another approximation argument, that if $\Lambda \in U({}^Q\mathfrak{r}_k)$ then

$$(4.14.2) \quad [F, F]_\Lambda^{(k)} \geqq 0, \ |[F, F']^{(k)}|^2 \leqq [F, F]_\Lambda^{(k)}[F', F']_\Lambda^{(k)}.$$

Finally, define the space $\mathcal{T}_k(V^{(i)})$ to be the space of functions

$$F = \oplus F_s: U({}^Q\mathfrak{r}_k) \to \bigoplus_{s \in {}^QW^i(\mathfrak{r}_k, C_k)} \prod_\omega \mathrm{Hom}(S((\mathfrak{r}_k)_s), V_\omega{}^i)$$

satisfying the following conditions

(i) $[F(\Lambda), F']_{\Lambda}{}^{(k)}$ is measurable for each $F' \in \mathrm{Hom}_k(V^{(i)})$

(ii) $F_{se_kw}(\Lambda) = F_s(e_kw\Lambda), \quad w \in {}^QW(\mathfrak{r}_k, \mathfrak{r}_k)$

(iii) $\dfrac{1}{\omega_k}\displaystyle\int_{U({}^Q\mathfrak{r})} [F(\Lambda), F(\Lambda)]_\Lambda^{(k)} d\Lambda = \|F\|^2 < \infty.$

Property (iii) means in particular that, modulo the usual equivalence relation, we can view $\mathcal{T}_k(V^{(i)})$ as a Hilbert space, and then

$$\mathcal{T}(V^{(i)}) = \bigoplus_k \mathcal{T}_k(V^{(i)})$$

is also a Hilbert space. It contains as a subspace the space consisting of sums

$$\bigoplus_k \bigoplus_s d\Phi(se_k\Lambda)$$

where $\Phi = r_i(\Lambda)\Phi'(\Lambda)$. In fact this subspace is dense, c.f. the argument in 4.8.

Consequently the map

$$\Phi(\Lambda) \to \mathcal{Q}\phi^\wedge$$

extends to an isometric map

$$F(\Lambda) \to f^\wedge$$

$$\mathcal{T}(V^{(i)}) \to {}^Q\mathcal{L}_S(\{P\}, \chi\delta_Q)$$

and then $(f^\wedge, g^\wedge)$ is equal to

$$\sum_k \frac{1}{\omega_k}\int_{U({}^Q\mathfrak{r})} [F(\Lambda), G(\Lambda)]_\Lambda^{(k)} d\Lambda.$$

On the other hand suppose that $\oplus F_k$ satisfies condition (ii) above, but not necessarily (i) and (iii), but that if $G = \oplus G_k$ lies in the aforementioned dense subspace then

$$[F_k(\Lambda), G_k(\Lambda)]_\Lambda{}^{(k)}$$

is measurable (for each k), and

$$\sum_k \frac{1}{\omega_k}\int_{U({}^Q\mathfrak{r})} [F_k(\Lambda), G_k(\Lambda)]_\Lambda^k d\Lambda \leqq c\|G\|.$$

Then we can conclude that $F = \oplus F_k$ lies in $\mathcal{T}(V^{(i)})$ with norm at most c.

Apply this remark to F given by

$$F_k(\Lambda) = \oplus d\Phi(se_k\Lambda), \ \Phi(\Lambda) \in \mathcal{H}^{(i)}(R) \otimes V^{(i)}$$

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and $c = \|\phi^\wedge\|$, $\phi^\wedge$ corresponding to Φ. Consequently if $P^{(j)} \in {}^Q\{P\}$, $\Psi \in \mathscr{H}^{(j)}(R) \otimes V^{(j)}$ then using the second inequality of (4.14.2) and the fact that always $[F, F]_\Lambda \leqq \|F\|^2$ if $F \in \mathscr{T}(V^{(i)})$ we see that

$$\sum_t \sum [\Phi, \Psi, s, t]_\Lambda^{(k)}$$

is integrable on $U({}^Q\mathfrak{r}_k)$. But this is meromorphic on this axis, with singularities of form $\alpha(X) = \mu$. Therefore if it is integrable along the axis $U({}^Q\mathfrak{r}_k)$ it must be analytic (i.e., regular) on that set. This proves part (iii) of the proposition.

To prove part (iv), let $\phi'(g)$ be that element of ${}^Q\mathscr{L}_S(\{P\}, \chi\delta_Q)$ corresponding to $F = \bigoplus_k \bigoplus_s d\Phi(se_k\Lambda)$ under the map $F \to f^\wedge$. Let

$$\Psi(\cdot) = r_j(\cdot)\Psi'(\cdot).$$

Then $(\mathscr{Q}\psi^\wedge, \phi^\wedge)$ is simply the expression (4.14.1) with the roles of Ψ and Φ, and the roles of i and j, interchanged. This however is simply $(\mathscr{Q}\psi^\wedge, \phi')$. We conclude that the projection of $\phi^\wedge$ in ${}^Q\mathscr{L}_S(\{P\}, \chi\delta_Q)$ is ϕ' and this proves part (iv).

It remains to prove parts (i) and (ii). To prove the first we use the fact that if $\phi^P \sim 0$ for all P, then ϕ is itself zero. Indeed, suppose the sum in the statement of part (i) is not analytic along $U({}^Q\mathfrak{r})$; call this sum $E(\cdot, \Lambda)$ for argument's sake, and let $\mathfrak{t}$ be a singular hyperplane of $E(\cdot, \Lambda)$, meeting $U({}^Q\mathfrak{r}_k)$.

Let $\psi \in {}^Q\mathscr{C}_0(P', \chi\delta_Q)$, then $\psi \to \psi^\wedge$ and ψ has a Fourier transform

$$\Psi(g, \zeta) = \int_{Z_M(F)\backslash Z_M(\mathbf{A})} \zeta^{-1}(z)\psi(zg)dz$$

for $\zeta \in {}^QD_M(\chi\delta_Q)$. Let $g(\Lambda)$ be any analytic function on ${}^Q\mathfrak{r}_k$. Then

$$\int_{Q(F)N_Q(\mathbf{A})Z_Q(\mathbf{A})\backslash G(\mathbf{A})} \operatorname{Res}_{Q\mathfrak{t}} g(\Lambda)E(x, \Lambda)\bar{\psi}^\wedge(x)dx$$
$$= \operatorname{Res}_{Q_\mathfrak{t}} \int g(\Lambda)E(x, \Lambda)\bar{\psi}^\wedge(x)dx.$$

The integral is shown by a standard calculation, and rearrangement as in (4.14.1) to be equal to

$$g(\Lambda)[\Phi, \Psi]_\Lambda^{(k)}.$$

The latter term has no singularities which meet $U({}^Q\mathfrak{r}_k)$ by part (iii), so the residue is zero. We conclude that the cuspidal component of $\operatorname{Res}_{Q\mathfrak{t}} (g(\Lambda)E(\cdot, \Lambda))^{P'}$ is zero for all $P' \in {}^Q\{P\}$, and hence all P'. Thus

$$\operatorname{Res}_{Q\mathfrak{t}} g(\Lambda)E(\cdot, \Lambda) \equiv 0.$$

This is a contradiction, because if $f(\Lambda)$ is meromorphic and ${}^Q\mathfrak{t}$ is a

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singular hyperplane then by taking Laurent expansions one sees that it is easy to choose $g(\Lambda)$ so that

$$\mathrm{Res}_{Q\mathfrak{t}}\, g(\Lambda)f(\Lambda) \not\equiv 0.$$

As for part (ii), one knows from part (iv) that if ϕ' is the projection of $\phi^{\wedge}$ onto ${}^{Q}\mathscr{L}_{S}(\{P\}, \chi\delta_{Q})$ then for each $\mathscr{Q}\psi^{\wedge} \in {}^{Q}\mathscr{L}_{S}(\{P\}, \chi\delta_{Q})$ the inner product $(\mathscr{Q}\psi^{\wedge}, \phi')$ is given by the expression in part (iv). But it is a straightforward exercise to see that this is simply the expression in part (ii) integrated against $\psi^{\wedge}$, and this concludes the proof.

4.15. There is an important corollary to this proposition which will be needed in Section 5. We shall not prove it here because the proof is more or less the same as the number field case to which we refer the reader. In [**3**] it occurs as the corollary to Lemma 7.6; a rather nice proof is given in [**7**] Proposition 5.11. We do however take the liberty of stating it.

COROLLARY. *Suppose* ${}^{Q}\{P\}$ *is non empty, and let* $\mathfrak{r} \in {}^{Q}S^{(i)}$, $F \in$ Hom $(S(\mathfrak{r}), V^{(i)})$. *Let* $\mathfrak{b}$ *be the largest standard subspace contained in* $\mathfrak{r}^{v}$. *Suppose that* $\mathfrak{t}$ *is the inverse image in* $\mathfrak{r}$ *of a singular hyperplane of* $E(\cdot, F, \Lambda)$ *on* ${}^{Q}\mathfrak{r}$ *which meets* $U({}^{Q}\mathfrak{r})$. *Then* $\mathfrak{t} \supseteq \mathfrak{b}$.

Remarks. In particular if $\mathfrak{r}$ is standard then the corollary implies that $E(\cdot, F, \Lambda)$ has no singular hyperplanes which meet $U({}^{Q}\mathfrak{r})$. In proving the corollary one treats this case first, and then passes to the general case by means of Langlands' Lemma 7.4 (see Section 7).

4.16. This concludes the list of properties of Eisenstein systems that are of interest to us for Section 5. The reader may have wondered where axiom (iii) of Section 3.3 was ever used; the truth is that it is used in the proof of Langlands' Lemma 7.4 (hence any results which use that lemma), and since we need an analogue of Langlands' Lemma 7.4 we need axiom (iii). It will also be used later.

5. The main theorem. In this section we establish the main theorem of this paper. The proof can be regarded almost as a simplified version of the number field case, which is due to Langlands. In writing out the proof, which is an induction argument, we have found the approach of [**7**] to be very helpful in places, and have used it here.

5.1. We begin with some remarks on affine hyperplanes and subspaces in $\iota L_{Z_M}\backslash\mathfrak{a}_M(\mathbf{C})$.

LEMMA. *Let* $\mathfrak{t}$ *be a hyperplane in* $\mathfrak{r}$, *defined by the (additional) equation* $\alpha^{v}(\Lambda) = c$. *Suppose that* $\Lambda_0 \in X(\mathfrak{r}) + \mathfrak{r}^{v} \cap \mathfrak{a}^{(i)}$. *Then*

$$\mathfrak{t} \cap U(\mathfrak{r}, \Lambda_0) \neq \emptyset$$

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if and only if

$$\operatorname{Re}\,(c - \alpha^v(X(\mathfrak{r}))) = \alpha^v(\Lambda_0 - X(\mathfrak{r})).$$

Proof. Suppose $\mathfrak{t}$ meets $U(\mathfrak{r}, \Lambda_0)$, then there exists $\Lambda \in \mathfrak{t}$ such that $\Lambda = \Lambda_0 + \iota Z$, Z real. Then $\Lambda - X(\mathfrak{r})$ is equal to $\Lambda_0 - X(\mathfrak{r}) + \iota Z$, so that $\operatorname{Re} \alpha^v(\Lambda - X(\mathfrak{r}))$ is equal to $\alpha^v(\Lambda_0 - X(\mathfrak{r}))$, i.e.,

$$\operatorname{Re}\,(c - \alpha^v(X(\mathfrak{r}))) = \alpha^v(\Lambda_0 - X(\mathfrak{r})). \tag{5.1.1}$$

Conversely, suppose (5.1.1) holds. Then for any $\Lambda \in \mathfrak{t}$, one has

$$\operatorname{Re} \alpha^v(\Lambda - X(\mathfrak{r})) = \alpha^v(\Lambda_0 - X(\mathfrak{r})).$$

Writing $\Lambda - X(\mathfrak{r}) = A + \iota B$ one finds

$$\operatorname{Re} \alpha^v(\Lambda - X(\mathfrak{r})) = \alpha^v(A)$$

so that

$$\alpha^v(A - (\Lambda_0 - X(\mathfrak{r})) = 0.$$

Thus

$$A = \Lambda_0 - X(\mathfrak{r}) + C,\ C \in \mathfrak{t}^v \cap \mathfrak{a}^{(i)}, \quad \text{and}$$

$$\Lambda - X(\mathfrak{r}) = (\Lambda_0 - X(\mathfrak{r})) + C + \iota B$$

so $\Lambda = \Lambda_0 + C + \iota B$, and $\Lambda - C = \Lambda_0 + \iota B$. But $\Lambda - C \in \mathfrak{t}$ and we are done.

5.2. Corollary. *Suppose* $\Lambda_1, \Lambda_2 \in X(\mathfrak{r}) + \mathfrak{r}^v \cap \mathfrak{a}^{(i)}$ *satisfy*

$$\mathfrak{t} \cap U(\mathfrak{r}, \Lambda_i) \neq \emptyset \text{ for } i = 1, 2.$$

Then $\mathfrak{t} \cap U(\mathfrak{r}, \Lambda) \neq \emptyset$ *for some* $\Lambda \in t\Lambda_1 + (1 - t)\Lambda_2$ $(0 \leqq t \leqq 1)$ *if and only if*

$$\alpha^v(\Lambda_1 - X(\mathfrak{r})) - \operatorname{Re}\,(c - \alpha^v(X(\mathfrak{r})) \quad \textit{and}$$

$$\alpha^v(\Lambda_2 - X(\mathfrak{r})) - \operatorname{Re}\,(c - \alpha^v(X(\mathfrak{r})))$$

have opposite signs.

Remark. We shall frequently make use of a segment of the form $t\Lambda_1 + (1 - t)\Lambda_2$ for $\Lambda_1, \Lambda_2 \in X(\mathfrak{r}) + \mathfrak{r}^v \cap \mathfrak{a}^{(i)}$. By definition, it consists of points $X(\mathfrak{r}) + tY_1 + (1 - t)Y_2$ $0 \leqq t \leqq 1$, where

$$\Lambda_j = X(\mathfrak{r}) + Y_j, \quad Y_j \in \mathfrak{r}^v \cap \mathfrak{a}^{(i)}, \quad j = 1, 2.$$

5.3. Lemma. *Suppose* $\mathfrak{t} \subseteq \mathfrak{r}$ *as above. There is a unique point* $\Lambda_R(\mathfrak{t})$ *in* $\mathfrak{r}^v \cap \mathfrak{a}^{(i)}$ *such that if* $\Lambda \in X(\mathfrak{r}) + \mathfrak{r}^v \cap \mathfrak{a}^{(i)}$ *and* $\mathfrak{t} \cap U(\mathfrak{r}, \Lambda) \neq \emptyset$, *then* $\Lambda + \iota\Lambda_R(\mathfrak{t})$ *is the element of* $\mathfrak{t} \cap U(\mathfrak{r}, \Lambda)$ *closest to* Λ.

Proof. Write $X(\mathfrak{t}) = X(\mathfrak{r}) + Y(\mathfrak{t})$, $Y(\mathfrak{t}) \in \mathfrak{r}^v$, and suppose $Y(\mathfrak{t}) = RY(\mathfrak{t}) + \iota IY(\mathfrak{t})$. Then $X(\mathfrak{t}) = X_R(\mathfrak{t}) + \iota IY(\mathfrak{t})$ with $X_R(\mathfrak{t}) = X(\mathfrak{r}) + RY(\mathfrak{r})$. Suppose Λ has the property stated in the lemma, $\Lambda \in X(\mathfrak{r}) +$

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$\mathfrak{r}^v \cap \mathfrak{a}^{(i)}$. Then $\Lambda = X(\mathfrak{r}) + \mathrm{B}$, with $B \in \mathfrak{r}^v \in \mathfrak{a}^{(i)}$, and there is $Z \in \mathfrak{r}^v \cap \mathfrak{a}^{(i)}$ so that

$$X(\mathfrak{r}) + \mathrm{B} + \iota Z = X(\mathfrak{r}) + RY(t) + \iota IY(\mathfrak{t}) + A, \quad A \in \mathfrak{t}^v.$$

Thus $B = RY(\mathfrak{t}) + RA$, $Z = IY(\mathfrak{t}) + IA$, $RA \in \mathfrak{t}^v \cap \mathfrak{a}^{(i)}$. Hence $\Lambda = X_R(\mathfrak{t}) + RA$, and $Z = IY(\mathfrak{t}) + IA$. Setting $IY(\mathfrak{t}) = \Lambda_R(\mathfrak{t})$, the result follows.

5.4. COROLLARY (of the proof). *Let $\mathfrak{t}_R$ be the set of $\Lambda \in X(\mathfrak{r}) + \mathfrak{r}^v \cap \mathfrak{a}^{(i)}$ with the property stated in Lemma 5.3. Then*

$$\mathfrak{t}_R = X_R(\mathfrak{t}) + \mathfrak{t}^v \cap \mathfrak{a}^{(i)}$$
$$X(\mathfrak{t}) = X_R(\mathfrak{t}) + \iota\Lambda_R(\mathfrak{t})$$
$$X(\mathfrak{t}) + \mathfrak{t}^v \cap \mathfrak{a}^{(i)} = \mathfrak{t}_R + \iota\Lambda_R(\mathfrak{t}).$$

5.5. We now come to the main theorem of this section.

THEOREM. *There are $(q + 1)$ collections $S_m = \bigcup_{i=1}^{r} S_m^{(i)}$ of affine spaces of dimension m, such that to each element of S_m there belongs a spectral Eisenstein system, and if ${}^Q\{P\}$ is non-empty, then*

$${}^Q\mathscr{L}(\{P\}, \chi\delta_Q) = \bigoplus_{m=0}^{q-{}^*q} {}^Q\mathscr{L}_m$$

where ${}^Q\mathscr{L}_m$ is the space associated to S_m by Proposition 4.14 (i.e., ${}^Q\mathscr{L}_m = {}^Q\mathscr{L}_{S_M}(\{P\}, \chi\delta_Q)$), with ${}^Q\mathscr{L}_{m_1}$ orthogonal to ${}^Q\mathscr{L}_{m_2}$ if $m_1 \neq m_2$. Here ${}^*q$ = dimension of $\mathfrak{a}_Q(\mathbf{C})$.

The proof of this theorem is by induction, and to facilitate this we shall break it into a sequence of lemmas and definitions. First we shall construct the machinery by which the induction step is effected, and then show how to start the induction. Assuming that the induction step can be carried out, we shall prove the theorem. Finally, we show how to carry out the induction step.

5.6. For the induction step, we are going to suppose that we are given a finite collection $S^{(i)}$ of distinct affine subspaces of $\mathfrak{a}\mathbf{c}^{(i)}$ of dimension m such that each element of $S^{(i)}$ has a geometric Eisenstein system belonging to it for $1 \leqq i \leqq r$. We are also going to suppose given a finite collection $T^{(i)}$, $1 \leqq i \leqq r$, of not necessarily distinct affine subspaces of dimension $m - 1$, such that each element of $T^{(i)}$ has a geometric Eisenstein system belonging to it; furthermore the collection $T^{(i)}$ will be supposed divided into two disjoint classes

$$T^{(i)} = T^{(i)}(B) \cup T^{(i)}(C).$$

To each element $\mathfrak{r}$ of $S^{(i)}$ or $T^{(i)}(B)$ will be supposed given a non-empty convex cone $V(\mathfrak{r})$; to each element $\mathfrak{t}$ of $T^{(i)}(C)$ will be supposed given a non-empty convex open set $W(\mathfrak{t})$.

We shall suppose that this collection of affine spaces and convex open sets satisfies the following geometric conditions (bar denoting closure):

GI For each $\mathfrak{r} \in S^{(i)}$ or $T^{(i)}$, $\operatorname{Re} X(\mathfrak{r}) \in \overline{{}^+\mathfrak{a}(\mathfrak{r})}$ where $\mathfrak{a}(\mathfrak{r})$ is the orthogonal complement of $\mathfrak{r}^v \cap \mathfrak{a}^{(i)}$ in $\mathfrak{a}^{(i)}$.

GII For each $\mathfrak{r} \in S^{(i)}$ or $T^{(i)}(B)$, $V(\mathfrak{r})$ is a cone $V(\mathfrak{r}, X(\mathfrak{r}))$ with vertex $X(\mathfrak{r})$.

For each $\mathfrak{t} \in T^{(i)}(C)$, $\operatorname{Re} W(\mathfrak{t})$ is contained in the interior of the convex hull of $(\mathfrak{a}^{(i)})^+$ and $\overline{{}^+\mathfrak{a}(\mathfrak{t})}$.

We shall also make some assumptions concerning the location of the singularities of the associated Eisenstein systems.

SIII For each $\mathfrak{r} \in S^{(i)}$, and $\Lambda \in V(\mathfrak{r}, X(\mathfrak{r}))$, no singular hyperplane of the associated Eisenstein system meets $U(\mathfrak{r}, \Lambda)$.

For each $\mathfrak{t} \in T^{(i)}$, $\Lambda \in V(\mathfrak{t})$ or $W(\mathfrak{t})$, no singular hyperplane of the associated Eisenstein system meets $U(\mathfrak{t}, \Lambda)$.

SIV For any $\mathfrak{r} \in S^{(i)}$ or $T^{(i)}(B)$, any singular hyperplane of the associated Eisenstein system which meets $C(\mathfrak{r}, \epsilon_{\mathfrak{r}})$ also meets $U(\mathfrak{r}, X(\mathfrak{r}))$. Here

$$C(\mathfrak{r}, \epsilon_{\mathfrak{r}}) = \{\Lambda \in \mathfrak{r} \mid \|\operatorname{Re}(\Lambda - X(\mathfrak{r}))\| < \epsilon_{\mathfrak{r}}\}.$$

Remark. By a cone $V(\mathfrak{r}, X(\mathfrak{r}))$, we mean the set of points

$$\{tX(\mathfrak{r}) + (1 - t)\Lambda \mid 0 < t < 1,\ \Lambda \in B\}$$

where B is a ball (of say, radius $\epsilon_{\mathfrak{r}}$) on $\mathfrak{r} \cap \mathfrak{a}^{(i)}$.

SV (cf. 4.15) If $\mathfrak{t} \in T^{(i)}(B)$ and $\mathfrak{r}$ is a singular hyperplane (defined on ${}^Q\mathfrak{t}$) of the associated Eisenstein system which meets $U(\mathfrak{t}, X(\mathfrak{t}))$, then the inverse image $\mathfrak{w}$ in $\mathfrak{t}$ of $\mathfrak{r}$ is such that $\mathfrak{w}^v$ contains the largest distinguished subspace in $\mathfrak{t}^v$.

Finally, we make the following assumption:

AVI Suppose $Q \supset P^{(i)}$, $P^{(j)} \in {}^Q\{P\}$. Then there exists an orthogonal projection

$$\mathcal{Q}_Q : {}^Q\mathcal{L}(\{P\}, \chi\delta_Q) \to {}^Q\mathcal{L}(\{P\}, \chi\delta_Q)$$

onto a closed subspace, such that if $\Phi \in \mathcal{H}^{(i)}(R) \otimes V^{(i)}$, $\Psi \in \mathcal{H}^{(j)}(R) \otimes V^{(j)}$, and $\phi\hat{}$, $\psi\hat{}$ are the associated elements of ${}^Q\mathcal{L}(\{P\}, \chi\delta_Q)$, then

$$(\mathcal{Q}_Q\phi\hat{}, \psi\hat{}) = \sum_{\mathfrak{r}} \sum_{s \in {}^QW^{(j)}(\mathfrak{r})} \times \int_{U({}^Q\mathfrak{r}, \Lambda_{\mathfrak{r}})} (N(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\bar{\Lambda}))d\Lambda + \sum_{\mathfrak{t}} \sum_{s \in {}^QW^{(j)}(\mathfrak{t})} \int_{U({}^Q\mathfrak{t}, \Lambda)} (N(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\bar{\Lambda}))d\Lambda$$

where $\mathfrak{r} \in {}^QS^{(i)}$, $\mathfrak{t} \in {}^QT^{(i)}$, $\Lambda_{\mathfrak{r}} \in V(\mathfrak{r})$, $\Lambda \in V(\mathfrak{t})$.

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A collection $(S^{(i)}, T^{(i)}(B), T^{(i)}(C))$, together with associated Eisenstein systems and non-empty convex open sets, which satisfies the above conditions will be called an *amalgamation.*

5.7. Our first example of an amalgamation will be the one which starts the induction proof of the theorem. Put $m = \dim \mathfrak{a}^{(i)} + 1$, and let $S^{(i)} = \emptyset$, $T^{(i)}(B) = \emptyset$, $T^{(i)}(C) = \{\mathfrak{a}_{\mathbf{C}}{}^{(i)}\}_{1 \leqq i \leqq r}$. The Eisenstein system associated to $\mathfrak{a}_{\mathbf{C}}{}^{(i)}$ is that given as the first example of an Eisenstein system (3.11.1). The non-empty convex open set for $\mathfrak{a}_{\mathbf{C}}{}^{(i)}$ will just be $(C_{P(i)} + \delta_{P(i)})$ intersected with $\mathfrak{a}^{(i)}(R)$ for R suitably large, viz.

$$R > \sup \{\|\mathrm{Re}\, X(\mathfrak{t})\| \mid \mathfrak{t} \text{ a singular hyperplane}\}.$$

The projection $\mathcal{Q}_Q$ is the identity map. This set of data then defines an amalgamation.

5.8. LEMMA. *The Eisenstein systems associated to the $S^{(i)}$ figuring in the definition of an amalgamation are spectral.*

Proof. The only thing to do is to choose trigonometric polynomials $r_i(\Lambda)$ so that $N(s, \Lambda)dr_i$ vanishes identically on each ${}^Q\mathfrak{t}$ for $\mathfrak{t} \in {}^QT^{(i)}$, and $s \in {}^QW^{(i)}(\mathfrak{t})$, but such that $r_i \not\equiv 0$ on ${}^Q\mathfrak{r}$ for $\mathfrak{r} \in {}^QS^{(i)}$. Once this is done, the orthogonal projection $\mathcal{Q}_Q$ figuring in the definition of amalgamation will do as the projection $\mathcal{Q}_{Q_S}$ figuring in the definition of spectral Eisenstein system. Indeed, choose $r_i(\Lambda)$ satisfying these conditions, and so that $N(s, \Lambda)dr_i(\Lambda)$ is holomorphic on ${}^Q\mathfrak{r}$, then replacing Φ by $r_i(\cdot)\Phi(\cdot)$ in AVI, we see that the terms involving $\mathfrak{t} \in {}^QT^{(i)}$ disappear, and we are left with

$$\sum_{\mathfrak{r} \in {}^QS^{(i)}} \sum_{s \in {}^QW^{(j)}(\mathfrak{r})} \int_{U({}^Q\mathfrak{r}, \Lambda_\mathfrak{r})} (N(s, \Lambda)d(r_i(\Lambda)\Phi(\Lambda)), d\Psi(-s\bar{\Lambda}))d\Lambda.$$

Since $N(s, \Lambda)d(r_i(\Lambda)\Phi(\Lambda))$ is holomorphic, a straightforward application of the theory of residues (in the form of Cauchy's theorem) then implies the result. Since this type of argument will occur in more detail later, we spare the reader the details for the present.

5.9. The induction step will be taken care of by the following

PROPOSITION. *For each amalgamation $(S^{(i)}, T^{(i)}(B), T^{(i)}(C))$ of dimension m there exists an amalgamation*

$$\mathrm{Res}\,(S^{(i)}, T^{(i)}(B), T^{(i)}(C)) \quad \textit{of dimension } (m-1)$$

with orthogonal projection

$$\mathrm{Res}\, \mathcal{Q}_Q = \mathcal{Q}_Q - \mathcal{Q}_{Q_S}$$

where $\mathcal{Q}_{Q_S}$ is the orthogonal projection associated to the (spectral) Eisen-

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stein system belonging to $^Q S$ *by Lemma* 5.8 (*we shall often write simply* $\mathcal{Q}_S$ *rather than* $\mathcal{Q}_{QS}$).

Remark. The objects in the residual amalgamation will be denoted by placing Res in front of the corresponding object in the original amalgamation.

5.10. Before beginning the proof of 5.9, let us see how it implies the main theorem.

First, for each positive integer m, define inductively an amalgamation Amal (m) of dimension $(q + 1) - m$

Amal (o) is the standard amalgamation of 5.7.

Amal (m) is to be Res $\{\text{Amal } (m - 1)\}$.

In particular, Amal (m) will be trivial if $m > \dim \mathfrak{a}^{(i)} + 1$. We shall denote the objects of Amal (m) by $S_m{}^{(i)}$, $T_m{}^{(i)}$, $\mathcal{Q}_{Q,m}, \ldots$.

Secondly, the proposition says that

$$\mathcal{Q}_Q = \text{Res } \mathcal{Q}_Q + \mathcal{Q}_S$$

so that

$$\text{Image } \mathcal{Q}_Q = \text{Image } (\text{Res } \mathcal{Q}_Q) \oplus {}^Q\mathcal{L}_m$$

where ${}^Q\mathcal{L}_m$ is as in the statement of the main theorem.

Let us show that

$$^Q\mathcal{L}(\{P\}, \chi\delta_Q) = \text{Im}(\mathcal{Q}_{Q,m}) \oplus \bigoplus_{j=0}^{m-1} {}^Q\mathcal{L}_j \qquad (\text{Proj}(m))$$

where ${}^Q\mathcal{L}_j$ is of course the closed subspace associated to $\cup_i {}^Q S_j{}^{(i)}$.

For $m = 0$, $\mathcal{Q}_{Q,o}$ is the identity map, and the right hand sum is empty.

In general, the remark above tells us that

$$\text{Im } (\mathcal{Q}_{Q,m}) = \text{Im } (\mathcal{Q}_{Q,m-1}) \oplus {}^Q\mathcal{L}_m$$

so that Proj $(m + 1)$ is true if Proj (m) is true. On the other hand, the sums in AVI are empty if $m > q - {}^*q$, so that

$$\mathcal{Q}_{Q,q-{}^*q} = \{0\},$$

and hence

$$^Q\mathcal{L} = \bigoplus_{j=0}^{q-{}^*q} {}^Q\mathcal{L}_j$$

(note: ${}^Q\mathcal{L}_0 = \{0\}$) as claimed in the theorem.

5.11. We must now prove the proposition. The first thing to notice is that Res $\mathcal{Q}_Q$ must be equal to $\mathcal{Q}_Q - \mathcal{Q}_S$. Given this, let $\Phi \in \mathcal{H}^{(i)}(R) \otimes$

$V^{(i)}$, $\Psi \in \mathscr{H}^{(j)}(R) \otimes V^{(j)}$, then

$$(\text{Res } \mathscr{Q}_Q\phi\hat{\ }, \psi\hat{\ }) = (\mathscr{Q}_Q\phi\hat{\ }, \psi\hat{\ }) - (\mathscr{Q}_S\phi\hat{\ }, \psi\hat{\ }).$$

Now according to Proposition 4.14 the second expression on the right side is equal to

$$\sum_{k=1}^{u} \frac{1}{\omega_k} \int_{U(^Q\mathfrak{r}_k)} [\Phi, \Psi]_\Lambda{}^{(k)} d\Lambda.$$

Thus using AVI, we see that $(\text{Res } \mathscr{Q}_Q\phi\hat{\ }, \psi\hat{\ })$ is equal to the sum of

$$\sum_{\mathfrak{r} \in {}^QS^{(i)}} \sum_{s \in {}^QW^{(j)}(\mathfrak{r})} \int_{U(^Q\mathfrak{r}, \Lambda\mathfrak{r})} (N(s, \Lambda)d\Phi(\Lambda), d\Psi(-s\bar{\Lambda}))d\Lambda$$

and

$$\sum_{\mathfrak{t} \in {}^QT^{(i)}} \sum_{t \in {}^QW^{(j)}(\mathfrak{t})} \int_{U(^Q\mathfrak{t}, \Lambda)} (N(t, \Lambda)d\Phi(\Lambda), d\Psi(-t\bar{\Lambda}))d\Lambda$$

less

$$\sum_{k=1}^{u} \frac{1}{\omega_k} \int_{U(^Q\mathfrak{r}_k)} [\Phi, \Psi]^{(k)} d\Lambda.$$

We can rewrite all this as

$$(5.11.1) \quad \sum_{k=1}^{u} \frac{1}{\omega_k} \int_{U(^Q\mathfrak{r}_k, \Lambda\mathfrak{r}_k)} - \int_{U(^Q\mathfrak{r}_k)} ([\Phi, \Psi]_\Lambda{}^{(k)})d\Lambda + \sum_{\mathfrak{t} \in {}^QT^{(i)}} I(\mathfrak{t})$$

where

$$I(\mathfrak{t}) = \sum_{t \in {}^QW^{(i)}(\mathfrak{t})} \int_{U(^Q\mathfrak{t}, \Lambda)} (N(t, \Lambda)d\Phi(\Lambda), d\Psi(-t\bar{\Lambda}))d\Lambda.$$

The point to notice here is that over $U(^Q\mathfrak{r}_k, \Lambda_{\mathfrak{r}_k})$ each of the summands

$$(N(te_ks^{-1}, se_k\Lambda)d\Phi(se_k\Lambda), d\Psi(-te_k\Lambda))$$

is holomorphic (by the assumption S III).

On $U(^Q\mathfrak{r})$, however, this is not so; indeed by Proposition 4.14 it is only $[\Phi, \Psi]^{(k)}$ which is holomorphic on $U(^Q\mathfrak{r})$. We emphasize that this is a possibility which does occur in general, cf. the remarks in the introduction, and Langlands' example on $\mathbf{SL}_4$ in appendix III of [**3**].

The next thing for us to do, then, is to shift the region of integration so that the summation signs may be interchanged with the integral sign.

5.12. Let $\mathfrak{r} \in S^{(i)}$ and define $F_\mathfrak{r}$ to be the subset of the co-roots corresponding to $P^{(i)}$ consisting of those co-roots which vanish on the largest standard subspace of $\mathfrak{r}^\vee$. Suppose that $\mathfrak{t} \subseteq \mathfrak{r}$ is a hyperplane in $\mathfrak{r}$ which meets $U(^Q\mathfrak{r})$: if $^Q\mathfrak{t}$ is a singular hyperplane for the Eisenstein system attached to $\mathfrak{r}$, then Corollary 4.15 implies that $\mathfrak{t}^v$ contains the largest standard subspace of $\mathfrak{r}^v$ as well. (Remember: we now know that the Eisenstein systems belonging to the $\mathfrak{r} \in S^{(i)}$ are spectral, so that 4.15 is applicable.) This implies that

$$\mathfrak{t} = \{\Lambda \in \mathfrak{r} | \alpha^v(\Lambda) = \alpha^v(X(\mathfrak{r}))\}$$

for some $\alpha^v \in F_{\mathfrak{r}}$.

Given $\alpha^v \in \Delta_P{}^v(i)$, and $\epsilon > 0$, let

$$RB_\epsilon(\alpha^v) = RB_\epsilon(X(\mathfrak{r})) \cap (X(\mathfrak{r}) + \ker \alpha^v)$$

provided $\ker \alpha^v \not\supseteq \mathfrak{r}^v$, and define $RB_\epsilon(\mathfrak{r})$ to be the complement in $RB_\epsilon(X(\mathfrak{r}))$ of the union of the $RB_\epsilon(\alpha^v)$ as α^v runs over $\Delta_P{}^v(i)$. Here

$$RB_\epsilon(X(\mathfrak{r})) = \{X(\mathfrak{r}) + \Lambda | \Lambda \in \mathfrak{r}^v \cap \mathfrak{a}^{(i)}, \|\Lambda\| < \epsilon\}.$$

We define $RB_\epsilon(\mathfrak{r}, F_{\mathfrak{r}})$ the same way except that α^v is constrained to lie in $F_{\mathfrak{r}}$.

Each of $RB_\epsilon(\mathfrak{r})$ and $RB_\epsilon(\mathfrak{r}, F_{\mathfrak{r}})$ admits a decomposition into a finite disjoint union of convex open sets which we shall call *chambers*, when there is no danger of confusion with the usual chambers previously defined. Thus we can write

$$RB_\epsilon(\mathfrak{r}) = \bigcup C_c(\epsilon, \mathfrak{r})$$

where the number of c is $c(\mathfrak{r}) = c_k$, say, a number which depends only on C_k if $\mathfrak{r} \in C_k$.

5.13. For each equivalence class C_k in ${}^QS$ fix a standard space $r_k \in C_k$, and for each $C_c(\epsilon, \mathfrak{r})$, $\mathfrak{r} \in C_k$, choose a general point $\Lambda_c(\epsilon, \mathfrak{r}) \in C_c(\epsilon, \mathfrak{r})$.

LEMMA. (Res $\mathcal{Q}_Q\phi\hat{}, \psi\hat{}$) *is equal to the sum of*

$$\sum_k \frac{1}{c_k} \sum_{1 \leqq c \leqq c_k} \sum_{\mathfrak{r}} \sum_t \left\{\int_{U(Q_{\mathfrak{r}}, \Lambda_{\mathfrak{r}})} - \int_{U(Q_{\mathfrak{r}}, \Lambda_c(\epsilon, \mathfrak{r}))}\right\}$$

of

$$(N(t, \Lambda)d\Phi(\Lambda), d\Psi(-t\bar{\Lambda}))d\Lambda$$

plus

$$\sum_{\mathfrak{t}} I(\mathfrak{t})$$

where $\mathfrak{r}$ *runs through* ${}^QS^{(i)} \cap C_k$, *and* t *runs through* ${}^QW^{(j)}(\mathfrak{r})$.

Proof. Consider the first term of the expression (5.11.1), and in particular the part

$$\sum_{k=1}^{u} \frac{1}{\omega_k} \int_{U(Q_{\mathfrak{r}_k})} [\Phi, \Psi]^{(k)} d\Lambda$$
$$= \sum_{k=1}^{u} \frac{1}{\omega_k} \int_{U(Q_{\mathfrak{r}_k})} \sum_t \sum (N(te_k s^{-1}, se_k\Lambda)d\Phi(se_k\Lambda), d\Psi(-te_k\bar{\Lambda}))$$

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where

$$t \in {}^QW^{(j)}(\mathfrak{r}_k, C_k), \quad s \in {}^QW^{(i)}(\mathfrak{r}_k, C_k).$$

The expression inside the summation sign is equal to the integral of $U({}^Q\mathfrak{r}_k)$ of

$$(5.13.1) \quad \sum_{\mathfrak{r}} \sum_{t} \sum_{\mathfrak{w}} \sum_{s} (N(te_k s^{-1}, se_k \Lambda) d\Phi(se_k \Lambda), d\Psi(-te_k \bar{\Lambda}))$$

where

$$\mathfrak{r} \in C_k \cap {}^QS^{(i)}, \quad \mathfrak{w} \in C_k \cap {}^QS^{(j)}.$$

Now consider the expression

$$(5.13.2) \quad \frac{1}{c_k} \sum_{1 \leqq c \leqq c_k} \int_{U({}^Q\mathfrak{r}_k, \Lambda_c(\epsilon, k))} \sum_{\mathfrak{r}} \sum_{t} \sum_{\mathfrak{w}} \sum_{s} N(t; s; \Lambda)$$

with $\Lambda_c(\epsilon, k) = \Lambda_c(\epsilon, \mathfrak{r}_k)$.

The inner sums are the same as for (5.12.1) above, and we have written (somewhat imprecisely) $N(t; s; \Lambda)$ for

$$(N(te_k s^{-1}, se_k \Lambda) d\Phi(se_k \Lambda), d\Psi(-te_k \bar{\Lambda})).$$

The integrand in (5.13.1) is holomorphic over $U({}^Q\mathfrak{r}_k)$ by Corollary 4.15. In (5.13.2) each of the summands in the integrand is holomorphic over the region of integration: this is by construction, and assumption SIV.

On the other hand, the integral of (5.13.1) taken over $U({}^Q\mathfrak{r}_k)$ is equal to (5.13.2): this follows by the residue argument (Cauchy's theorem) of 5.8. Multiply (5.13.2) through by $1/\omega_k$ and sum over k, $1 \leqq k \leqq u$. On making the change of variable $\Lambda' = se_k \Lambda$ we find the result to be equal to the sum from $k = 1$ to $k = u$ times $1/\omega_k c_k$ of

$$\sum_{1 \leqq c \leqq c_k} \sum_{\mathfrak{r}} \sum_{t} \sum_{s} \int_{U({}^Q\mathfrak{r}, se_k \Lambda_c(\epsilon, k))} (N(t, \Lambda) d\Phi(\Lambda), d\Psi(-t\bar{\Lambda}))$$

where now $t \in {}^QW^{(j)}(\mathfrak{r})$.

For s as above, a given chamber in $RB_\epsilon(\mathfrak{r})$ contains exactly one of the points $se_k \Lambda_c$, so that the above expression becomes

$$\sum_{k=1}^{u} \frac{1}{c_k} \sum_{1 \leqq c \leqq c_k} \sum_{\mathfrak{r}} \sum_{t} \int_{U({}^Q\mathfrak{r}, \Lambda_c(\epsilon, \mathfrak{r}))} (N(t, \Lambda) d\Phi(\Lambda), d\Psi(-t\bar{\Lambda}))$$

by the remark on holomorphicity above, and using the fact that

$$\#({}^QW(\mathfrak{r}_k, \mathfrak{r}_k)) = \#({}^QW(\mathfrak{r}_k, \mathfrak{r})).$$

Putting all this together and using (5.11.1) again, we find the statement of the lemma.

5.14. Next we look at $RB_\epsilon(\mathfrak{r}, F_\mathfrak{r}) = \cup_c C_c(\epsilon, F_\mathfrak{r})$, as before, except that the number of components will now be a number (let us call it $c(F_\mathfrak{r})$)

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which depends on $\mathfrak{r}$ and not simply C_k if $\mathfrak{r} \in C_k$. Suppose that the chamber $C_c(\epsilon, F_{\mathfrak{r}})$ contains C_c chambers of $RB_\epsilon(\mathfrak{r})$, and choose general points $\Lambda_c(\epsilon, F_{\mathfrak{r}}) \in C_c(\epsilon, F_{\mathfrak{r}})$. If $\mathfrak{r} \in C_k$ set $c_k = c(\mathfrak{r})$. Then referring to Lemma 5.13 we see that $(\mathcal{Q}_Q\phi\hat{\ }, \psi\hat{\ })$ is equal to

$$\sum_{\mathfrak{r}} c(\mathfrak{r})^{-1} \sum_{1 \leqq c \leqq c(\mathfrak{r})} \sum_{t} \left\{ \int_{U(^Q\mathfrak{r}, \Lambda_\mathfrak{r})} - \int_{U(^Q\mathfrak{r}, \Lambda_c(\epsilon, \mathfrak{r}))} (N(t, \Lambda) \ldots) \right\}$$

where $\mathfrak{r}$ now runs over ${}^QS^{(i)}$, together with the sum

$$\sum_{\mathfrak{t} \in {}^QT^{(i)}} I(\mathfrak{t}).$$

The residue argument then implies that this in turn is equal to

$$(5.14.1) \quad \sum_{\mathfrak{r}} c(\mathfrak{r})^{-1} \sum_{1 \leqq c \leqq c(F_\mathfrak{r})} C_c \times \sum_{t} \left\{ \int_{U(^Q\mathfrak{r}, \Lambda_\mathfrak{r})} - \int_{U(^Q\mathfrak{r}, \Lambda_c(\epsilon, F_\mathfrak{r}))} (N(t, \Lambda) \ldots) \right\}$$

plus

$$\sum_{\mathfrak{t}} I(\mathfrak{t}).$$

5.15. We are now in a position to begin defining the residual amalgamation. Let $T_1^{(i)}$ be the following set: an element of $T_1^{(i)}$ is a triple $(\mathfrak{r}, c, \mathfrak{w})$ with $\mathfrak{w}$ a hyperplane in $\mathfrak{r}$ so that $\Lambda_\mathfrak{r}$ and $\Lambda_c(\epsilon, F_\mathfrak{r})$ are on opposite sides of $\mathfrak{w} \cap RB_\epsilon(\mathfrak{r}, F_\mathfrak{r})$. To each such object we attach the residual Eisenstein system

$$\frac{C_c}{c(\mathfrak{r})} \operatorname{Res}_\mathfrak{w} E(., ., ., .).$$

If this is trivial, we remove the corresponding object from $T_1^{(i)}$. There is a unit normal implicit in this construction: we choose it to be the one which points from $\mathfrak{w}$ to $\Lambda_\mathfrak{r}$.

To each element $(\mathfrak{r}, c, \mathfrak{w}) \in T_1^{(i)}$ we shall associate a non-empty open convex subset $V_\mathfrak{w}^{(1)}$ of $X(\mathfrak{w}) + \mathfrak{w} \cap \mathfrak{a}^{(i)}$ as follows: Choose a polygonal path in $RB_\epsilon(\mathfrak{r})$ which starts at $\Lambda_c(\epsilon, F_\mathfrak{r})$ and stops at $\Lambda_\mathfrak{r}$ which crosses Weyl hyperplanes and α^v-hyperplanes in such a way that each hyperplane is crossed orthogonally at most once, and then only if the two points above are on opposite sides of it. Let $V_\mathfrak{w}^{(1)}$ be the connected component in $\mathfrak{w} \bigcup_{\mathfrak{t}' \neq \mathfrak{t}} RB_\epsilon(\mathfrak{r}) \cap \mathfrak{t}'$ determined by the crossing point of the path at $\mathfrak{w}$. Here $\mathfrak{t}'$ runs through the Weyl hyperplanes and α^v-hyperplanes.

5.16. Let $\mathfrak{r} \in {}^QS^{(i)}$; in the sum below we shall consider only those $(\mathfrak{r}, c, \mathfrak{w})$ such that $\mathfrak{w}$ contains the largest standard subspace of C_k, $\mathfrak{r} \in C_k$. We now come to a residue argument which will be dealt with in more detail than has hitherto been the case.

LEMMA. *If* $\Lambda_{\mathfrak{w}} \in V_{\mathfrak{w}}^{(1)}$, *then*

$$\sum_{t \in \Omega_{W^{(j)}}(\mathfrak{w})} \int_{U(\Omega_{\mathfrak{r}}, \Lambda_{\mathfrak{r}})} - \int_{U(\Omega_{\mathfrak{r}}, \Lambda_c(\epsilon, F_{\mathfrak{r}}))} (N(t, \Lambda) \ldots) d\Lambda$$

is equal to

$$\sum_{\mathfrak{w}} \sum_{t \in \Omega_{W^{(j)}}(\mathfrak{w})} \int_{U(\Omega_{\mathfrak{w}}, \Lambda_{\mathfrak{w}})} (N(t, \Lambda) d\Phi(\Lambda), d\Psi(-t\Lambda)) d\Lambda.$$

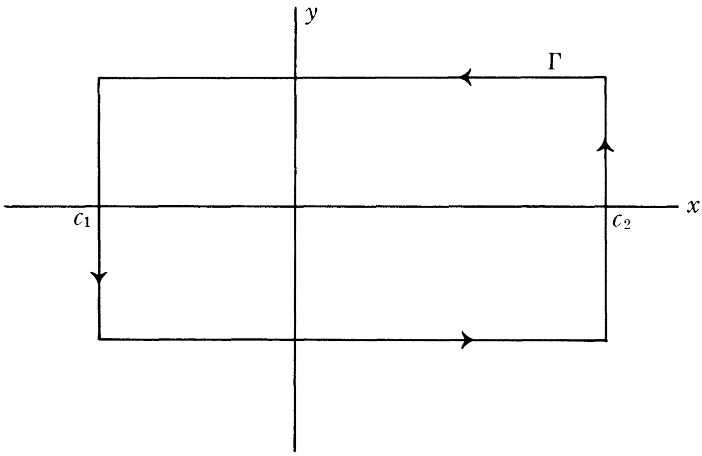
Proof. Observe that in the second expression the integrand is holomorphic over the region of integration. Denote the path in 5.15 by γ, and let Λ_1, Λ_2 be two points on γ which are at either end of the straight line segment which crosses $\mathfrak{w}$; we suppose that Λ_1 is the same side as Λ_c and Λ_2 is the same side as $\Lambda_{\mathfrak{r}}$. Then the first expression is equal to

$$\sum_{t} \sum_{\mathfrak{w}} \left\{ \int_{U(\Omega_{\mathfrak{r}}, \Lambda_2(\mathfrak{w}))} - \int_{U(\Omega_{\mathfrak{r}}, \Lambda_1(\mathfrak{w}))} (N(t, \Lambda) \ldots) d\Lambda. \right.$$

Let H be a real unit normal pointing towards $\Lambda_2(\mathfrak{w})$ from $\mathfrak{w}$ and define c_1, c_2 by $\Lambda_1(\mathfrak{w}) = \Lambda_0 + c_1 H$, $\Lambda_2(\mathfrak{w}) = \Lambda_0 + c_2 H$ where Λ_0 is the crossing point of γ with $\mathfrak{w}$ and $c_1 < 0$, $c_2 > 0$ (of course everything here depends on $\mathfrak{w}$). From the definition of $V_{\mathfrak{w}}^{(1)}$ we only have to prove the result with $\Lambda_{\mathfrak{w}} \in V_{\mathfrak{w}}^{(1)}$ replaced by Λ_0. Applying Fubini's theorem we see that the difference above is equal to

$$-\iota \sum_{t} \int_{U(\Omega_{\mathfrak{w}}, \Lambda_0)} \int_{\Gamma} (N(t, \Lambda) \ldots) d\Lambda$$

where Γ is the contour (in the fundamental domain picture) depicted below.



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By periodicity, of course, all the values of the functions on the top horizontal line are equal to corresponding values on the bottom horizontal line, and we can always arrange for both horizontal lines to contain no singularities of $N(t, \Lambda)$. Summation is still over ${}^QW^{(j)}(\mathfrak{r})$. Taking into account the residue theorem and the definition of Res $N(t, \Lambda)$, the lemma follows.

5.17. Taking into account the definitions of the Eisenstein systems associated to $\mathfrak{w} \in T_1^{(i)}$, we may now say that $(\mathcal{Q}_Q\phi^\wedge, \psi^\wedge)$ is equal to

$$(5.17.1) \quad \sum_{\mathfrak{w}} \sum_t \int_{U({}^Q\mathfrak{w}, \Lambda_{\mathfrak{w}})} (N(t, \Lambda)d\Phi(\Lambda), d\Psi(-t\Lambda))d\Lambda$$

where $\mathfrak{w}$ runs over $T^{(i)} \cup T_1^{(i)}$, and t runs over ${}^QW^{(j)}(\mathfrak{w})$, with $\Lambda_{\mathfrak{w}} \in V(\mathfrak{w})$, $W(\mathfrak{w})$, or $V_{\mathfrak{w}}^{(1)}$ as the case may be.

We now start to define Res $S^{(i)}$: $S_{m+1}^{(i)}$ will be the set of all distinct affine subspaces $\mathfrak{r}$ of dimension $m - 1$ such that there is a $\mathfrak{w} \in T^{(i)} \cup T_1^{(i)}$ with $\mathfrak{r} = \mathfrak{w}$. We associate an Eisenstein system to $\mathfrak{r}$ by

$$E_{\mathfrak{r}}(.\,,\,.\,,\,.) = \sum_{\mathfrak{w}} E_{\mathfrak{w}}(.\,,\,.\,,\,.)$$

where $\mathfrak{w}$ runs over those $\mathfrak{w} = \mathfrak{r}$. The Eisenstein systems in the sum are, of course, different in general. If the resulting $E_{\mathfrak{r}}$ is trivial, we omit $\mathfrak{r}$ from $S_{m+1}^{(i)} = \text{Res } S^{(i)}$.

5.18. The next thing to do is to show that the elements of Res $S^{(i)}$ satisfy the conditions of the induction step. That the associated Eisenstein systems are geometric is clear: the only case in doubt is an element coming from $T_1^{(i)}$, and that follows immediately because if $\mathfrak{w} \in T_1^{(i)}$, then $\mathfrak{w}$ meets a $U({}^Q\mathfrak{r})$, $\mathfrak{r} \in S^{(i)}$ so that Re $X(\mathfrak{w}) = X(\mathfrak{r})$ (which is real in fact by Proposition 4.2). Furthermore, the conditions of GI are satisfied except again if $T_1^{(i)}$. In this case we know that Re $X(\mathfrak{w}) \in {}^+\mathfrak{a}(\mathfrak{r})$ by the above argument, hence

$$\text{Re } X(\mathfrak{w}) = \sum c_\alpha \alpha, \quad c_\alpha \geqq 0$$

with $\alpha \in \Delta_P(i)$ and $\alpha^v(\mathfrak{r}^v) = 0$. Since $\mathfrak{r}^v \supseteq \mathfrak{w}^v$, the result follows.

The remaining conditions depend on the existence of $\epsilon_{\mathfrak{r}}$ and Res $V(\mathfrak{r})$ which we now construct. For $\epsilon_{\mathfrak{r}}$ observe that the number of singular hyperplanes of the Eisenstein system attached to $\mathfrak{r}$ is finite, so we can find $\epsilon_{\mathfrak{r}}$ with the property stated in SIV. For Res $V(\mathfrak{r})$ take a connected component of $RB_\epsilon(\mathfrak{r})$ with $\epsilon = \epsilon_{\mathfrak{r}}$, then GII is satisfied. If $\Lambda_{\mathfrak{r}} \in \text{Res } V(\mathfrak{r})$, suppose $\mathfrak{w}$ (a singular hyperplane of the associated Eisenstein system) meets $U(\mathfrak{r}, \Lambda_{\mathfrak{r}})$. Then $\mathfrak{w}$ is of the form

$$X(\mathfrak{r}) + (\ker \alpha^v \cap \mathfrak{r}^v) \quad \text{for some } \alpha^v$$

and, from 5.1,

$$0 \neq \alpha^v(\Lambda_{\mathfrak{r}} - X(\mathfrak{r})) = \operatorname{Re}(c_{\mathfrak{w}} - \alpha^v(X(\mathfrak{r})) = \operatorname{Re} c_{\mathfrak{w}} - \operatorname{Re}\alpha^v(X(\mathfrak{r})).$$

Thus

$$|\alpha^v(\Lambda_{\mathfrak{r}} - X(\mathfrak{r}))| \leqq \|\alpha^v\| \, \|\Lambda_{\mathfrak{r}} - X(\mathfrak{r})\| < \|\alpha^v\|\epsilon_{\mathfrak{r}}(\mathfrak{r}) \leqq \|\alpha^v\| \frac{\|\alpha^v(\Lambda_{\mathfrak{r}} - X(\mathfrak{r}))\|}{\|\alpha^v\|} = \|\alpha^v(\Lambda_{\mathfrak{r}} - X(\mathfrak{r}))\|$$

by construction, which is impossible. Thus SIII holds for Res $S^{(i)}$.

5.19. Our next task is to reexamine (Res $\mathscr{D}_Q\phi\hat{}$, $\psi\hat{}$) so as to push it into the form AVI. The expression (5.17.1) can be written (tautologically) as

$$\sum_{\mathfrak{r}}\sum_{t}\int_{U(^Q\mathfrak{r},\Lambda_{\mathfrak{r}})}(N(t, \Lambda)\ldots)d\Lambda$$

plus

$$\sum_{\mathfrak{w}}\sum_{t}\left\{\int_{U(^Q\mathfrak{w},\Lambda')} - \int_{U(^Q\mathfrak{w},\Lambda_{\mathfrak{w}})}(N(t, \Lambda)\ldots)d\Lambda\right\}.$$

In the first expression $\mathfrak{r} \in$ Res $S^{(i)}$, $t \in {}^QW^{(j)}(\mathfrak{r})$; in the second expression $\mathfrak{w} \in T^{(i)} \cup T_1^{(i)}$, $t \in {}^QW^{(j)}(\mathfrak{w})$; if $\mathfrak{w} \in T^{(i)}(C)$, then $\Lambda' \in W(\mathfrak{w})$ (convex open set), if $\mathfrak{w} \in T^{(i)}(B)$, then $\Lambda' \in V(\mathfrak{w})$ (cone), and if $\mathfrak{w} \in T_1^{(i)}$, then $\Lambda' \in V_{\mathfrak{w}}^{(1)}$. Moreover, $\Lambda_{\mathfrak{r}} \in$ Res $V(\mathfrak{r})$ in the first expression, while in the second $\Lambda_{\mathfrak{w}} \in$ Res $V(\mathfrak{r})$ if $\mathfrak{w} = \mathfrak{r} \in$ Res $S^{(i)}$.

Let us now proceed to define Res $T^{(i)}(B)$: the objects will consist of those singular hyperplanes $\mathfrak{u}$ of the Eisenstein systems associated to elements $\mathfrak{w}$ of $T^{(i)}(B) \cup T_1^{(i)}$ which meet $U(^Q\mathfrak{w})$ such that $\mathfrak{u} \cap (X(\mathfrak{w}) + \mathfrak{w}^v \cap \mathfrak{a}^{(i)})$ separates Λ and $\Lambda_{\mathfrak{w}}$. To each such hyperplane one associates the corresponding residual Eisenstein system: if this is trivial omit $\mathfrak{u}$.

One can now argue as before to find $\epsilon_{\mathfrak{u}}$, Res $V_B(\mathfrak{u})$ for Res $T^{(i)}(B)$, and to see that Res $(\mathscr{D}_Q\phi\hat{}, \psi\hat{})$ is equal to

$$\begin{aligned}(5.19.1)\quad &\sum_{\mathfrak{r}}\sum_{s}\int_{U(^Q\mathfrak{r},\Lambda_{\mathfrak{r}})}(N(s, \ )\ldots)d\Lambda \\ &+ \sum_{\mathfrak{u}}\sum_{s}\int_{U(^Q\mathfrak{u},\Lambda')}(N(s, \Lambda)\ldots)d\Lambda \\ &+ \sum_{\mathfrak{w}}\sum_{s}\left\{\int_{U(^Q\mathfrak{w},\Lambda'')} - \int_{U(^Q\mathfrak{w},\Lambda_{\mathfrak{w}})}(N(s, \Lambda)\ldots)d\Lambda\right\}\end{aligned}$$

where $\mathfrak{r} \in$ Res $S^{(i)}$, $\mathfrak{u} \in$ Res $T^{(i)}(B)$, $\mathfrak{u} \in T^{(i)}(C)$, and $s \in {}^QW^{(j)}(\mathfrak{r})$, ${}^QW^{(j)}(\mathfrak{u})$ or ${}^QW^{(j)}(\mathfrak{w})$ as the case may be, and $\Lambda_{\mathfrak{r}} \in$ Res $V(\mathfrak{r})$, $\Lambda' \in$ Res $V_B(\mathfrak{u})$ (in the second expression), while $\Lambda'' \in W(\mathfrak{w})$ and $\Lambda_{\mathfrak{w}} \in$ Res $V(\mathfrak{w})$. One point to remark here is that obtaining this result one uses condition SV in place of Corollary 4.15; this is why we need SV.

In (5.19.1) the first two expressions are in the form suitable for induction: it remains to deal with the last expression and to construct the elements of Res $T^{(i)}(C)$. This will be done more or less simultaneously. We also remark in passing that in starting the induction in practice, this is the natural point of departure, because one starts with elements of $T^{(i)}(C)$ and shifts the contour of integration to the imaginary axis.

5.20. Let $\mathfrak{w}$ be an element of $T^{(i)}(C)$; by assumption there is an associated non-empty convex open set $W(\mathfrak{w})$. We have also constructed the cone Res $V(\mathfrak{w})$ as well. Suppose $\mathfrak{u}$ is a singular hyperplane of the Eisenstein system associated to $\mathfrak{w}$ and let Λ_1, Λ_2 be two points: $\Lambda_1 \in W(\mathfrak{w})$, $\Lambda_2 \in$ Res $V(\mathfrak{w})$. Then S III implies that $\mathfrak{u}$ does not meet $U({}^Q\mathfrak{u}, \Lambda_i)$ for $i = 1$ or 2. It follows from Corollary 5.2 that $\mathfrak{u}$ meets $U({}^Q\mathfrak{w}, \Lambda)$ for some Λ on the line segment joining Λ_1 and Λ_2 if and only if

$$\alpha^v(\Lambda_i - X(\mathfrak{w})) - \text{Re}\,(c - \alpha^v(X(\mathfrak{w}))$$

have opposite signs (α^v corresponding to $\mathfrak{u}$). On the other hand, S III implies that

$$\alpha^v(\Lambda - X(\mathfrak{w})) - \text{Re}\,(c - \alpha^v(X(\mathfrak{w})) \neq 0$$

for $\Lambda \in$ Res $V(\mathfrak{w})$ or $\Lambda \in W(\mathfrak{w})$: each of these two sets is connected, so this number is of constant sign. This implies the following

LEMMA. *Let* $\mathfrak{w} \in T^{(i)}(C)$, $\mathfrak{u}$ *a singular hyperplane of the Eisenstein system belonging to* $\mathfrak{w}$. *If for some* $\Lambda_1 \in W(\mathfrak{w})$, $\Lambda_2 \in$ Res $V(\mathfrak{w})$ *it is true that* $\mathfrak{u}$ *meets* $U({}^Q\mathfrak{w}, \Lambda)$ *for some* Λ *on the line segment joining* Λ_1 *to* Λ_2, *then* $\mathfrak{u}$ *meets* $U({}^Q\mathfrak{w}, \Lambda)$ *for some* Λ *on the line segment joining* Λ_1 *to* Λ_2 *for any such* $\Lambda_1 \in W(\mathfrak{w})$, $\Lambda_2 \in$ Res $(V(\mathfrak{w}))$.

We denote by Sing $\mathfrak{w}$ the set of such $\mathfrak{u}$ in Lemma 5.20.

5.21. We take Res $T^{(i)}(C)$ to be the set of all Sing $\mathfrak{w}$ for $\mathfrak{w} \in T^{(i)}(C)$ (note that the $\mathfrak{w}$ are not necessarily distinct). If $\mathfrak{u} \in$ Res $T^{(i)}(C)$, we attach to it the residual Eisenstein system coming from $\mathfrak{w}$: if this is trivial, we omit $\mathfrak{u}$ from Res $T^{(i)}(C)$. The real unit normal in this construction is chosen to point from $\mathfrak{u}_R$ to $W(\mathfrak{w})$ where $\mathfrak{u}_R$ is in Corollary 5.4. The next step is to construct the non-empty convex open set $W(\mathfrak{u})$ for $\mathfrak{u} \in$ Res $T^{(i)}(C)$. Unfortunately this is not so easy as before and we have to be a bit more careful.

5.22. First, let $\text{Sing}_R\,(\mathfrak{w})$ denote the set of all $\mathfrak{u}_R$ as $\mathfrak{u}$ runs through Sing $(\mathfrak{w})$, and let $\text{Sing}_R = \bigcup_{\mathfrak{w}} \text{Sing}_R\,(\mathfrak{w})$. Let Int $(\mathfrak{w})$ denote the set

$$\{\mathfrak{u}_R \cap \mathfrak{v}_R | \mathfrak{v}_R, \mathfrak{u}_R \in \text{Sing}_R\,(\mathfrak{w})\}.$$

Thus Int $(\mathfrak{w})$ consists of affine subspaces of codimension 2. Write

$$\begin{aligned} \text{Int}\,(\mathfrak{w}) &= X\,\text{Int}\,(\mathfrak{w}) \,\dot\cup\, G\,\text{Int}\,(\mathfrak{w}) \\ X\,\text{Int}\,(\mathfrak{w}) &= \{\mathfrak{u} \in \text{Int}\,(\mathfrak{w}) | X(\mathfrak{w}) \in \mathfrak{u}\} \\ G\,\text{Int}\,(\mathfrak{w}) &= \{\mathfrak{u} \in \text{Int}\,(\mathfrak{w}) | X(\mathfrak{w}) \notin \mathfrak{u}\} \quad (G = \text{good}). \end{aligned}$$

Let Λ^* be any point in $W(\mathfrak{w})$, and let G Int $(\mathfrak{w})^*$ be the set of elements of G Int $(\mathfrak{w})$ which meet the line segment joining $X(\mathfrak{w})$ to Λ^*. We choose $RB(\mathfrak{w})$, $RB(\mathfrak{w})^*$ to be real balls with centres $X(\mathfrak{w})$, Λ^* respectively such that the only $\mathfrak{u} \in G$ Int $(\mathfrak{w})$ which meet the convex hull of $RB(\mathfrak{w})$ and $RB(\mathfrak{w})^*$ are those in G Int $(\mathfrak{w})^*$. If $\mathfrak{u} \in G$ Int $(\mathfrak{w})^*$, then $X(\mathfrak{w}) \notin \mathfrak{u}$ so $X(\mathfrak{w})$, $\mathfrak{u}$ determine a hyperplane in $X(\mathfrak{w}) + \mathfrak{w}^v \cap \mathfrak{a}^{(i)}$ passing through Λ^*, because $\mathfrak{u}$ meets the line segment joining $X(\mathfrak{w})$ and Λ^*. In this way $RB(\mathfrak{w})^*$ is cut up into "chambers", each of which meets $W(\mathfrak{w})$ non-trivally. We pick one of these chambers and write $W(\mathfrak{w})^*$ for its intersection with $W(\mathfrak{w})$.

Next we adjust Res $V(\mathfrak{w})$ by asking that Re (Res $V(\mathfrak{w})$) lie in the interior of the convex hull H of $\mathfrak{a}^{(i)+}$ and ${}^+\mathfrak{a}(\mathfrak{w})$. This is possible: first, the convex hull meets $X(\mathfrak{w}) + \mathfrak{w}^v \cap \mathfrak{a}^{(i)}$ because each contains $W(\mathfrak{w})$ by G II. Secondly, Re $X(\mathfrak{w}) \in \overline{{}^+\mathfrak{a}(\mathfrak{w})}$ by GI. Thus if Λ is in $H \cap (W(\mathfrak{w}) + \mathfrak{w}^v \cap \mathfrak{a}^{(i)})$, then $X(\mathfrak{w}) + t(\Lambda - X(\mathfrak{w}))$ lies in $H \cap (X(\mathfrak{w}) + \mathfrak{w} \cap \mathfrak{a}^{(i)})$ again for $0 < t \leqq 1$. Consequently we can choose Res $V(\mathfrak{w})$ (a chamber of $RB_\epsilon(\mathfrak{w})$) as desired.

5.23. There is still X Int $(\mathfrak{w})$ to deal with. Choose $\Lambda_\mathfrak{w}^* \in V(\mathfrak{w}) \cap H$ (H as above), not lying in any $\mathfrak{u} \in X$ Int $(\mathfrak{w}) \cup G$ Int $(\mathfrak{w})^*$. Then $\Lambda_\mathfrak{w}^*$, $\mathfrak{u}$ generate a hyperplane in $X(\mathfrak{w}) + \mathfrak{w}^v \cap \mathfrak{a}^{(i)}$ for $\mathfrak{u} \in X$ Int $(\mathfrak{w})$. The collection of these divides $W(\mathfrak{w})^*$ into "components". We fix one such and denote it by $W(\mathfrak{w})^*$ as well. In what follows we write

$$\Lambda_\mathfrak{w}^*(t) = (1-t)X(\mathfrak{w}) + t\Lambda_\mathfrak{w}^*.$$

LEMMA. *Let Ω be a compact subset of $W(\mathfrak{w})^*$. We can choose $\tau > 0$ such that if $0 < t < \tau$, then the line segment joining $\Lambda_\mathfrak{w}^*(t)$ to Λ misses each* $\mathfrak{u} \in$ Int $(\mathfrak{w})$, *every* $\Lambda \in \Omega$.

Proof. Choose τ' so small that $\Lambda_\mathfrak{w}^*(t) \in RB(\mathfrak{w})$ for $0 < t < \tau'$.

Now suppose $\mathfrak{u} \in X$ Int $(\mathfrak{w})$: the pair $\Lambda_\mathfrak{w}^*(t)$, $\mathfrak{u}$ generates the same hyperplane as the pair $\Lambda_\mathfrak{w}^*$, $\mathfrak{u}$, so this hyperplane must miss $W(\mathfrak{w})^*$, and thus $\mathfrak{u}$ cannot meet the line segment joining $\Lambda_\mathfrak{w}^*(t)$ and Λ (any $\Lambda \in W(\mathfrak{w})^*$).

If $\mathfrak{u} \in G$ Int $(\mathfrak{w}) \backslash$G Int $(\mathfrak{w})^*$, then $\mathfrak{u}$ cannot meet the line segment joining $\Lambda_\mathfrak{w}^*(t)$ and Λ, by the choice of $RB(\mathfrak{w})$ and $RB(\mathfrak{w})^*$, provided $t < \tau'$.

Finally, if $\mathfrak{u} \in G$ Int $(\mathfrak{w})^*$, the definition of $W(\mathfrak{w})^*$ implies that the line segment joining $X(\mathfrak{w})$ to Λ must miss $\mathfrak{u}$. I claim there is a small

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ball B with centre $X(\mathfrak{w})$ contained in $RB(\mathfrak{w})$ such that $\Lambda' \in B$ implies the line segment joining Λ' to Λ misses each $\mathfrak{u} \in G$ Int $(\mathfrak{w})^*$, each $\Lambda \in \Omega$.

Indeed suppose not, then there is a sequence $\Lambda_n' \rightarrow X(\mathfrak{w})$ such that for each Λ_n' there is a line segment to some $\Lambda_n \in \Omega$ and that line segment cuts some $\mathfrak{u}_n$. Because the collection G Int $(\mathfrak{w})^*$ is finite, one of the $\mathfrak{u}_n$ is cut infinitely often: say the sequence $(1 - t_n)\Lambda_n' + t_n\Lambda_n$ meets $\mathfrak{u}$ for $0 \leqq t_n \leqq 1$. Because Ω is compact, the set of Λ_n has a limit point $\Lambda_0 \in \Omega$. Similarly the set of t_n has a limit point $0 \leqq t \leqq 1$. Consider

$$X = (1 - t)X(\mathfrak{w}) + t\Lambda_0:$$

it is the limit of a sequence of points on $\mathfrak{u}$, hence belongs to $\mathfrak{u}$. Hence $t \neq 0$, $t \neq 1$ by the definition of $\mathfrak{u}$ and Λ_0 respectively, and X lies on the line segment joining Λ_0 and $X(\mathfrak{w})$ and $X \in \mathfrak{u}$. This is a contradiction, and completes the proof of the lemma, because all we need to do is make sure that τ' is less than the radius of B as well.

5.24. Given $\Lambda_1 \in W(\mathfrak{w})^*$, choose $\tau_1 > 0$ as in Lemma 5.23. We can suppose that $\Lambda_{\mathfrak{w}}^*(t) \in$ Res $V(\mathfrak{w}) \cap RB(\mathfrak{w})$ for $0 < t < \tau_1$. If we fix any such t, the line segment joining $\Lambda_{\mathfrak{w}}^*(t)$ to Λ_1 meets $\mathfrak{u}_R$ if $\mathfrak{u} \in$ Sing $(\mathfrak{w})$ because by definition $\mathfrak{u}_R$ separates $W(\mathfrak{w})$ and Res $V(\mathfrak{w})$.

Hence by the lemma the point of intersection determines a connected path component of $\mathfrak{u}_R \backslash \bigcup \mathfrak{u}_R \cap \mathfrak{v}_R$ where the union is over all $\mathfrak{v} \in$ Sing $(\mathfrak{w})$, $\mathfrak{u} \neq \mathfrak{v}$, and this path component does not depend on the choice of Λ_1, τ_1 or t. Indeed suppose Λ_2, τ_2, t' are also given. Then Λ_1, Λ_2 determine a compact line segment L, so choose τ_L as in Lemma 5.23. We let $\langle x, y \rangle$ denote the line joining x, y. Then the following union determines a path connecting the respective crossing points in $\mathfrak{u}_R \backslash \bigcup \mathfrak{u}_R \cap \mathfrak{v}_R$:

$$\{ \langle \Lambda_{\mathfrak{w}}^*(t), \Lambda_1 \rangle \cap \mathfrak{u}_R | \tau_L/2 \leqq t \leqq t_1 \}$$
$$\{ \langle \Lambda_{\mathfrak{w}}^*(\tau_L/2), \Lambda_L \rangle \cap \mathfrak{u}_R | \Lambda_L \in L \}$$
$$\{ \langle \Lambda_{\mathfrak{w}}^*(t), \Lambda_2 \rangle \cap \mathfrak{u}_R | \tau_L/2 \leqq t \leqq t_2 \}$$

where we make sure $\tau_L \leqq 2t_1, 2t_2$.

Let $\tilde{W}(\mathfrak{u})$ be the intersection of $\mathfrak{a}^{(i)}(R)$ with this path component; this is a non-empty convex open subset of $\mathfrak{u}_R$. To obtain Res $W(\mathfrak{u})$, we simply take

$$\iota\Lambda_R(\mathfrak{u}) + \tilde{W}(\mathfrak{u}) \cap H_{\mathfrak{u}}$$

where $H_{\mathfrak{u}}$ is the interior of the convex hull of $\mathfrak{a}^{(i)+}$ and $\overline{+\mathfrak{a}(\mathfrak{u})}$.

The intersection in this definition is non-empty; indeed $\Lambda_{\mathfrak{w}}^*(t) \in$ Res $V(\mathfrak{w}) \cap H$ and we can choose Λ_1 in the definition of the component above to lie in $H_{\mathfrak{w}}$, from the definition of $W(\mathfrak{w})$. The result follows.

5.25. Our final job is to verify that the relevant conditions are satisfied for Res $T^{(i)}(C)$: we must verify that the associated Eisenstein systems are geometric as well as GI, GII, SIII (the other conditions are empty). When we have done this, we shall then verify AVI for the entire residual amalgamation and this will finish the induction step (i.e., the proof of Proposition 5.9).

We first verify that $\operatorname{Re} X(\mathfrak{u}) \in {}^{+}\mathfrak{a}^{(i)}$ and the conditions GI, GII. In fact, GII is clear from the definition of Res $W(\mathfrak{u})$.

As for GI, observe that the interior of the convex hull of $\mathfrak{a}^{(i)+}$ and $\overline{+\mathfrak{a}(\mathfrak{u})}$ (i.e., $H_{\mathfrak{u}}$) meets Re $\mathfrak{u}$ in a non-empty open set, by the construction of $W(\mathfrak{u})$. If Λ belongs to this intersection, then the projection of Re Λ to $+\mathfrak{a}(\mathfrak{u})$ is just $\operatorname{Re} X(\mathfrak{u})$ so that $\operatorname{Re} X(\mathfrak{u})$ must belong to the interior of the convex hull of the projection of $(\mathfrak{a}^{(i)})^{+}$ to $\mathfrak{a}(\mathfrak{u})$ and $\overline{+\mathfrak{a}(\mathfrak{u})}$.

On the other hand, the projection of $\mathfrak{a}^{(i)+}$ to $\mathfrak{a}(\mathfrak{u})$ is contained in $+\mathfrak{a}(\mathfrak{u})$; indeed it is contained in $\mathfrak{a}(\mathfrak{u})^{+} \subseteq {}^{+}\mathfrak{a}(\mathfrak{u})$, as is well known. Thus $\operatorname{Re} X(\mathfrak{u})$ lies in ${}^{+}\mathfrak{a}(\mathfrak{u})$ and this is GI.

5.26. Lemma. ${}^{+}\mathfrak{a}(\mathfrak{u}) \subseteq {}^{+}\mathfrak{b}$ *where* $\mathfrak{b}$ *is the orthogonal complement of the largest distinguished subspace contained in* $\mathfrak{u}^{v}$.

Assuming the lemma we see that $\operatorname{Re} X(\mathfrak{u})$ lies in ${}^{+}\mathfrak{a}(\mathfrak{u})$ from 5.25, and hence in ${}^{+}\mathfrak{b}$ by the lemma. This implies that the collection Res $T^{(i)}(C)$ is geometric.

Proof of 5.26. Let $\{\epsilon_i\}$ be the basis dual to the simple roots $\alpha \in \Delta_P(i)$, then $\langle \epsilon_i, \alpha_j \rangle = \delta_{ij}$. Each element of ${}^{+}\mathfrak{a}(\mathfrak{u})$ can be written as a non-negative linear combination of the α_i, and the coefficient of α_i is zero if and only if $\epsilon_i \in (\mathfrak{u}^{v} \cap \mathfrak{a}^{(i)})^{*}$. The space spanned by these ϵ_i is the dual of the largest standard subspace contained in $\mathfrak{u}^{v}$, so that by definition of $\mathfrak{b}$ we see that ${}^{+}\mathfrak{a}(\mathfrak{u}) \subseteq {}^{+}\mathfrak{b}$.

5.27. We now check SIII: we want to show that if $\Lambda \in$ Res $W(\mathfrak{u})$, then no singular hyperplane of $\operatorname{Res}_{\mathfrak{u}} E(.\,,.\,,.\,,.)$ meets $U({}^{Q}\mathfrak{u}, \Lambda)$; we are implicitly assuming $\mathfrak{u}$ is a hyperplane in $\mathfrak{u} \in T^{(i)}(C)$.

Suppose Res $(\mathfrak{u}) \subseteq \mathfrak{u}$ is such a singular hyperplane: there is a hyperplane $\mathfrak{v}$ in $\mathfrak{w}$ meeting $U({}^{Q}\mathfrak{u}, \Lambda)$, $\Lambda = \iota\Lambda_R(\mathfrak{u}) + Z$, $Z \in \tilde{W}(\mathfrak{u})$, and $\mathfrak{v}$ is a singular hyperplane of the Eisenstein system associated to $\mathfrak{w}$. Hence $\mathfrak{v}$ meets $U({}^{Q}\mathfrak{w}, \Lambda')$ with $\Lambda' \in \tilde{W}(\mathfrak{u})$. But $\tilde{W}(\mathfrak{u})$ is contained in $\mathfrak{u}_R \backslash \bigcup \mathfrak{u}_R \cap \mathfrak{v}_{R'}$ the union being taken over all singular hyperplanes $\mathfrak{v}$ of the Eisenstein system associated to $\mathfrak{w}$ ($\mathfrak{u} \neq \mathfrak{v}$). This is a contradiction, as follows from the definitions.

5.28. We now deal with AVI. Recall that (Res $\mathscr{Q}_Q\phi^{\wedge}, \psi^{\wedge}$) is equal to the expression (5.19.1). In this expression we note that Λ'' and $\Lambda_{\mathfrak{w}}$ may be chosen anywhere in $W(\mathfrak{w})$, and Res $V(\mathfrak{w})$ respectively. We shall choose Λ'' to be in $W(\mathfrak{w})^{*} \cap H_{\mathfrak{w}}$; let B be a small ball about Λ''. Choose

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$\tau > 0$ so that if $0 < t < \tau$ the line joining $\Lambda_{\mathfrak{w}}{}^*$ to any point in $\bar{B}$ misses each $\mathfrak{h} \in \operatorname{Int}(\mathfrak{w})$ (Lemma 5.23). We suppose τ is chosen so that $0 < t < \tau$ implies that

$$\Lambda_{\mathfrak{w}}{}^*(t) \in \operatorname{Res} V(\mathfrak{w}) \cap RB(\mathfrak{w})$$

(where $RB(\mathfrak{w})$ is as in 5.22). Fix such a t and take $\Lambda_{\mathfrak{w}} = \Lambda_{\mathfrak{w}}{}^*(t)$. Let γ be a polygonal path from $\Lambda_{\mathfrak{w}}$ to Λ'' lying in the convex hull of $\Lambda_{\mathfrak{w}}$ and B such that γ crosses each $\mathfrak{u}_R$ ($\mathfrak{u} \in \operatorname{Sing}(\mathfrak{w})$) orthogonally. For each such $\mathfrak{u}_R$ let $\Lambda_1{}^{\mathfrak{u}}$, $\Lambda_2{}^{\mathfrak{u}}$ be the points at the ends of the straight line segment crossing $\mathfrak{u}_R$, numbered so that Λ_1 lies on the same side of $\mathfrak{u}_R$ as $\Lambda_{\mathfrak{w}}$, and Λ_2 on the same side as Λ''. Let Λ_0 be the crossing point of γ and $\mathfrak{u}_R$. Then

$$(5.28.1)\quad \int_{U(Q\mathfrak{w},\Lambda'')} - \int_{U(Q\mathfrak{w},\Lambda_{\mathfrak{w}})} (N(s,\Lambda)\ldots)d\Lambda = \sum_{\mathfrak{u}_R} \left\{ \int_{U(Q\mathfrak{w},\Lambda_2{}^{\mathfrak{u}})} - \int_{U(Q\mathfrak{w},\Lambda_1{}^{\mathfrak{u}})} (N(s,\Lambda)\ldots)d\Lambda \right\}.$$

In the sum $\mathfrak{u}$ is supposed to contain the largest standard subspace in $\mathfrak{w}^v$: otherwise $\mathfrak{u}$ is not a singular hyperplane by assumption SV, and this difference is zero.

5.29. LEMMA.

$$\sum_{s\in Q_{W^{(j)}}(\mathfrak{w})} \left\{ \int_{U(Q\mathfrak{w},\Lambda_2{}^{\mathfrak{u}})} - \int_{U(Q\mathfrak{w},\Lambda_1{}^{\mathfrak{u}})} (N(s,\Lambda)\ldots)d\Lambda \right\}$$

is equal to

$$\sum_{\mathfrak{u}'} \sum_{s\in Q_{W^{(j)}}(\mathfrak{u}')} \int_{U(Q\mathfrak{u},\Lambda_0,\mathfrak{u}'+\iota\Lambda_R(\mathfrak{u}'))} (N(s,\Lambda)\ldots)d\Lambda$$

where the first sum in the second expression runs over all $\mathfrak{u}'$ *in* Res $T^{(i)}(C)$ *such that* $\mathfrak{u}_R{}' = \mathfrak{u}_R$, $\mathfrak{u}'$ *containing the largest standard subspace of* $\mathfrak{w}^v$.

Assume the lemma for the moment: then

$$\Lambda_{0'\mathfrak{u}'} + \iota\Lambda_R(\mathfrak{u}') \in \operatorname{Res} W(\mathfrak{u}')$$

for a given $\mathfrak{u}'$ in the sum above, and this point is as good as any other in Res $W(\mathfrak{u}')$. Hence we find the sum over s of the left side of (5.28.1) is equal to

$$\sum_{\mathfrak{u}\in \operatorname{Res} T^{(i)}(C)} \sum_{s\in Q_{W^{(j)}}(\mathfrak{u})} \int_{U(Q\mathfrak{u},\Lambda_{\mathfrak{u}})} (N(s,\Lambda)\ldots)d\Lambda$$

with $\Lambda_{\mathfrak{u}} \in \operatorname{Res} W(\mathfrak{u})$.

Substituting this into the expression for $(\operatorname{Res} \mathcal{Q}_Q\phi\hat{}, \psi\hat{})$ we see that we have the condition AVI, and this completes the induction step.

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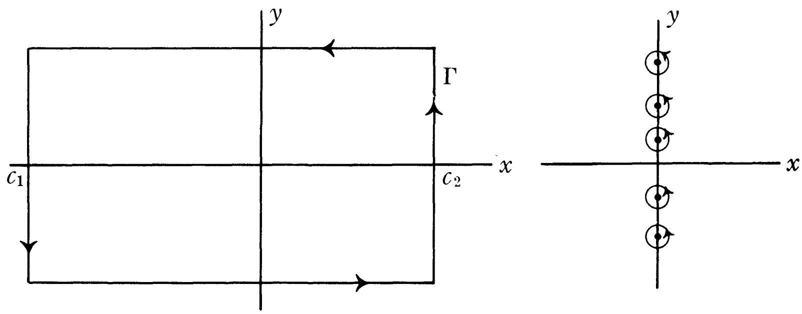
Proof of 5.29. We proceed in a fashion similar to that of Lemma 5.16. Choose a real unit normal H from $\mathfrak{u}$ pointing to $\Lambda_2^{\mathfrak{u}}$, and let $c_2 > 0$, $c_1 < 0$ be so that

$$\Lambda_1^{\mathfrak{u}} = \Lambda_{0,\mathfrak{u}} + c_1 H$$
$$\Lambda_2^{\mathfrak{u}} = \Lambda_{0,\mathfrak{u}} + c_2 H.$$

Applying Fubini's theorem, we find the difference in the lemma is

$$-\iota \sum_s \int_{U(Q\mathfrak{u}_R, \Lambda_{0,\mathfrak{u}})} \int_\Gamma (N(s, \Lambda) \ldots) d\Lambda$$

with Γ depicted below.



where

$$\mathfrak{u}_R = X_R(\mathfrak{u}) + \mathfrak{u}_R^v, \quad \mathfrak{u}_R^v = \mathfrak{u}' \cap \mathfrak{a}^{(i)}, \quad \text{any } \mathfrak{u}' = \mathfrak{u}.$$

Put $\Lambda_R(\mathfrak{u}') = c'H$. Then the integral above over Γ is equal to the integral over the contour depicted on the right, where the circles are centred about the various c', taken anticlockwise.

Using the definition of Res $N(s, \Lambda)$, this last can be put in the form stated in the lemma. As in 5.16 we can always arrange for Γ to miss any singularities.

PART II

6. Eisenstein series.

6.1. In [**5**] 2.3.4 we gave an orthogonal decomposition

$$\mathscr{L}(\xi) = \bigoplus_{\{P\}} \mathscr{L}(\{P\}, \xi).$$

Our eventual aim (Section 8) is to decompose $\mathscr{L}(\{P\}, \xi)$ further. For this we first set $\mathscr{L}(G, \{P\}, \xi)$ to be the closure of the sum of the irreducible invariant subspaces of $\mathscr{L}(\{P\}, \xi)$.

Suppose that $\{R\}$ is another association class, and write $\{R\} \geqq \{P\}$ if ${}^R\{P\}$ is non empty for some $R \in \{R\}$. For purposes of induction we need an "R-analogue" of the space $\mathscr{L}(G, \{P\}, \xi)$. To this end let $\mathfrak{P}_R = \{P(M_R)\}$ denote an association class of parabolics in M_R of the form ${}^RP = P \cap M_R$, $P \in \{P\}$. We let

$$\mathscr{C}_0(\{P(M_R)\}, \xi) = \oplus \mathscr{C}_0(P(M_R), \xi)$$

where the sum is over the elements of $\{P(M_R)\}$, and where by definition the space $\mathscr{C}_0(P(M_R), \xi)$ consists of those functions

$$\phi : {}^RP(F){}^RN(\mathbf{A})\backslash M_R(\mathbf{A}) \rightarrow \mathbf{C}$$

satisfying the conditions analogous to those of [**5**] 2.1.5. The only point here is that $Z_G(\mathbf{A})$ transform by ξ, and that in particular such a ϕ satisfies

$$\int_{Z(\mathbf{A}){}^RP(F){}^RN(\mathbf{A})\backslash M_R(\mathbf{A})} |\phi(m)|^2 dm < \infty.$$

We can then define the space $\mathscr{L}(\{{}^RP\}, \xi)$ which will be the closure in $\mathscr{L}_{M_R}(\xi)$ (see [**5**] 2.1.4, for the definition of this space) of functions of type

$$\phi^\wedge(m) = \sum_{{}^RP(F)\backslash M_R(F)} \phi(\gamma m)$$

where $\phi \in \mathscr{C}_0(P(M_R), \xi)$.

It then follows from the Fourier transform that we can write $\mathscr{L}(\{{}^RP\}, \xi)$ as a direct integral

$$\mathscr{L}(\{{}^RP\}, \xi) = \int_{D^0 M_R(\xi)}^{\oplus} \mathscr{L}(\{{}^RP\}, \zeta) d\zeta$$

or again as

$$\int_{\mathrm{Re}\zeta=\zeta_0}^{\oplus} \mathscr{L}(\{{}^RP\}, \zeta) d\zeta.$$

The "R analogue" referred to above will then be the closure in $\mathscr{L}_{M_R}(\zeta)$ of the sum of the irreducible invariant subspaces of $\mathscr{L}(\{{}^RP\}, \zeta)$; we denote it by $\mathscr{L}(M_R, \{{}^RP\}, \zeta)$.

Finally, set

$$\mathscr{L}(M_R, \{{}^RP\}, \xi) = \int_{D^0 R(\xi)}^{\oplus} \mathscr{L}(M_R, \{{}^RP\}, \zeta) d\zeta).$$

6.2. *Reminder*. The space $\mathscr{L}_{M_R}(\zeta)$ consists of measurable functions

$$\phi : M_R(F)\backslash M_R(\mathbf{A}) \rightarrow \mathbf{C},$$

transforming by ζ, for which

$$\int_{Z_{M_R}(\mathbf{A})M_R(F)\backslash M_R(\mathbf{A})} |\zeta|^{-2}(m) |\phi(m)|^2 dm < \infty.$$

6.3. With these minor inductive details out of the way, the first space we shall want to consider will consist of functions

$$\phi : N_R(\mathbf{A})R(F)\backslash G(\mathbf{A}) \to \mathbf{C}$$

satisfying the following conditions:

(i) ϕ is K-finite (on the right).

(ii) For each $g \in G(\mathbf{A})$, the function $m \to \phi(mg)$ has compact support modulo $M_R{}^0$.

(iii) There is an invariant subspace $V \subseteq \mathcal{L}(M_R, \{^R P\}, \xi)$ such that for each g, $m \to \phi(mg) \in V$.

(iv) $\phi(zg) = \xi(z)\phi(g)$, $z \in Z(\mathbf{A})$.

We write this space as $\mathcal{C}_0(R, \{^R P\}, \xi)$, so that as before

$$\mathcal{C}_0(R, \{^R P\}, \xi) = \int_{\mathrm{Re}\zeta=\zeta_0}^{\oplus} \mathcal{C}_0(R, \{^R P\}, \zeta)d\zeta.$$

6.4. Let $\Phi \in \mathcal{C}_0(R, \{^R P\}, \zeta\delta_R)$, where as usual δ_R is the modular function associated to R. We can formally define

$$E(g, \Phi) = \sum_{\gamma \in R(F)\backslash G(F)} \Phi(\gamma g). \tag{6.4.1}$$

From [**5**] 2.2.2 we see that this series converges absolutely for Re $\zeta > \delta_R$, and uniformly on subsets of $G(\mathbf{A})$ which are compact modulo $Z(\mathbf{A})$. The first question of importance to us is whether $E(g, \Phi)$ can be analytically continued to a meromorphic function. We already know this in case $\{R\} = \{P\}$ for then we are simply dealing with Eisenstein series arising from cusp forms. To carry out the analytic continuation in general will be a simple matter once we have related our present framework to that of the preceding sections.

6.5. According to Theorem 5.9 of Section 5 (the main theorem) there is a collection of affine spaces $S_M = \cup S_m{}^{(i)}$ so that if $^Q\{P\}$ is non empty then

$$^Q\mathcal{L}(\{P\}, \chi\delta_Q) = \oplus\, ^Q\mathcal{L}_m(\{P\}, \chi\delta_Q)$$

where the direct sum is over those m for which rk $Q \leqq m \leqq$ rk P, and, for each m, $^Q\mathcal{L}_m$ is the closed subspace associated with $^Q S_m$.

LEMMA. *Let* q = rank Q. *Then*

$$^Q\mathcal{L}_q \supseteq \mathcal{C}_0(Q, \{^Q P\}, \chi\delta_Q).$$

To see this, one argues as follows: pick a finite dimensional representation (irreducible) σ of K, and an irreducible invariant subspace V of $\mathcal{L}(M_Q, \{^Q P\}, \chi\delta_Q)$. The space

$$\mathcal{C}_0(Q, \{^Q P\}, \chi\delta_Q, V, \sigma) \subseteq \mathcal{C}_0(Q, \{^Q P\}, \chi\delta_Q)$$

consisting of those ϕ such that for each $g \in G(\mathbf{A})$

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(i) $m \to \phi(mg) \in V$, $m \in M_R(\mathbf{A})$

(ii) $k \to \phi(kg)$ is a matrix coefficient of σ, $k \in K$, is then finite dimensional, cf. [**2**] 5.8.

Now pick a prime v such that σ_v is trivial, and consider the local Hecke algebra $\mathscr{H}_v$ with respect to the reductive group $M_{R,v}$. It acts on the space

$$\mathscr{C}_0(Q, \{{}^QP\}, \chi\delta_Q, V, \sigma)$$

as well as the space

$${}^Q\mathscr{L}(\{P\}, \chi\delta_Q)$$

and one sees easily that ${}^Q\mathscr{L}_q$ corresponds to the discrete part of the spectrum of $\mathscr{H}_v$. Hence

$$\mathscr{C}_0(Q, \{{}^QP\}, \chi\delta_Q, V, \sigma) \subseteq {}^Q\mathscr{L}_q$$

and since

$$\bigcup_V \bigcup_\sigma \mathscr{C}_0(Q, \{{}^QP\}, \chi\delta_Q, V, \sigma) = \mathscr{C}_0(Q, \{{}^QP\}, \chi\delta_Q)$$

the result follows.

6.6. We are now ready to analytically continue our Eisenstein series. First, observe that the space ${}^R\mathscr{L}_r$ $(r = \text{rk}R)$ can be described in terms of Eisenstein systems: this was used implicitly in the proof of Lemma 6.5; indeed $\Phi \in {}^R\mathscr{L}_r$ can be written as

$$\Phi(g) = \sum_{\mathfrak{s}} E(g, F, X(\mathfrak{s}))$$

where $\mathfrak{s}$ runs through ${}^RS_r$. Consequently

$$E(g, \Phi_\zeta) = \sum_{\mathfrak{s}} \sum_{R(F)\backslash G(F)} E(\gamma g, F, X(\mathfrak{s}))e\langle\zeta, \bar{H}_{M_R}(\gamma g)\rangle$$

where as usual $\Phi_\zeta = T_\zeta\Phi$, $\Phi \in \mathscr{C}_0(Q, \{{}^QP\}, \chi\delta_Q)$, χ is a character.

According to assumption (iii) for Eisenstein systems each

$$\sum_{R(F)\backslash G(F)} E(\gamma g, F, X(\mathfrak{s}))e\langle\zeta, \bar{H}_{M_R}(\gamma g)\rangle$$

(c.f. 3.9) is also an Eisenstein system with respect to $G \supseteq P$. The result then follows from what we know about Eisenstein systems. We summarize this as a lemma.

LEMMA. *The function $E(g, \Phi)$ initially defined by* (6.4.1) *can be analytically continued to a rational function, whose singularities all lie on hyperplanes.*

We remark that the continued function $E(g, \Phi)$ is invariant by the lattice ιL_{Z_R}: indeed the series (6.4.1) is so, hence by the principle of analytic continuation so is the continued function.

7. Intertwining operators.

7.1. The next step is to define intertwining operators $M(w, \Lambda)$ for $w \in W(M_{R'}M_{R'})$ if R and R' are associate, to analytically continue them and to show that they satisfy functional equations. For this it will be necessary to remind ourselves of Langlands' Lemma 7.4, which plays a key role in what follows.

Suppose Q with Lie algebra $\mathfrak{a}_Q(\mathbf{C})$ is given such that ${}^Q\{P\} \neq \phi$. Let $S = \bigcup S^{(i)}$ be a collection of affine spaces as in 4.1–4.2. Suppose that $\mathfrak{r} \in {}^QS^{(i)}$, $\mathfrak{t} \in {}^QS^{(j)}$, and $s \in {}^QW(\mathfrak{r}, \mathfrak{t})$, then there is a linear transformation,

$$N(s, \Lambda)\colon \operatorname{Hom}(S(\mathfrak{r}), V^{(i)}) \rightarrow S(\mathfrak{t}) \otimes V^{(j)}.$$

As explained in 4.1 the collection ${}^QS$ breaks into equivalence classes, and as proved in 4.2, each equivalence class C_k possesses at least one standard subspace $\mathfrak{r}_k$. As before we set

$${}^QW(\mathfrak{r}_k, C_k) = \bigcup_{\mathfrak{t} \in C_k} {}^QW(\mathfrak{r}_k, \mathfrak{t})$$

and define, in addition,

$${}^QW_0(\mathfrak{r}_k, C_k) = \{s \in W(\mathfrak{r}_k, C_k) | s \text{ fixes } r_k{}^v \text{ pointwise}\}.$$

We let e_k be the unique element of ${}^QW(\mathfrak{r}_k, \mathfrak{r}_k)$ which is the identity on $\mathfrak{r}_k{}^v$.

Given $\mathfrak{t}_1$, $\mathfrak{t}_2$ in C_k we observe that every element of ${}^QW(\mathfrak{t}_1, \mathfrak{t}_2)$ can be written in the form te_ks^{-1} with $s \in {}^QW(\mathfrak{r}_k, \mathfrak{t}_1)$ and $t \in {}^QW(\mathfrak{r}_k, \mathfrak{t}_2)$.

7.2. Let us consider ${}^Q\mathfrak{r}_k$ as a submanifold ${}^Q\mathfrak{r}_k(\chi)$ of $D_M(\chi\delta_Q)$. Fix a component of the submanifold with base point $\omega\delta_P$, say. Then we have the function $N(w, \Lambda)$ which we can view as a (meromorphic) operator-valued function on the component

$$\{\omega\delta_P\} \times {}^Q\mathfrak{r}_k \quad ({}^Q\mathfrak{r}_k \subseteq {}^Q\mathfrak{a}_M(\mathbf{C})).$$

Let P_k be the standard parabolic corresponding to $\mathfrak{r}_k$. There are then projection maps

$$\begin{array}{ccc} {}^QD_M(\chi\delta_Q) & \xrightarrow{\pi_k} & {}^QD_k(\chi\delta_Q) \\ & \searrow \quad \swarrow & \\ & D_Q(\xi) & \end{array}$$

For $\Lambda \in {}^Q\mathfrak{r} \subseteq {}^Q\mathfrak{a}_M(\mathbf{C})$ form the matrices

$$M_Q(\Lambda) = (N(te_ks^{-1}, se_k\Lambda);\ s, t \in {}^QW(\mathfrak{r}_k, C)).$$
$$M_Q{}^k(\Lambda) = (N(te_ks^{-1}, se_k\Lambda);\ s, t \in {}^QW_0(\mathfrak{r}_k, C)).$$

If $s \in {}^QW_0^{(j)}(\mathfrak{r}_k, \mathfrak{s})$ and $t \in {}^QW_0(\mathfrak{r}_k, \mathfrak{t})$ say, then

$$\mathfrak{s} = -s\bar{X}_k + \mathfrak{r}_k^v, \quad \mathfrak{t} = -t\bar{X}_k + \mathfrak{r}_k^v,$$

and te_ks^{-1} takes $\mathfrak{s}$ to $\mathfrak{t}$, fixing $\mathfrak{r}_k^v$ in the process. Applying the transitivity axiom (iii) for Eisenstein systems, we have immediately the following result:

LEMMA. *The matrix $M_Q^k(\Lambda)$ is a function only of $\pi_k\Lambda$. It is even holomorphic in $\pi_k\Lambda$, in an obvious sense.*

7.3. If $s \in {}^QW(\mathfrak{r}_k, C)$, let $j(s)$ be the unique integer for which $(\mathfrak{r}_k)_s \in {}^QS^{j(s)}$. Then the matrix $M_Q(\Lambda)$ can be regarded as a linear transformation

$$M_Q(\Lambda): \bigoplus_{s \in {}^QW(\mathfrak{r}_k, C)} \operatorname{Hom}(S((\mathfrak{r}_k)_s), V^{j(s)}) \to \bigoplus_{s \in {}^QW(\mathfrak{r}_k, C)} S((\mathfrak{r}_k)_s) \otimes V^{j(s)}$$

which is a meromorphic function in Λ.

The matrix $M_Q^k(\Lambda)$ can also be interpreted in a similar fashion; each of these matrices has a finite dimensional range: we define the *rank* of $M_Q(\Lambda)$ or $M_Q^k(\Lambda)$ to be the dimension of its range. In [**3**] Langlands proves the following remarkable fact:

7.4. THEOREM (Langlands [**3**], Lemma 7.4). *Suppose that the collection S satisfies the following property:*

$$N(s, \Lambda) = N^*(s^{-1}, -s\bar{\Lambda})$$

for any Q and P, P′ ∈ ${}^Q\{P\}$ (this is so if S is spectral, by Proposition 4.8).

Then rank $M_Q(\Lambda)$ = rank $M_Q^k(\Lambda)$ *whenever $M_Q(\Lambda)$ is defined at* Λ.

The proof of this is long and relies ultimately on the transitivity axiom (iii) for Eisenstein systems and the fact that an element s can be written as a product of simple "reflections" (see e.g. [**5**], 1.3.4); a proof is also given in [**7**] Proposition 5.4. The proof in the function field case is essentially the same and we feel justified in omitting it.

In the applications we shall want to consider submatrices of $M_Q(\Lambda)$, determined by subsets of ${}^QW(\mathfrak{r}_k, C)$. Let A, B be sets of ${}^QW(\mathfrak{r}_k, C)$. We shall write $M_Q(A, B; \Lambda)$ for the submatrix of $M_Q(\Lambda)$ whose domain is

$$\bigoplus_{s \in A} \operatorname{Hom}(S((\mathfrak{r}_k)_s), V^{j(s)})$$

and whose range lies in

$$\bigoplus_{t \in B} S((\mathfrak{r}_k)_t) \otimes V^{j(t)}.$$

In case $A = B = {}^QW(\mathfrak{r}_k, C)$ we recover $M_Q(\Lambda)$, and in case $A = B = {}^QW_0(\mathfrak{r}_k, C)$ we recover $M_Q^k(\Lambda)$.

7.5. In the above, take Q to be G, so that C will be an equivalence class in ${}^G S$; if we wish to emphasize the subscript k and (or) G we shall write $C_k(G)$.

Let Q, Q' be associate parabolic subgroups with $\{Q\} \geqq \{P\}$. A subset of ${}^G W(\mathfrak{r}_k, C)$ of particular interest to us can be constructed as follows: Let $w \in W({}^Q\mathfrak{a}, {}^{Q'}\mathfrak{a})$ and consider those $s \in {}^G W(\mathfrak{r}_k, C)$ for which $(\mathfrak{r}_k)_s$ is standard with

$$(\mathfrak{r}_k)_s{}^v = {}^{Q'}\mathfrak{a}_{\mathbf{C}}$$
$$s|\mathfrak{r}_k{}^v = w \quad (\text{thus } \mathfrak{r}_k{}^v = {}^Q\mathfrak{a}_{\mathbf{C}}).$$

We write $|w|$ for the set of such s. The next lemma is a straightforward consequence of decomposing w into a product of elementary reflections ([**5**] 1.3.4).

LEMMA. *$|w|$ is non-empty.*

7.6. Let R be a parabolic subgroup of rank r for which ${}^R\{P\} \neq \emptyset$; we then have the collection S from the main theorem of Section 5. For the parabolic Q we shall take G itself. We want to apply Proposition 7.4 to the collection $S' = S_r$, with respect to the equivalence relation induced by G. Thus ${}^G S'$ breaks into equivalence classes C', where $\mathfrak{r}$, $\mathfrak{t}$ are equivalent if there is an element s of ${}^G W(\mathfrak{r}, \mathfrak{t})$ for which $\mathfrak{r}_s = \mathfrak{t}$.

To each equivalence class C' there is canonically associated an association class of parabolic subgroups $\{C'\}$: one takes the association class determined by any standard element of C'. Because each C' consists of affine subspaces it is possible that distinct C' will give the same association class.

Accordingly if we set ${}^R\mathscr{L}_r(\{P\}, \chi\delta_R, \mathfrak{t})$ to consist of the functions $E(g, F, X(\mathfrak{t}))$ where $F \in \operatorname{Hom}(S(\mathfrak{t}), V^{(i)})$ with $P^{(i)} \in {}^R\{P\}$, then with $r = \operatorname{rank} R$,

$${}^R\mathscr{L}_r(\{P\}, \chi\delta_R) = \bigoplus_{\mathfrak{t}\in {}^R S_r} {}^R\mathscr{L}_r(\{P\}, \chi\delta_R, \mathfrak{t}).$$

Set, for C' as above,

$${}^R\mathscr{L}_r(\{P\}, \chi\delta_R, C') = \bigoplus_{\mathfrak{t}\in C'} {}^R\mathscr{L}_r(\{P\}, \chi\delta_R, \mathfrak{t})$$

so that

$${}^R\mathscr{L}_r(\{P\}, \chi\delta_R) = \bigoplus_{C'} {}^R\mathscr{L}_r(\{P\}, \chi\delta_R, C').$$

The only C' which can occur in this sum are those for which $\{C'\} = \{R\}$, and $\mathfrak{t} \in C'$ contributes if and only if $\mathfrak{t}^v \leftrightarrow R$.

We want to apply Propoposition 7.4: for the collection S we shall take $S' = S_r$ (as in the main theorem, Section 5). For Q we shall take G itself, for C we shall take an equivalence class C' such that $\{C'\} = \{R\}$. According to 7.4 the rank of $M_G(\Lambda)$ is equal to the rank of $M_G = M_G{}^k(\chi\delta_Q)$, whenever $M_G(\Lambda)$ is defined.

7.7. *Remark*. We pointed out that distinct C' can give the same $\{C'\}$. Equally one could have $\mathfrak{r}$, $\mathfrak{s}$ in C', both distinct and standard, corresponding to the same element in $\{C'\}$. Lemma 7.5 implies that at least each element of $\{C'\}$ does correspond to a standard subspace in C'.

7.8. Now let $R, R' \in \{R\}$; we are going to construct a linear transformation $M(w, \Lambda)$, for each $w \in W(\mathfrak{a}_{R'}\mathfrak{a}_{R'})$, and $\Lambda \in \mathfrak{a}_R(\mathbf{C})$

$$\mathscr{C}_0(R, \{^R P\}, \chi\delta_R) \subseteq \bigoplus_{C'} {}^R\mathscr{L}_r(\{P\}, \chi\delta_R, C')$$

we shall construct a transformation $M(w, \Lambda)$

$$M(w, \Lambda): {}^R\mathscr{L}_r(\{P\}, \chi\delta_R, C') \rightarrow {}^{R'}\mathscr{L}_r(\{\mathrm{P}\}, \chi^{w-1}\delta_{R'}, C').$$

Let $C' = C_G$ be an equivalence class, as in 7.6, with respect to G. Let C_{st} be the set of standard elements in C_G; if $\mathfrak{r} \in C_{st}$, then $\mathfrak{r}^v$ corresponds to a standard parabolic R for which $^R\{P\} \neq \emptyset$, and then $\mathfrak{r}$ lies in an equivalence class $\{\mathfrak{r}\}_R$ with respect to R: if $\mathfrak{t} \in \{\mathfrak{r}\}_R$, then $\mathfrak{t}^v = \mathfrak{r}^v$ and $\mathfrak{t} \in C_{st}$. In this way

$$C_G \supseteq C_{st} = \amalg C_R$$

where each C_R is an equivalence class $\{\mathfrak{r}\}_R$ as above.

Reminder. If $\mathfrak{r} \in C_{st}$ is a member of $^R S_r$, then $\mathfrak{r}^v$ must be of the form $X(\mathfrak{r}^v) + \mathfrak{a}_R(\mathbf{C})$.

We may then write, with $C_R \leftrightarrow R$ in each case

$${}^R\mathscr{L}_r(\{P\}, \chi\delta_R, C_G) = \bigoplus_{C_R} {}^R\mathscr{L}_r(\{P\}, \chi\delta_R, C_R)$$

(cf. Remark 7.7) where the spaces on the right are defined in the obvious way, we see that we need only define $M(w, \Lambda)$ on each space on the right.

7.9. Now suppose that $\mathfrak{r} \in C_R$ as above, and let $R' \in \{R\}$, $w \in W(\mathfrak{a}_{R'}\mathfrak{a}_{R'})$. Choose $w_0 \in |w|$ as in 7.5 and set $\mathfrak{t} = \mathfrak{r}_{w_0}$. Then $\mathfrak{t}$ will be a member of $C_{R'}$ say, another equivalence class $\{\mathfrak{t}\}_{R'}$ in C_G.

We then have matrices

$$M_G({}^R W(\mathfrak{r}, C_R), |w|; \Lambda + X(\mathfrak{r})), \ \Lambda \in \mathfrak{r}^v$$
$$M_G(|w^{-1}|, {}^{R'}W(\mathfrak{t}, C_{R'}); \Lambda' + X(\mathfrak{t}))$$

with domains and ranges denoted respectively by

$$\mathrm{Hom}_R \quad , \quad S_R \quad .$$
$$\mathrm{Hom}_{R'} \quad , \quad S_{R'} \quad .$$

There are natural isomorphisms, defined component wise,

$$T_H(\Lambda): \mathrm{Hom}_R \rightarrow \mathrm{Hom}_{R'}$$
$$T_S(\Lambda'): S_R \rightarrow S_{R'}$$

such that the diagram

$$(7.9.1)\qquad \begin{array}{ccc} \mathrm{Hom}_R & \xrightarrow{M_G({}^RW(\mathfrak{r},\, C_R),\, |w|;\, \Lambda + X(\mathfrak{r}))} & S_R \\ {\scriptstyle T_H(\Lambda)}\downarrow & & \downarrow{\scriptstyle T_S(w\Lambda)} \\ \mathrm{Hom}_{R'} & \xrightarrow{M_G(|w^{-1}|,{}^{R'}W(\mathfrak{t},\, C_{R'}),\, w\Lambda + X(\mathfrak{t}))} & S_{R'} \end{array}$$

is commutative.

Thus

$$\begin{aligned} &\text{Image } (T_S(w\Lambda)\,.\, M_G({}^RW(\mathfrak{r},\, C_R)\, \ldots .)) \\ &= \text{Image } (M_G(|w^{-1}|,\, {}^{R'}W(\mathfrak{t},\, C_{R'})\, \ldots .)) \subseteq \text{Image } M_G(\mathfrak{t}) \end{aligned}$$

by Proposition 7.4 (we are writing $M_G(\mathfrak{t})$ to denote the dependence on $\mathfrak{t}$).

There are surjective (by definition) maps

$$\begin{aligned} &E_{\mathfrak{r}}\colon \text{Domain } (M_G(\mathfrak{r})) \to {}^R\mathcal{L}_r(\{P\},\, \chi\delta_R,\, C_R) \\ &E_{\mathfrak{r}}\colon \text{Domain } M_G(\mathfrak{t}) \to {}^{R'}\mathcal{L}_r(\{P\},\, \chi^{w-1}\delta_{R'},\, C_{R'}) \end{aligned}$$

given by the following rule: if, for example

$$F = \oplus\, F_s \in \bigoplus_s \mathrm{Hom}(S(\mathfrak{r}_s),\, V^{j(s)})$$

$s \in {}^RW(\mathfrak{r},\, C_R)$ $(\subseteq {}^GW_0(\mathfrak{r},\, C_G)\,!)$ then

$$E_{\mathfrak{r}}(F) = \bigoplus_s E(\cdot,\, F_s, X(\mathfrak{r}_s)).$$

7.10. We now tentatively define $M(w,\, \Lambda)$. Suppose that

$$\Phi \in {}^R\mathcal{L}_r(\{P\},\, \chi\delta_R,\, C_R).$$

Choose F_Φ in Domain $M_G(\mathfrak{r})$ such that $E_{\mathfrak{r}}(F_\Phi) = \Phi$. From the commutativity of (7.9.1) we see that

$$G_\Phi = T_S(w\Lambda) \cdot M_G({}^RW(\mathfrak{r},\, C_R),\, |w|,\, \Lambda + X(\mathfrak{r}))F_\Phi$$

is an element of Image $(M_G(\mathfrak{t}))$. Choose a $\Phi_{\mathfrak{t}}$ in Domain $(M_G(\mathfrak{t}))$ so that $M_G(\mathfrak{t})\Phi_{\mathfrak{t}}$ is this element, and set

$$M(w,\, \Lambda)\Phi = E_{\mathfrak{t}}(\Phi_{\mathfrak{t}}).$$

Of course we must verify that the left side really is independent of the choices made above.

7.11. If $\Phi \in {}^R\mathcal{L}_r(\{P\},\, \chi\delta_R,\, C_R)$ we know that

$$\Phi = \sum_{\mathfrak{s} \in C_R} E(g,\, F_{\mathfrak{s}}, X(\mathfrak{s}))$$

so that $E(g,\, \Phi,\, \Lambda)$ then makes sense by axiom (iii) for Eisenstein systems

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and is, in fact, a rational function; c.f. the proof of Lemma 6.6. This is the sense in which the following lemma is to be interpreted.

LEMMA. *Let* $\Phi \in {}^R\mathscr{L}_\tau(\{P\}, \chi\delta_R, C_R)$. *Then always*

$$E(g, \Phi, \Lambda) = E(g, M(w, \Lambda)\Phi, w\Lambda).$$

Proof. By a well-known theorem of Langlands (see [**5**] 2.2.9 for example) it is enough to compare the constant terms of each side for $P' = P^{(j)} \in \{P\}$, and show them to be equal. Writing out the definitions of $E^{P'}(g, \Phi, \Lambda)$ and $E^{P'}(g, M(w, \Lambda)\Phi, w\Lambda)$ respectively, and making the appropriate change of variables with respect to Weyl group elements, we see that this amounts to showing that the elements $T_H(\Lambda)F_\Phi$ and Φ_{t} (notation of 7.10) have the same images under the matrices

$$M_G(|w^{-1}|, {}^GW(\mathrm{t}, C_G); w\Lambda + X(\mathrm{t}))$$

and

$$M_G({}^{R'}W(\mathrm{t}, C_{R'}), {}^GW(\mathrm{t}, C_G); w\Lambda + X(\mathrm{t}))$$

respectively.

Now Φ_{t} and $T_H(\Lambda)F_\Phi$ can obviously be regarded as elements in the domain of the matrix

$$M_G(w\Lambda + X(\mathrm{t}))$$

and then we simply have to show that their difference lies in the kernel of this latter matrix.

On the other hand, Φ_{t} is such that

$$\begin{aligned} M_G(\mathrm{t})\Phi_{\mathrm{t}} &= G_\Phi \quad \text{(notation of 7.10)} \\ &= M_G({}^{R'}W(\mathrm{t}, C_{R'}), {}^GW(\mathrm{t}, C_G); w\Lambda + X(\mathrm{t}))\Phi_{\mathrm{t}}, \end{aligned}$$

hence $\Phi_{\mathrm{t}} - T_H(\Lambda)F_\Phi$ lies in the kernel of

$$M_G({}^{R'}W(\mathrm{t}, C_{R'}), {}^GW(\mathrm{t}, C_G); w\Lambda + X(\mathrm{t}))$$

by the commutativity of the diagram (7.9.1). The result follows, because

$$M_G(w\Lambda + X(\mathrm{t}))$$

and

$$M_G({}^{R'}W(\mathrm{t}, C_{R'}), {}^GW(\mathrm{t}, C_G); w\Lambda + X(\mathrm{t}))$$

have the same rank by 7.4 and hence the same kernel.

7.12. Suppose now that $M(w, \Lambda)\Phi$ and $M'(w, \Lambda)\Phi$ are constructed as in 7.10 but corresponding to different choices. By Lemma 7.11 we find that

$$E(g, M(w, \Lambda)\Phi, w\Lambda) = E(g, M'(w, \Lambda)\Phi, w\Lambda).$$

The result we seek now follows from

7.13. LEMMA. *Let $\Phi(\Lambda)$ be meromorphic with values in*

$$^{R'}\mathscr{L}_r(\{P\}, C_{R'}, \nu\delta_{R'}).$$

If $E(g, \Phi(\Lambda), \Lambda) \equiv 0$, *then* $\Phi(\Lambda) \equiv 0$.

Proof. We have

$$E(g, \Phi(\Lambda), \Lambda) = E(g, F_\Phi(\Lambda), \Lambda) \equiv 0$$

and this implies that

$$M_G(^{R'}W(\mathfrak{t}, C_{R'}), {}^GW(\mathfrak{t}, C_G), \Lambda + X(\mathfrak{t}))F_\Phi = 0$$

so that $F_\Phi \in \ker (M_G(\mathfrak{t}))$. But by construction

$$\Phi(\Lambda) = E_\mathfrak{t}(F_\Phi)$$

and taking constant terms as in Lemma 7.11, the claim follows.

7.14. The arguments of 7.11–7.13 show also that $M(w, \Lambda)$ is a function on $\iota L_{Z_R}\backslash\mathfrak{a}_R(\mathbf{C})$. We leave to the reader to show that the $M(w, \Lambda)$ really do give rise to

$$M(w, \Lambda): \mathscr{C}_0(R, \{^RP\}, \chi\delta_R) \rightarrow \mathscr{C}_0(R', \{^{R'}P\}, \chi^{w^{-1}}\delta_{R'}).$$

In any case, one way of doing this is by 8.7 below.

The next item for us is to show that the $M(w, \Lambda)$ satisfy the expected functional equations.

PROPOSITION. *Let* $w_1 \in W(\mathfrak{a}_R, \mathfrak{a}_{R'})$, $w_2 \in W(\mathfrak{a}_{R'}, \mathfrak{a}_{R''})$. *Then*

$$M(w_2w_1, \Lambda) = M(w_2, w_1\Lambda)M(w_1, \Lambda).$$

Proof. Indeed, let $\Phi \in {}^R\mathscr{L}_r(\{P\}, C_R, \chi\delta_R)$, and consider

$$E(g, M(w_2w_1, \Lambda)\Phi, w_2w_1\Lambda)$$

and

$$E(g, M(w_2, w_1\Lambda)M(w_1, \Lambda)\Phi, w_2w_1\Lambda).$$

According to Lemma 7.11, these are equal, and applying 7.13 the result follows.

7.15. The next few remarks are pertinent for the lemma that will follow. Consider the map

$$M_G(\mathfrak{r}): \mathrm{Hom}_R \rightarrow S_R.$$

This is the composition (by definition) of the map

$$E_\mathfrak{r}: \mathrm{Hom}_R \rightarrow {}^R\mathscr{L}_r(\{P\}, \chi\delta_R, C_R)$$

with the constant term map

$$C_\mathfrak{r}: {}^R\mathscr{L}_r(\{P\}, \chi\delta_R, C_R) \rightarrow S_R.$$

On the other hand, $E_{\mathfrak{r}}$ is a map of finite order, hence from Section 2 it has an adjoint $E_{\mathfrak{r}}^*$

$$^{R}\mathcal{L}_{r}(\{P\}, \chi\delta_R, C_R) \rightarrow S_R.$$

Proposition 4.14 implies that if $F, H \in S_R$, then

$$(E_{\mathfrak{r}}(F), E_{\mathfrak{r}}(H))_{\mathcal{L}} = (M_G(\mathfrak{r})F, H)_{S_R}$$

where $(\ , \ )_{\mathcal{L}}$ denotes the inner product in $^{R}\mathcal{L}_{r}(\{P\}, \chi\delta_R, C_R)$. But this inner product is also equal to

$$(C_{\mathfrak{r}} \cdot E_{\mathfrak{r}}(F), H)_{S_R}.$$

Take $E_{\mathfrak{r}}(F) = \Phi$; we then have

$$(\Phi, E_{\mathfrak{r}}(H)) = (C_{\mathfrak{r}}\Phi, H)$$

and since $E_{\mathfrak{r}}$ is surjective, this implies that

$$E_{\mathfrak{r}}^* = C_{\mathfrak{r}}.$$

7.16. LEMMA. *The adjoint of* $M(w, \Lambda)$ *is* $M(w^{-1}, -w\bar{\Lambda})$.

Proof. We only need to check this for

$$M(w, \Lambda): {}^{R}\mathcal{L}_{r}(\{P\}, \chi\delta_R, C_R) \rightarrow {}^{R'}\mathcal{L}_{r}(\{P\}, \chi^{w^{-1}}\delta_{R'}, C_{R'}).$$

Let $F \in \mathrm{Hom}_R$; then

$$\begin{aligned} &C_{\mathfrak{t}} \cdot M(w, \Lambda)E_{\mathfrak{r}}(F) \\ &= T_S(w\Lambda) \cdot M_G({}^{R}W(\mathfrak{r}, C_R), |w|, \Lambda + X(\mathfrak{r}))F. \end{aligned}$$

Indeed $C_{\mathfrak{t}} \cdot M(w, \Lambda)E_{\mathfrak{r}}(F)$ is equal to $C_{\mathfrak{t}} \cdot E_{\mathfrak{t}}(T)$ where $M_G(\mathfrak{t})T$ is equal to

$$T_S(w\Lambda) \cdot M_G({}^{R}W(\mathfrak{r}, C_R), |w|, \Lambda + X(\mathfrak{r}))F.$$

Thus it is equal to $M_G(\mathfrak{t})T$ (since $M_G(\mathfrak{t}) = C_{\mathfrak{t}} \cdot E_{\mathfrak{t}}$), i.e., to

$$T_S(w\Lambda) \cdot M_G({}^{R}W(\mathfrak{r}, C_R), |w|, \Lambda + X(\mathfrak{r}))F.$$

(We know that $M(w, \Lambda)\Phi$ is independent of the choice of F_Φ such that $E_{\mathfrak{t}}(F_\Phi) = \Phi$, so for $\Phi = E_{\mathfrak{r}}(F)$, we choose $F = F_\Phi$.)

Taking adjoints we find that

$$\begin{aligned} &C_{\mathfrak{r}} \cdot M(w, \Lambda)^* \cdot E_{\mathfrak{t}} \\ &= M_G(|w|, {}^{R}W(\mathfrak{r}, C_R), -\bar{\Lambda} + X(\mathfrak{r})) \cdot T_H(-w\bar{\Lambda}) \\ &= T_S(-\bar{\Lambda}) \cdot M_G(|w^{-1}|, {}^{R'}W(\mathfrak{t}, C_{R'}), -w\bar{\Lambda} + X(\mathfrak{t})). \end{aligned}$$

By the same argument as above, this is equal to

$$C_{\mathfrak{r}} \cdot M(w^{-1}, -w\bar{\Lambda}) \cdot E_{\mathfrak{t}}.$$

Since $C_{\mathfrak{r}}$ is injective and $E_{\mathfrak{t}}$ is surjective, the lemma follows.

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7.17. COROLLARY. *$M(w, \Lambda)$ is holomorphic for* Re $\Lambda = 0$, *and hence* $E(g, \Phi, \Lambda)$ *is as well.*

The first assertion is obvious, the second follows from 4.14.

7.18. *Remark.* We saw in 4.14 that the matrix $M_G(\Lambda)$ may have singularities along the axis $U(\mathfrak{r})$ and that one must be content with a weaker statement, viz. that of 4.14 (iii). The results above imply that certain submatrices of $M_G(\Lambda)$ are however analytic along this axis.

8. The spectral decomposition.

8.1. We employ the same notation as that of Section 7. In particular C_G denotes an equivalence class under G in S_r, and $\mathfrak{r}$ is a standard element in C_G; we suppose that $\{C_G\} = \{R\}$.

LEMMA. *Suppose* F_R: $\iota L_{Z_R}\backslash\iota\mathfrak{a}_R \to U$ *is a continuous function,* U *a finite dimensional subspace of* ${}^R\mathcal{L}_r(\{P\}, \chi\delta_R, C_G)$. *Then*

$$\int_{\operatorname{Re}\Lambda=0} E(g, F_R(\Lambda), \Lambda)d\Lambda$$

exists, and lies in $\mathcal{L}(\xi)$.

Proof. The first assertion is clear because $E(\cdot, F_R(\Lambda), \Lambda)$ is regular on the (compact) imaginary axis.

To prove the second assertion, argue as follows: Firstly, we can replace the $C_G = \amalg C_R$ in ${}^R\mathcal{L}_r(\{P\}, \chi\delta_R, C_G)$ by a C_R. Secondly, writing

$$F_R(\Lambda)(g) = \sum \Phi_i(\Lambda)F_i(g)$$

where $\{F_i\}$ is a finite basis for U, we can reduce the problem to showing that

$$\int_{\operatorname{Re}\Lambda=0} E(g, F, \Lambda)\Phi(\Lambda)d\Lambda$$

is an element of $\mathcal{L}(\xi)$ when Φ: $\iota L_{Z_R}\backslash\iota\mathfrak{a}_R \to \mathbf{C}$ is continuous. Finally, we may suppose $F = E_\mathfrak{r}(F')$.

Set

$$\operatorname{Hom}(C_G) = \bigoplus_w \operatorname{Hom}(S(\mathfrak{r}_w), V(\{P\}))$$

where

$$V(\{P\}) = \bigoplus_{\{P\}} \bigoplus_\omega \mathcal{E}_0(P, \omega\delta_P) \quad \text{and} \quad w \in {}^GW(\mathfrak{r}, C_G),$$

and define a function

$$F_{C_G}: U(\mathfrak{r}) \to \operatorname{Hom}(C_G)$$

by setting $F_{C_G} = \bigoplus_w F_w$, where

$$F_w(\Lambda + X(\mathfrak{r})) = \begin{cases} \Phi(w\Lambda)F' & \text{if } \ w \in {}^GW(\mathfrak{r}, \mathfrak{r}) \\ 0 & \text{otherwise.} \end{cases}$$

Consider

$$(8.1.1) \qquad \int_{U(\mathfrak{r})} (M_G(\Lambda)F_{C_G}, F_{C_G})_{\mathfrak{s}} d\Lambda.$$

By definition this is equal to

$$\int_{U(\mathfrak{r})} \sum_t \sum_s \Phi(t\Lambda')\overline{\Phi(s\Lambda')}(N_G(te_k s^{-1}, se_k\Lambda)F', F')d\Lambda$$

where

$$\Lambda' = \Lambda - X(\mathfrak{r}) \quad \text{and} \quad t, s \in {}^G W(\mathfrak{r}, \mathfrak{r})$$

so that everything comes down to the behaviour of

$$(N_G(te_k s^{-1}, se_k\Lambda)F', F')$$

on $U(\mathfrak{r})$. Now $te_k s^{-1} \in |w|$ where $w = te_k s^{-1}|\mathfrak{r}^v$ and then $N_G\ (te_k s^{-1}, se_k\Lambda)$ forms an entry in the matrix

$$M_G({}^R W(\mathfrak{r}, C_R), |w|, \Lambda)$$

which we know to be holomorphic on $U(\mathfrak{r})$ (cf. remark 7.18). Thus the integral (8.1.1) exists.

On the other hand, up to a constant of integration,

$$\int_{\mathrm{Re}\,\Lambda=0} E(g, F_R(\Lambda), \Lambda)d\Lambda = \frac{1}{\#(C_G)}\int_{U(\mathfrak{r})} E(g, F_{C_G}, \Lambda')d\Lambda'$$

where $\#(C_G) = |{}^G W(\mathfrak{r}, \mathfrak{r})|$, and the fact that (8.1.1) exists amounts to saying that F_{C_G} is an element of the space $\bigoplus_i \mathscr{T}_k(V^{(i)})$ (cf. the proof of Proposition 4.14). But then the function

$$\int_{U(\mathfrak{r})} E(g, F_{C_G}, \Lambda')d\Lambda'$$

is an element of $\mathscr{L}(\xi)$ (ibid), which is what we want.

8.2. Let F be as in Lemma 8.1, and suppose in addition that

$$H: {}_\iota L_{Z_{R'}} \backslash {}_\iota \mathfrak{a}_{R'} \rightarrow V$$

is a continuous function where

$$V \subseteq \bigoplus_{w \in W(R,R')} {}^{R'}\mathscr{L}_r(\{P\}, \chi^{w^{-1}}\delta_{R'}, C_G')$$

is a finite dimensional subspace, and $R, R' \in \{R\}$.

LEMMA. *The inner product in* $\mathscr{L}(\xi)$ *of*

$$\int_{\mathrm{Re}\,\Lambda=0} E(g, F(\Lambda), \Lambda)d\Lambda$$

and

$$\int_{\mathrm{Re}\,\Lambda'=0} E(g, H(\Lambda'), \Lambda')d\Lambda'$$

is equal to

$$\sum_{w\in W(\mathfrak{a}_{R'},\mathfrak{a}_{R'})} \int_{\mathrm{Re}\,\Lambda=0} (M(w, \Lambda)F(\Lambda), H(w\Lambda))d\Lambda.$$

Proof. As before, we can replace $C_G(C_G')$ by a $C_R(C_{R'}')$ as the case may be, and reduce to the case where

$$F(\Lambda) = \Phi(\Lambda)F_R$$
$$H(\Lambda') = \Psi(\Lambda')H_{R'}$$

where $\Phi(\Lambda)$, F_R(resp. $\Psi(\Lambda)$, $H_{R'}$) are as in 8.1 (resp. the analogues of the corresponding objects in 8.1). Define F_{C_G} as in 8.1, and the function $H_{C_G} = \oplus H_w$ by

$$H_w(\Lambda + X(\mathfrak{r})) = \begin{cases} 0 & \text{if } w' \notin {}^GW(\mathfrak{r}, \mathfrak{f}) \\ \Psi(w'\Lambda)H' & \text{if } w' \in {}^GW(\mathfrak{r}, \mathfrak{f}) \end{cases}$$

where $\mathfrak{f}$ is a standard element for $C_{R'}'$.

Using the proof of Proposition 4.14 again, we see that the inner product in question is equal to

$$(8.2.1) \qquad \frac{1}{\#(C_G)}\int_{U(\mathfrak{r})} (M_G(\Lambda)F_{C_G}, H_{C_G})d\Lambda$$

where $\#(C_G)$ is as before. The integrand is a sum over the various s, t as before, and each summand is holomorphic on the axis $U(\mathfrak{r})$. Consequently (8.2.1) is equal to (up to a constant of integration)

$$(8.2.2) \qquad \sum_{s\in {}^GW(\mathfrak{r},\mathfrak{f})} \int_{\mathrm{Re}\,\Lambda=0} \Phi(\Lambda)\overline{\Psi(s\Lambda)}(T_S(s\Lambda)N_G(s, \Lambda + X(\mathfrak{r})F_R', H_{R'}').$$

Let

$${}^GW(\mathfrak{r}, \mathfrak{f}) = \amalg |w| \cap {}^GW(\mathfrak{r}, \mathfrak{f})$$

with w running through $W(a_R, a_{R'})$: the sets on the right have 1 or 0 elements in them. Let $W(\mathfrak{r}, \mathfrak{f}; R, R')$ be the subset of $W(\mathfrak{a}_R, \mathfrak{a}_{R'})$ consisting of w for which the corresponding term on the right is non-empty, and let s_w be the corresponding element in $|w| \cap {}^GW(\mathfrak{r}, \mathfrak{f})$.

Then $(T_S(s_w\Lambda)N_G(s_w, \Lambda + X(\mathfrak{r}))F_{R'}'H_{R'}')$ is simply

$$(T_S(s_w\Lambda)M_G({}^RW(\mathfrak{r}, C_R), |w|, \Lambda + X(\mathfrak{r}))F_R', H_{R'}')$$

so (8.2.2) becomes (cf. 7.10, and proof of 7.16)

$$\sum_{{}^GW(\mathfrak{r},\mathfrak{f},R,R')} \int (C_{\mathfrak{f}} \cdot M(w, \Lambda) \cdot E_{\mathfrak{f}}(F_R'), H_{R'}')\Phi(\Lambda)\overline{\Psi(w\Lambda)}d\Lambda$$

$$= \sum \int_{\mathrm{Re}\,\Lambda=0} (M(w, \Lambda) \cdot E_{\mathfrak{r}}(F_R'), E_{\mathfrak{f}}(H_{R'}'))\Phi(\Lambda)\overline{\Psi(w\Lambda)}$$

(remember that $C_{\mathfrak{t}}$ is the adjoint of $E_{\mathfrak{t}}$, cf. Section 7.15)

$$= \sum_{w} \int_{\operatorname{Re}\Lambda=0} (M(w, \Lambda)F_R, H_{R'})\Phi(\Lambda)\overline{\Psi(w\Lambda)}d\Lambda$$

$w \in {}^G W(\mathfrak{r}, \mathfrak{f}, R, R')$. Now

$${}^G W(\mathfrak{r}, \mathfrak{f}, R, R') \subseteq W(\mathfrak{a}_R, \mathfrak{a}_{R'}),$$

so to complete the proof we only need to show that if

$$w \notin {}^G W(\mathfrak{r}, \mathfrak{f}, R, R'),$$

then it contributes nothing to the integral, i.e.,

$$(M(w, \Lambda)F(\Lambda), H(w\Lambda)) = 0$$

but this follows because then $M(w, \Lambda)F(\Lambda)$ lies in

$${}^{R'}\mathcal{L}_{\mathfrak{r}}(\{P\}, \chi^{w-1}\delta_{R'}, C_{R'}{}')$$

where $C_{R'}{}' \leftrightarrow \mathfrak{r}_w$ and this is orthogonal to

$$\bigoplus_w {}^{R'}\mathcal{L}_{\mathfrak{r}'}(\{P\}, \chi^{w-1}\delta_{R'}, C_{R'}),$$

by construction, since then $C_{R'}{}' \neq C_{R'}$ as a moment's reflection shows.

8.3. Our final task is to exhibit the spectral decomposition of $\mathcal{L}(\{P\}, \xi)$ using the results on Eisenstein series and intertwining operators. For this, we introduce the space

$$\mathcal{H}(\{R\}, \{P\}, C_G, K')$$

consisting of collections $\{F_R\}$ of measurable functions

$$F_R: \iota L_{Z_R}\backslash\iota\mathfrak{a}_R \rightarrow \bigoplus_{\{\chi\}} {}^R\mathcal{L}_{\mathfrak{r}}(\{P\}, \chi\delta_R, C_G, K')$$

(where χ runs through a set of (unitary) representatives for the components of $D_{M_R}(\xi)$, and ${}^R\mathcal{L}_{\mathfrak{r}}(\{P\}, \chi\delta_R, C_G, K')$ is defined in the usual way; the direct sum above is then finite, cf. [**5**] II, 1.2, II, 1.7) which satisfy the following conditions:

(i) $\quad F_{R'}(w\Lambda) = M(w, \Lambda)F_R(\Lambda), \quad w \in W(\mathfrak{a}_R, \mathfrak{a}_{R'}).$

Here $M(w, \Lambda)$ is viewed as an operator

$$\oplus\, {}^R\mathcal{L}_{\mathfrak{r}}(\{P\}, \chi\delta_R, C_G, K') \rightarrow \bigoplus_{\{\chi'\}} {}^{R'}\mathcal{L}_{\mathfrak{r}}(\{P\}, \chi'\delta_{R'}, C_G, K')$$

in the obvious manner.

(ii) $\displaystyle \frac{1}{\#\{R\}} \sum_R \int_{\operatorname{Re}\Lambda=0} (F_R(\Lambda), F_R(\Lambda))d\Lambda < \infty$

where $\#\{R\}$ is the number of elements in $W(\mathfrak{a}_R, \mathfrak{a}_{R'})$ (any $R' \in \{R\}$) multiplied by the number of elements in $\{R\}$. Note that (ii) then furnishes an inner product for $\mathcal{H}(\{R\}, \{P\}, C_G, K')$.

Our previous results imply

8.4. LEMMA. *If* $F = \{F_R\} \in \mathscr{H}(\{R\}, \{P\}, C_G, K')$, *then*

$$T(F) = \frac{1}{\#\{R\}} \sum_P \int_{\operatorname{Re}\Lambda_R=0} E(g, F_R(\Lambda_R), \Lambda_R) d\Lambda_R$$

is an element of $\mathscr{L}(\xi)$, *and* $\|T(F)\| = \|F\|$.

Proof. We showed square integrability in 8.1 in case the F_R were continuous; the general case follows by a density argument. As for the statement about norms, set $\tilde{F}_R(g)$ to be equal to the integral above, then (8.2) implies that $(\tilde{F}_R, \tilde{F}_{R'})$ is equal to

$$\sum_{w \in W(\mathfrak{a}_R, \mathfrak{a}_{R'})} \int_{\operatorname{Re}\Lambda=0} (M(w, \Lambda)F_R(\Lambda), F_{R'}(w\Lambda)) d\Lambda$$

$$= \sum_w \int_{\operatorname{Re}\Lambda=0} (M(w, \Lambda)F_R(\Lambda), M(w, \Lambda)F_R(\Lambda)) d\Lambda$$

$$= \sum \int_{\operatorname{Re}\Lambda=0} (F_R(\Lambda), F_R(\Lambda)) d\Lambda$$

($M(w, \Lambda)$ is unitary on $\operatorname{Re} \Lambda = 0$)

$$= \#(W(\mathfrak{a}_R, \mathfrak{a}_{R'})) \int_{\operatorname{Re}\Lambda=0} (F_R(\Lambda), F_R(\Lambda)) d\Lambda$$

and from this the result follows.

8.5. We now have an injective isometry

$$\mathscr{H}(\{R\}, \{P\}, C_G, K') \to \mathscr{L}(\xi).$$

On the other hand, we also have the space

$${}^G\mathscr{L}_\tau(\{P\}, \xi) = \oplus\, {}^G\mathscr{L}_\tau(\{P\}, \xi, C_G'),$$

the sum taken over all equivalence classes C_G' in S_τ (under G).

LEMMA. *The image of* $\cup_{K'} \mathscr{H}(\{R\}, \{P\}, C_G, K')$ *in* $\mathscr{L}(\xi)$ *is precisely the space*

$${}^G\mathscr{L}_\tau(\{P\}, \xi. C_G).$$

Remarks. (i) Note that $\{C_G\} = \{R\}$.

(ii) $\quad {}^G\mathscr{L}_\tau(\{P\}, \xi, C_G) = {}^G\mathscr{L}_{S_\tau}(\{P\}, \xi, C_G)$

is a closed subspace of $\mathscr{L}(\{P\}, \xi)$.

Proof. We need only consider elements in ${}^G\mathscr{L}_\tau(\{P\}, \xi, C_G)$ of the form

$$\int_{\operatorname{Re}\Lambda=0} E_\tau(g, F(\Lambda), \Lambda) d\Lambda$$

where

$$F(\Lambda): \iota L_{Z_M}\backslash \mathfrak{a}_M(\mathbf{C}) \rightarrow \bigoplus_{w \in W^{(i)}(\mathfrak{r}, C_G)} \operatorname{Hom}(S(\mathfrak{r}_w) \ldots)$$

is at least analytic in some tube containing the imaginary axis and

$$(8.5.1) \qquad E_{\mathfrak{r}}(g, F(\Lambda), \Lambda) = \sum_w E(g, F(we_k\Lambda), we_k\Lambda)$$

$w \in {}^GW^{(i)}(\mathfrak{r}, C_G)$, $\mathfrak{r} = \mathfrak{r}_k$, etc. (cf. the statement of Proposition 4.14).

Let us show that such an element lies in the image of the isometry above. It is primarily a matter of going back to the definitions, and for this we had better be careful.

Let $\{F_R\}$ be a collection satisfying the definition of $\mathscr{H}(\{R\}, \{P\}, C_G, K')$; each such F_R is a measurable function

$$F_R: \iota L_{Z_R}\backslash \iota\mathfrak{a}_R \rightarrow \bigoplus_{\{\chi\}} {}^R\mathscr{L}_{\tau}(\{P\}, \chi\delta_R, C_G, K')$$

(it is enough to consider only F_R which are continuous). Then each

$$F_R(\Lambda) = E_{\mathfrak{r}}(g, F_R'(\Lambda), X(r)), \quad \text{and}$$

$$E(g, F_R(\Lambda), \Lambda) = E(g, F_R'(\Lambda), X(\mathfrak{r}) + \Lambda)$$

by axiom (iii) for Eisenstein systems, and analytic continuation. Here

$$F_R'(\Lambda): \iota L_{Z_R}\backslash \iota\mathfrak{a}_R \rightarrow \bigoplus_w \operatorname{Hom}(S(\mathfrak{r}_w), V(\{P\})) \quad \text{and}$$

$$w \in \bigcup {}^RW(\mathfrak{r}, C_R)$$

where the union runs over $R \in \{R\}$ (remember that $\{C_G\} = \{R\}$).

We now see that the difference between the two spaces comes from the non-standard elements r_w, $w \in {}^GW(\mathfrak{r}, C_G)$. To fix this up we turn to Langlands' Proposition 7.4 once again.

Consider $E_{\mathfrak{r}}(g, F(\Lambda), \Lambda)$ in (8.5.1) with $F(\Lambda) \in \text{Domain } M_G(\Lambda)$. Then Proposition 7.4 implies that

$$M_G({}^GW(\mathfrak{r}, C_G), {}^GW(\mathfrak{r}, C_R), \Lambda)F(\Lambda) \in \text{Image } M_G(\mathfrak{r}).$$

Choose H_Λ such that $M_G(\mathfrak{r})H_\Lambda$ is this element, and form

$$\mathrm{E}(\cdot, H_\Lambda, X(\mathfrak{r}))$$

which is an element of

$${}^R\mathscr{L}_{\tau}(\{P\}, \chi\delta_R, C_R, K')$$

where $C_R \leftrightarrow \mathfrak{r}$. Then

$$E(g, H_\Lambda, \Lambda + X(\mathfrak{r}))$$

(cf. axiom (iii) again) is, we claim, equal to

$$E(g, F(\Lambda), \Lambda + X(\mathfrak{r})).$$

To see this, we shall apply the usual constant term argument and then it is sufficient to show that

$$H_\Lambda - F(\Lambda) \in \text{kernel } M_G(\Lambda)$$

but this follows exactly as in the proof of Lemma 7.11.

Now define

$$F_{R'}(\Lambda_{R'}) = \sum_{w\in W(R,R')} M(w, \Lambda)^{-1}E_{\mathfrak{r}}(\cdot, H_{w\Lambda}),$$

which gives a collection $\{F_{R'}\}$ in $\mathscr{H}(\{R\}, \{P\}, C_G, K')$. Then

$$\int_{\operatorname{Re}\Lambda=0} E(g, H_\Lambda, \Lambda + X(\mathfrak{r}))$$

is equal to

$$\sum_{R'}\int_{\operatorname{Re}\Lambda'=0} E(g, F_{R'}(\Lambda'), \Lambda')d\Lambda'$$

(at least to within a constant), so we are done.

8.6. Now put

$$\mathscr{H}(\{R\}, \{P\}, K') = \bigoplus_{C_G}\mathscr{H}(\{\mathrm{R}\}, \{P\}, C_G, K'), \{R\} = \{C_G\}$$
$$\mathscr{L}(\{R\}, \{P\}, K', \xi) = \bigoplus_{C_G}{}^G\mathscr{L}_r(\{P\}, \xi, C_G, K'), \{R\} = \{C_G\}.$$

The two spaces on the left can be identified with each other, by means of an isometry.

Now by the main theorem of Section 5, $\mathscr{L}(\{P\}, K', \xi)$ is equal to

$$\begin{aligned}{}^G\mathscr{L}(\{P\}, K', \xi) &= \bigoplus_r {}^G\mathscr{L}_r(\{P\}, K', \xi)\\ &= \bigoplus_{\{R\}\geqq\{P\}}\mathscr{L}(\{R\}, \{P\}, K', \xi).\end{aligned}$$

Using the definitions and the results obtained we then find the

THEOREM. $\mathscr{L}(\{P\}, \xi) = \bigoplus_{\{R\}\geqq\{P\}}\mathscr{L}(\{R\}, \{P\}, \xi)$.

8.7. Each of the spaces $\mathscr{L}(\{R\}, \{P\}, \xi)$ has been defined in terms of residues of Eisenstein systems. The theory of Eisenstein series and associated intertwining operators furnishes us with an alternative explicit description of these spaces, which we now describe.

In the first place, let us return to Section 6.2 where we had spaces

$$\mathscr{L}(\{{}^RP\}, \chi\delta_R) \supseteq \mathscr{L}(M_R, \{{}^RP\}, \chi\delta_R).$$

It is apparent after a moment's thought that ${}^R\mathscr{L}_r(\{P\}, \chi\delta_R)$ is a subspace of

$$\operatorname{Ind}_{M_R(\mathbf{A})}^{G(\mathbf{A})}(\mathscr{L}(M_R, \{{}^RP\}, \chi\delta_R) = \mathscr{C}(R, \{{}^RP\}, \chi\delta_R),$$

(the definition of this latter space). Indeed for this to be so we need to show that ${}^R\mathscr{L}_r(\{P\}, \chi\delta_R)$ is an invariant subspace, and this follows from the definitions.

Consequently (cf. 6.5) the space ${}^R\mathscr{L}_r(\{P\}, \chi\delta_R)$ contains $\mathscr{C}_0(R, \{{}^RP\}, \chi\delta_R)$ as a dense subspace.

Suppose that U is some subset of $\iota L_{Z_R}\backslash\mathfrak{a}_R(\mathbf{C})$ (or $\iota L_{Z_R}\backslash\iota\mathfrak{a}_R$), supposed open. A holomorphic map

$$\Phi: U \rightarrow \mathscr{C}_0(R, \{{}^RP\}, \chi\delta_R)$$

can be regarded as an analytic family of cross sections $\{\Phi_\zeta\}_\zeta$ with each

$$\Phi_\zeta \in \mathscr{C}_0(R, \{{}^RP\}, \chi\zeta\delta_R),$$

cf. the discussion in [**5**] II. 1.7–1.8. The function

$$E(g, \Phi) = E(g, \Phi_\zeta)$$

initially defined for $\operatorname{Re}(\chi\zeta) = \operatorname{Re}\zeta > \delta_R$, was then analytically continued to a rational function over the corresponding component of $D_{M_R}(\zeta)$. In particular, if Φ is as above and U is $\iota L_{Z_R}\backslash\iota\mathfrak{a}_R$ then

$$\int_U E(g, \Phi(\zeta))d\zeta$$

is an element of $\mathscr{L}(\xi)$ by 8.1; we need only consider $\Phi(\zeta)$ taking values in finite dimensional invariant subspaces of ${}^R\mathscr{L}_r(\{P\}, \chi\delta_R)$, cf. the discussion in Lemma 6.5.

In particular, if

$$\phi(g) = \int^{\oplus}_{D_{M_R}{}^0(\zeta)} \Phi(g; \zeta)d\zeta$$

is an element of

$$\mathscr{C}_0(R, \{{}^RP\}, \xi),$$

then

$$\phi \rightarrow \int_U E(g, \Phi(\zeta))d\zeta$$

is a linear transformation $\phi \rightarrow T\Phi$

$$\mathscr{C}_0(\{R\}, \{{}^RP\}, \xi) \rightarrow \mathscr{L}(\xi),$$

and it extends naturally to a $G(\mathbf{A})$-equivariant map

$$T: \bigoplus_{R\in\{R\}} \bigoplus_{\{{}^RP\}} \mathscr{C}_0(R, \{{}^RP\}, \xi) \rightarrow \mathscr{L}(\xi).$$

In fact, the image of this map is none other than $\mathscr{L}(\{R\}, \{P\}, \xi)$. Indeed, suppose $\phi = \oplus\,\phi_R$ is an element of the space on the left hand

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side above. Then

$$T\phi(g) = \bigoplus_R \int E(g, \Phi_R(\zeta), \zeta)$$

by definition. The functional equations for $E(g, \Phi(\zeta), \zeta)$ imply that this is also equal to

$$\bigoplus_R \int E\left(g, \frac{1}{\#} \sum M(w, \zeta)\Phi_R(\zeta), w\zeta\right) d\zeta$$

where the sum is over all $w \in \bigcup_{R' \in \{R\}} W(\mathfrak{a}_R, \mathfrak{a}_{R'})$, and $\#$ is the number of terms in the sum: $\# = \#\{R\}$ in the notation of 8.3). The map

$$\mathrm{U}: \oplus\ \Phi_R(\zeta) \rightarrow \oplus \frac{1}{\#\{R\}} \sum M(w, \zeta)\Phi(\zeta)$$

sends $\oplus\ \Phi_R(\zeta)$ into $\mathscr{H}(R, \{P\}, \zeta\delta_R)$ where this latter space is the "non-trivial" K-finite version of the space $\mathscr{H}(R, \{P\}, \chi\delta_R, K')$. For this one only has to check that the image $\mathrm{U}\ (\oplus\ \Phi_R(\zeta))$ satisfies the conditions defining $\mathscr{H}(R, \{P\}, \zeta\delta_R)$ which follows from the functional equations for $M(w, \zeta)$. Indeed the functional equations and 8.4 tell us that this map

$$\mathrm{U} : \mathscr{C}_0(R, \{^RP\}, \zeta\delta_R) \rightarrow \mathscr{H}(R, \{P\}, \zeta\delta_R)$$

is an orthogonal projection onto the right subspace. Thus T factors through U, and the claim is established.

Let us put all this together as a theorem:

THEOREM. *The space $\mathscr{L}(\{P\}, \xi)$ is a direct sum of invariant spaces*

$$\mathscr{L}(\{P\}, \xi) = \bigoplus_{\{R\} \geqq \{P\}} \mathscr{L}(\{R\}, \{P\}, \xi).$$

The space $\mathscr{L}(\{R\}, \{P\}, \xi)$ is the image of the $G(\mathbf{A})$ equivariant map

$$T: \bigoplus_{R \in \{R\}} \bigoplus_{\{^RP\}} \mathscr{C}_0(R, \{^RP\}, \xi) \rightarrow \mathscr{L}(\xi)$$

$$\oplus\ \phi_R \rightarrow \bigoplus_R \int_{D_{M_R}{}^0(\zeta)} E(g, \Phi_R(\zeta))d\zeta.$$

It is a subspace of

$$\bigoplus_{R \in \{R\}} \bigoplus_{\mathfrak{p} = \{^RP\}} \mathrm{Ind}_{M_R(\mathbf{A})}^{G(\mathbf{A})}(\mathscr{L}(M_R, \{^RP\}, \xi)$$

and the map T factors through the orthogonal projection U.

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