On the Solutions of Mathieu's Differential Equation, and their Asymptotic Expansions.

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The following paper is a continuation of one read before the Society some years ago, and published in the *Proceedings*, Vol. XXXIV. (Part 2), Session 1915–1916. The results of that paper, more especially those summarised in Art. 14, and those of Arts. 17, 18 will be assumed.

The differential equation is

$$d^2u/d\alpha^2 + (\tfrac{1}{2}\kappa^2c^2 \cosh 2\alpha - s^2)u = 0.$$

Analytical representations were found for two independent solutions of this equation. In the present paper, asymptotic expansions of these solutions are obtained for the case when the real part of the variable α tends to infinity, positive or negative, the parameters κc and s remaining fixed. The expansions are deduced by direct transformation of the series already found for the solutions. In the present state of the theory of asymptotic series, this seems to be the only possible rigorous method.

The general term of the main solution was written in the form

$$(-1)^n \phi(n + \tfrac{1}{2}\nu)\, e^{(2n+\nu)\alpha},$$

$\phi(n+\frac{1}{2}\nu)$ being a particular value of a function $\phi(z)$ which was defined for all values of the complex variable z. Most of the developments now to be given are based on an expansion deduced for the function $\phi(z)$ in the form of a series of reciprocals of Π functions; this expansion can be expressed as the product of a single Π function by a series of inverse factorials.

By a slight change of notation, the above differential equation can be written in a form which reduces at once to Bessel's equation when a certain parameter is given the special value zero. The following analysis reduces, therefore, in a special case to a discussion of the solutions of Bessel's equation and their asymptotic expansions. The treatment of the simpler equation thus suggested is not without interest. It is akin in some respects to a method which has been given by Barnes.

1. The differential equation being

$$\frac{d^2u}{d\alpha^2}+(\tfrac{1}{2}\kappa^2c^2\cosh 2\alpha-s^2)u=0, \quad\ldots\ldots\ldots\ldots(1)$$

we have (I. Arts. 14, 18)* a solution

$$J(\nu,\ s,\ \kappa c,\ \alpha)=\sum_{n=-\infty}^{\infty}(-1)^n\,\phi\,(n+\tfrac{1}{2}\nu)\,e^{(2n+\nu)\alpha}, \quad\ldots\ldots(2)$$

and a second solution

$$G(\nu,\ s,\ \kappa c,\ \alpha)$$
$$=\frac{\pi}{2\sin\nu\pi}\{J(-\nu,\ s,\ \kappa c,\ \alpha)-e^{-i\nu\pi}J(\nu,\ s,\ \kappa c,\ \alpha)\}\ \ldots\ldots\ (3)$$

Instead of s, κc, ν, it is usually more convenient to use the constants r, λ, k, μ, where

$$r=\tfrac{1}{2}s,\ \ \lambda=\tfrac{1}{4}\kappa c,\ \ k=\tfrac{1}{2}\kappa c,\ \ \mu=\tfrac{1}{2}\nu.$$

We may write

$$\phi\,(z)\ =\frac{\lambda^{2z}}{\Pi\,(z+r)\,\Pi\,(z-r)}\,v_z, \quad\ldots\ldots\ldots\ldots(4)$$

where $v \rightarrow 1$, as $R\,(z)\rightarrow +\infty$.

The function ϕ satisfies the difference equation

$$\phi\,(z+1)+\phi\,(z-1)=\frac{z^2-r^2}{\lambda^2}\,\phi\,(z), \quad\ldots\ldots\ldots\ldots(5)$$

and we have the important relation (I. Arts. 6, 7)

$$\phi\,(z)\ \phi\,(-z-1)-\phi\,(-z)\ \phi\,(z+1)$$
$$=(1/\pi^2\lambda^2)\,\{\sin\,(z+r)\,\pi\sin\,(z-r)\,\pi-\lambda^4F_1(r)+\lambda^8F_2(r)-\ldots\},\ \ldots(6)$$

where $\quad F_1\,(r)=-\pi\sin 2r\pi/\{r\,(4r^2-1)\}$, (I. Art. 8)

$\qquad F_q\,(r)=C_q(r)\cos 2r\pi+S_q\,(r)\sin 2r\pi,$

where $C_q(r)$ and $S_q(r)$ are rational functions of r only; and $F_2(r)$ is given by I. (55), (68), (70).

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* References to the 1916 paper will be given in this form.

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The equation for $\mu(=\frac{1}{2}\nu)$ is (I. Art. 2)

$$\phi(\mu)\,\phi(-\mu-1)-\phi(-\mu)\,\phi(\mu+1)=0, \qquad \text{(7)}$$

or
$$\cos 2\mu\pi=\cos 2r\pi-2\lambda^4 F_1(r)+2\lambda^8 F_2(r)-\ldots, \qquad \text{(8)}$$

a holomorphic function of λ^4 and of r^2.

From (6) and (7)

$$\begin{aligned}\phi(z)\,\phi(-z-1)-\phi(-z)\,\phi(z+1)\\ &=(1/\pi^2\lambda^2)\{\sin(z+r)\pi\sin(z-r)\pi-\sin(\mu+r)\pi\sin(\mu-r)\pi\}\\ &=(1/\pi^2\lambda^2)\sin(z+\mu)\pi\sin(z-\mu)\pi, \qquad \text{(9)}\end{aligned}$$

Then, from I. (51), (53),

$$\begin{aligned}\underset{n=-\infty}{\text{Lt.}}\; v_{n+z} &= \frac{\sin(z+\mu)\pi\,\sin(z-\mu)\pi}{\sin(z+r)\pi\,\sin(z-r)\pi}\\ &=1-\frac{\sin(\mu+r)\pi\,\sin(\mu-r)\pi}{\sin(z+r)\pi\,\sin(z-r)\pi}, \qquad \text{(9a)}\end{aligned}$$

that is to say, this is the asymptotic form of v_z as z goes to $-\infty$ by steps of 1, the imaginary part of z remaining constant.

2. It will usually be convenient to work with a function $\chi(z)$ instead of with $\phi(z)$, the definition of the new function being

$$\chi(z)=\sqrt{(\tfrac{1}{2}\pi)}\,(2\lambda)^{-2z}\,\phi(z) \qquad \text{(10)}$$

The difference equation for $\chi(z)$ is, from (5),

$$k^4\chi(z+1)+\chi(z-1)=(4z^2-s^2)\,\chi(z); \qquad \text{(11)}$$

and, from (4), we have, when $\mathrm{R}(z)\to+\infty$,

$$\begin{aligned}\chi(z)&\approx\sqrt{(\tfrac{1}{2}\pi)}\,/\,\{2^{2z}\,\Pi(z+r)\,\Pi(z-r)\}\\ &\approx\sqrt{(\tfrac{1}{2}\pi)}\,/\,\{2^{2z}\,\Pi(z+\tfrac{1}{4})\,\Pi(z-\tfrac{1}{4})\}\\ &\approx 1\,/\,\Pi(2z+\tfrac{1}{2}), \qquad \text{(12)}\end{aligned}$$

where we use the relations $\Pi(z+a)\approx z^a\,\Pi(z)$,

and
$$\Pi(u)\,\Pi(u-\tfrac{1}{2})=\sqrt{\pi}\,2^{-2u}\,\Pi(2u). \qquad \text{(13)}$$

3. The form of (12) suggests an expansion of $\chi(z)$ in the form

$$\begin{aligned}\chi(z)=1/\Pi(2z+\tfrac{1}{2})+B_1/\Pi(2z+\tfrac{1}{2}+1)\\ +\ldots+B_n/\Pi(2z+\tfrac{1}{2}+n)+\ldots, \qquad \text{(14)}\end{aligned}$$

where the coefficients B_1, B_2, ... are independent of z.

Assuming the expansion provisionally, we may find a recurrence equation for the coefficients from (11), as follows. We have

$$\frac{4z^2-s^2}{\Pi(2z+\frac{1}{2}+n)}$$
$$=\frac{(2z+\frac{1}{2}+n)(2z-\frac{1}{2}+n)-2n(2z+\frac{1}{2}+n)+(n+\frac{1}{2})^2-s^2}{\Pi(2z+\frac{1}{2}+n)}$$
$$=\frac{1}{\Pi(2z+\frac{1}{2}+n-2)}-\frac{2n}{\Pi(2z+\frac{1}{2}+n-1)}+\frac{(n+\frac{1}{2})^2-s^2}{\Pi(2z+\frac{1}{2}+n)},$$

so that, by (14),

$$(4z^2-s^2)\chi(z)$$
$$=\frac{1}{\Pi(2z+\frac{1}{2}-2)}+\frac{\frac{1}{4}-s^2}{\Pi(2z+\frac{1}{2})}$$
$$+B_1\left\{\frac{1}{\Pi(2z+\frac{1}{2}-1)}-\frac{2}{\Pi(2z+\frac{1}{2})}+\frac{(1+\frac{1}{2})^2-s^2}{\Pi(2z+\frac{1}{2}+1)}\right\}$$
$$+B_2\left\{\frac{1}{\Pi(2z+\frac{1}{2})}-\frac{2\,.\,2}{\Pi(2z+\frac{1}{2}+1)}+\ldots\right\}$$
$$+\ \ldots,$$

$$(4z^2-s^2)\chi(z)-\chi(z-1)$$
$$=\sum_{n=0}\left[\{(n+\tfrac{1}{2})^2-s^2\}B_n-2(n+1)B_{n+1}\right]/\,\Pi(2z+\tfrac{1}{2}+n),$$

if we take $B_0=1$.

Also $k^4\chi(z+1)=\sum\limits_{n=0}k^4B_{n-2}/\,\Pi(2z+\frac{1}{2}+n)$,

if we also take $B_{-1}=0$, $B_{-2}=0$.
We thus require, by (11),

$$\{(n+\tfrac{1}{2})^2-s^2\}B_n-2(n+1)B_{n+1}-k^4B_{n-2}=0,\ \ldots\ldots\ldots \quad (15)$$

for $n=0,1,2,\ldots$; with $B_0=1$, $B_{-1}=0$, $B_{-2}=0$.
By taking $n=0,1,2$, we find

$$(\tfrac{1}{4}-s^2)-2B_1=0;\ B_1=\tfrac{1}{2}(\tfrac{1}{4}-s^2);$$
$$(\tfrac{9}{4}-s^2)B_1-2\cdot 2\,B_2=0;\ B_2=\tfrac{1}{2\cdot 4}(\tfrac{1}{4}-s^2)(\tfrac{9}{4}-s^2);$$
$$(\tfrac{25}{4}-s^2)B_2-2\cdot 3\,B_3-k^4=0;\ B_3=\tfrac{1}{2\cdot 4\cdot 6}(\tfrac{1}{4}-s^2)(\tfrac{9}{4}-s^2)(\tfrac{25}{4}-s^2)-\tfrac{1}{6}k^4;$$

and so on, so that the coefficients B_n are all defined, and are obviously polynomials in s and k.

4. The above process must now be justified by an examination of the convergence of the series in (14).

In (15) put

$$B_n = D_n\, \Pi\,(n - \tfrac{1}{2} + s)\, \Pi\,(n - \tfrac{1}{2} - s) \,/\, (2^n\, \Pi n)\,, \qquad \ldots\ldots\ldots (16)$$

so that the definition of D_n is

$$D_n = B_n\, 2^n\, \Pi n \,/\, \{\Pi\,(n - \tfrac{1}{2} + s)\, \Pi\,(n - \tfrac{1}{2} - s)\}\,. \qquad \ldots\ldots\ldots (17)$$

Then (15) becomes

$$D_n - D_{n+1}$$
$$= D_{n-2}\,.\,4n\,(n-1)\,k^4/\{(n+\tfrac{1}{2})^2 - s^2\}\{(n-\tfrac{1}{2})^2 - s^2\}\,\{(n-\tfrac{3}{2})^2 - s^2\} \quad \ldots (18)$$

Also $D_0 = D_1 = D_2 = 1/\Pi\,(-\tfrac{1}{2} + s)\,\Pi\,(-\tfrac{1}{2} - s) = \cos s\pi/\pi$.

Write (18) in the form

$$D_n - D_{n+1} = -\,c_{n-2}\, D_{n-2}\,,$$

or

$$D_n = D_{n-1} + c_{n-3}\, D_{n-3}\,.$$

Then $D_3 = D_2 + c_0\, D_0 = (1 + c_0)\, D_0\,,$

$D_4 = D_3 + c_1\, D_1 = (1 + c_0 + c_1)\, D_0\,,$

$D_5 = D_4 + c_2\, D_2 = (1 + c_0 + c_1 + c_2)\, D_0\,,$

$D_6 = D_5 + c_3\, D_3 = (1 + c_0 + c_1 + c_2 + c_3 + c_0\, c_3)\, D_0\,,$

$D_7 = D_6 + c_4\, D_4 = (1 + c_0 + c_1 + c_2 + c_3 + c_4 + c_0\, c_3 + c_0\, c_4 + c_1\, c_4)\, D_0\,.$

Clearly D_n is the sum of terms selected from the expansion of the product

$$(1 + c_0)\,(1 + c_1)\,(1 + c_2)\,\ldots\,(1 + c_{n-3})\, D_0\,:$$

the terms rejected being in fact those in which any two c's occur with suffixes differing by less than 3. If we write b_n for $|\, c_n \,|$ we have therefore

$$|\, D_n \,| < (1 + b_0)\,(1 + b_1)\,\ldots\,(1 + b_{n-3})\,|\, D_0 \,|\,.$$

But b_n is of the order $1/n^4$ when n is great, so that the infinite product $(1 + b_0)\,(1 + b_1)\,\ldots$ converges; D_n has therefore a finite upper limit L. Hence, from (18) and similar equations, by addition,

$$|\, D_n - D_{n+p} \,| < (b_{n-2} + b_{n-1} + \;\ldots\; + b_{n+p-3})\, L,$$

which tends to zero as $n \to \infty$, for every positive p.

It follows that D_n tends to a limit, D, say (for the value of D cf. Art. 8).

If s is half an odd integer, the above process fails, for the denominator on the right of (18) then vanishes for certain values of n. The conclusion still holds, however, as may easily be proved by a trifling modification of the process, starting from a sufficiently large value of n instead of the value 2. We now have from (16), as $n \rightarrow \infty$

$$B_n \approx D\, \Pi(n)\, n^{s-\frac{1}{2}}\, \Pi(n)\, n^{-s-\frac{1}{2}} / 2^n\, \Pi n,$$

or

$$B_n \approx 2^{-n}\, \Pi(n-1)\, D \quad \ldots\ldots\ldots\ldots \quad (19)$$

The general term of (14), viz. $B_n / \Pi(2z+\frac{1}{2}+n)$ has, therefore, the asymptotic form $2^{-n}\, \Pi(n-1)\, D / n^{2z+\frac{1}{2}}\, \Pi n$, or $2^{-n}\, n^{2z-\frac{3}{2}}\, D$, so that the series converges for every z: the convergence being similar to that of a hypergeometric series with fourth element equal to $\frac{1}{2}$. It will be seen later that in one interesting case the limit D is zero; since $B_n / \Pi(2z+\frac{1}{2}+n) \approx 2^{-n}\, n^{2z-\frac{3}{2}}\, D_n$, the convergence of (14) in this case is at least as strong as in the general case (*cf.* Art. 10).

5. With the values of the B's as now found, the processes of Art. 3 are obviously legitimate. It still remains to show that the function on the right of (14), w_z say, is the same as the function $\chi(z)$ already defined at (10). Now, w_z satisfies the difference equation

$$k^4 w_{z+1} + w_{z-1} = (4z^2 - s^2)\, w_z .$$

Eliminating $(4z^2 - s^2)$ between this equation and (11), we find

$$k^{4(z+1)} \{\chi(z+1)\, w_z - \chi(z)\, w_{z+1}\}$$
$$= k^{4z} \{\chi(z)\, w_{z-1} - \chi(z-1)\, w_z\};$$

that is, the function on the right is not altered by change of z into $z+1$, nor, therefore, by change of z into $z+n$, where n is any integer; by making n tend to $+\infty$, and noting the asymptotic forms of w_z and $\chi(z)$, we find that the function is zero. Hence we may write $w_z / \chi(z) = w_{z+1} / \chi(z+1)$, since obviously neither denominator vanishes for a z with sufficiently large modulus. Thus

$$w_z / \chi(z) = \text{Lt.}\ w_{z+n} / \chi(z+n) = 1,$$

so that

$$w_z = \chi(z).$$

6. The asymptotic form of $\chi(z)$, when $R(z) \rightarrow -\infty$, is important. In terms of $\chi(z)$, (9) is

$$\chi(z)\, \chi(-z-1) - k^4 \chi(-z)\, \chi(z+1)$$
$$= (2/\pi) \sin(z+\mu)\pi \sin(z-\mu)\pi \quad \ldots\ldots\ldots \quad (20)$$

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It follows from (4), (9*a*), and (12) that, when $R(z)$ is large and negative, the second term on the left is small in comparison with the first, provided neither $z+\mu$ nor $z-\mu$ is an integer. Hence

$$\chi(z) \approx (2/\pi) \sin(z+\mu)\pi \sin(z-\mu)\pi \,/\, \chi(-z-1) \,. \quad\ldots\ldots (21)$$

But, from (12), $\chi(-z-1) \approx / 1\, \Pi(-2z-\frac{3}{2})$;

hence, from the formula $\Pi(u)\,\Pi(-u-1) = -\pi / \sin u\pi$, ... (22)

$$\chi(z) \approx (2/\pi) \sin(z+\mu)\pi \sin(z-\mu)\pi\, \Pi(-2z-\tfrac{3}{2}), \quad\ldots. (23)$$

or $\chi(z) = -(2/\cos 2z\pi) \sin(z+\mu)\pi \sin(z-\mu)\pi / \Pi(2z+\frac{1}{2})$, ... (24)

if $R(z) \rightarrow -\infty$.

This result brings into prominence an important feature of the series (14) for $\chi(z)$. That series converges for every z, and, if $|z|$ is large, each of the early terms of the series is large in comparison with the succeeding term, whether the real part of z be positive or negative. But (24) shows that when $R(z)$ is large and *negative*, and the imaginary part of z is zero or not large, the first term of the series (14) does not give a first approximation to $\chi(z)$. The underlying reason for this peculiarity is that, in the case supposed, the moduli of the terms of (14), though they begin as a decreasing series, fall to a minimum, and rise again to a maximum, before they finally tend to zero; and the terms on either side of the maximum are not negligible in comparison with the early terms. If, however, we define $B(n, z)$ so that we have, identically,

$$\chi(z) = 1/\Pi(2z+\tfrac{1}{2}) + \ldots + B_{n-1}/\Pi(2z+\tfrac{1}{2}+n-1)$$
$$+ B(n, z) / \Pi(2z+\tfrac{1}{2}+n), \quad\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

then $B(n, z) = B_n + B_{n+1}/(2z+\frac{1}{2}+n+1)$

$$+ B_{n+2}/(2z+\tfrac{1}{2}+n+1)(2z+\tfrac{1}{2}+n+2) + \ldots$$

But, if $R(2z+\frac{1}{2}+n) > 0$, then $|2z+\frac{1}{2}+n+N| > N$ (where N is any positive integer), so that the terms of the series for $B(n, z)$ have moduli less than those of the convergent series

$$|B_n| + |B_{n+1}| / 1 + |B_{n+2}| / 1 \,.\, 2 + \ldots$$

The series for $B(n, z)$ therefore converges uniformly for every z within the region $R(2z+\frac{1}{2}+n) > 0$, so that its limit as $z \rightarrow \infty$ within this region may be found term by term, and is B_n.

If $p < n$, we may write

$$B(p, z) = B_p + B_{p+1} / (2z+\tfrac{1}{2}+n+p)$$
$$+ \ldots + B(n, z)\, \Pi(2z+\tfrac{1}{2}+p) / \Pi(2z+\tfrac{1}{2}+n),$$

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and clearly $B(p, z)$ converges to B_p when $|z| \to \infty$ if $B(n, z)$ converges to B_n, *i.e.* if $R(2z+\frac{1}{2}) > -n$, or if $R(z) >$ any assigned integer.

7. The asymptotic value of $\chi(z)$ in (24) is the first term of an asymptotic series of which some use will be made later. To find this series, put S for the right side of (20), and write that equation in the form

$$\chi(z)/\chi(-z) - k^4 \chi(z+1)/\chi(-z-1) = S/\chi(-z)\chi(-z-1).$$

Change z here into $z+1$, $z+2$, ..., $z+n$; multiply by k^4, k^8 ..., and add.

Thus $\chi(z)/\chi(-z) - k^{4n}\chi(z+n)/\chi(-z-n)$

$$= S\{1/\chi(-z)\chi(-z-1) + k^4/\chi(-z-1)\chi(-z-2) + \dots + k^{4n-4}/\chi(-z-n+1)\chi(-z-n)\} \quad \text{.........(26)}$$

By using (25) with z changed into $-z-1$, $-z-2$, ... $-z-n$ on the right side of (26), after multiplying (26) by $\chi(-z)$, we obtain the value of $\chi(z)$ as an expression which is the product of the right hand member of (24) by a factor which it is easy to see may be thrown into the form

$$1 + \beta_1/(2z+\tfrac{1}{2}+1) + \beta_2/(2z+\tfrac{1}{2}+1)(2z+\tfrac{1}{2}+2) + \dots + \beta(n, z)/(2z+\tfrac{1}{2}+1)\dots(2z+\tfrac{1}{2}+n),$$

where $\beta(n, z)$ tends to a finite limit when $z \to \infty$ in the region for which $R(z)$ is *less than* an assigned number,

Hence, if $R(z) < a$ (any assigned number),

$$\chi(z) = (-2/\cos 2z\pi)\sin(z+\mu)\pi \sin(z-\mu)\pi \times \{1/\Pi(2z+\tfrac{1}{2}) + \beta_1/\Pi(2z+\tfrac{1}{2}+1) + \dots + \beta(n, z)/\Pi(2z+\tfrac{1}{2}+n)\} \quad \text{.........(27).}$$

The periodic factor here is not changed by changing z into $z+1$; and a slight modification of the process of Art. 3 therefore shows that the coefficients β are simply the coefficients B; and it can then be shown in a moment, by using (27) with n changed into $n+1$, that the limit of $\beta(n, z)$, when $z \to \infty$ in the region $R(z) < a$, is B_n.

The argument fails if $z+\mu$ or $z-\mu$ is an integer; in this case (26) gives $\chi(z)/\chi(-z) = k^{4n}\chi(z+n)/\chi(-z-n)$, which is equivalent to the result of I., Art. 5,

$$\phi(-n-\mu)/\phi(n+\mu) = \phi(-\mu)/\phi(\mu). \quad \text{..............(28)}$$

Since the factor $(-2/\cos 2z\pi)\sin(z+\mu)\pi\sin(z-\mu)\pi$ in (24) and (27) is equal to $1-\cos 2\mu\pi/\cos 2z\pi$, which tends to the value 1 when $I(z)\to\infty$, it appears that in (25) $B(n, z)$ tends to the finite limit B_n when $|z|\to\infty$, with $-\pi+\delta>\arg z<\pi-\delta$, when δ is any small positive number. Briefly, we may say that as $|z|\to\infty$ in any direction except parallel to the negative direction of the real axis, (14) represents $\chi(z)$ *asymptotically* as well as *analytically.*

8. The value of the limit D which occurs in Art. 4 can now be determined. In (14) take $2z+\frac{1}{2}+n=0$, or $z=-\frac{1}{2}n-\frac{1}{4}$, where n is a positive integer. Then

$$\chi(-\tfrac{1}{2}n-\tfrac{1}{4})=B_n+B_{n+1}/1+B_{n+2}/1.2+\ldots \qquad \ldots\ldots(29)$$

But, from (19), $\quad B_n=2^{-n}\,\Pi(n-1)\,D(1+E_n)$,

where $E_n\to 0$, when $n\to\infty$. Thus

$$\chi(-\tfrac{1}{2}n-\tfrac{1}{4})=2^{-n}\,\Pi(n-1)\,D\{(1+E_n)$$
$$+\frac{n}{1}\tfrac{1}{2}(1+E_{n+1})+\frac{n(n+1)}{1.2}\frac{1}{2^2}(1+E_{n+2})+\ldots\}.$$

If E is the greatest value of $|En, |E_{n+1}|, \ldots$, it follows that the modulus of

$$\chi(-\tfrac{1}{2}n-\tfrac{1}{4})-2^{-n}\,\Pi(n-1)\,D\{1+\frac{n}{1}\tfrac{1}{2}+\frac{n(n+1)}{1.2}\frac{1}{2^2}+\ldots\}$$

is less than

$$E.2^{-n}\,\Pi(n-1)\,|D|\,\{1+\frac{n}{1}\tfrac{1}{2}+\frac{n(n+1)}{1.2}\frac{1}{2^2}+\ldots\};$$

or $\quad |\chi(-\tfrac{1}{2}n-\tfrac{1}{4})-2^{n}\,\Pi(n-1)\,D(1-\tfrac{1}{2})^{-n}|$
$$<E.2^{-n}\,\Pi(n-1)\,|D|\,(1-\tfrac{1}{2})^{-n}.$$

Hence $\quad \chi(-\tfrac{1}{2}n-\tfrac{1}{4})\approx\Pi(n-1)\,D$,

or $\quad D=\underset{n=+\infty}{Lt}\,\chi(-\tfrac{1}{2}n-\tfrac{1}{4})/\Pi(n-1)$.

But, by (23), when $R(z)\to-\infty$.

$$\chi(z)\approx(2/\pi)(\sin^2 z\pi-\sin^2\mu\pi)\,\Pi(-2z-\tfrac{3}{2}).$$

so that $\quad D=\underset{n=+\infty}{Lt}\,(2/\pi)(\tfrac{1}{2}-\sin^2\mu\pi)\,\Pi(n-1)/\Pi(n-1)$

or $\quad D=(1/\pi)\cos 2\mu\pi. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30)$

Then, from (19)

$$B_n\approx(1/\pi)\cos 2\mu\pi\,2^{-n}\,\Pi(n-1). \qquad \ldots\ldots\ldots\ldots\ldots(31)$$

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When k and r are so related that $\cos 2\mu\pi = 0$, or $\mu = N \pm \frac{1}{4}$, these results are nugatory; the asymptotic form of B_n for this case is given in Art. 10.

9. We have now to introduce a simple piece of analysis, the applications of which are by no means confined to the present subject.

Consider the function S_t defined by the series

$$(1+t)^{2z+\frac{1}{2}}/\Pi(2z+\tfrac{1}{2}) + B_1(1+t)^{2z+\frac{1}{2}+1}/\Pi(2z+\tfrac{1}{2}+1) + \dots$$
$$+ B_n(1+t)^{2z+\frac{1}{2}+n}/\Pi(2z+\tfrac{1}{2}+n) + \dots,$$

$1+t$ having the phase 0 at $t=0$.

The term in $B_n \approx (1/\pi)\cos 2\mu\pi\ 2^{-n}\ n^{-2z-\frac{3}{2}}\ (1+t)^{2z+\frac{1}{2}+n}$, so that the series has a circle of convergence of radius 2, round $t=-1$ as centre. The function can, therefore, be expanded in a series of ascending powers of t, with radius of convergence 1. The coefficients may be found by Maclaurin's Theorem.

The value of S_t for $t=0$ is $\chi(z)$. The p^{th} derivative of S_t is

$$(1+t)^{2z+\frac{1}{2}-p}/\Pi(2z+\tfrac{1}{2}-p)$$
$$+ B_1(1+t)^{2z+\frac{1}{2}+1-p}/\Pi(2z+\tfrac{1}{2}+1-p) + \dots,$$

the value of which for $t=0$ is $\chi(z-\frac{1}{2}p)$. Hence

$$(1+t)^{2z+\frac{1}{2}}/\Pi(2z+\tfrac{1}{2}) + B_1(1+t)^{2z+\frac{1}{2}+1}/\Pi(2z+\tfrac{1}{2}+1) + \dots$$
$$= \chi(z) + t\,\chi(z-\tfrac{1}{2}) + (t^2/1\,.\,2)\,\chi(z-1) + \dots$$
$$+ (t^n/n!)\,\chi(z-\tfrac{1}{2}n) + \dots, \qquad \text{(32)}$$

where $|t| < 1$.

By changing t into $-t$, we obtain also

$$(1-t)^{2z+\frac{1}{2}}/\Pi(2z+\tfrac{1}{2}) + B_1(1-t)^{2z+\frac{1}{2}+1}/\Pi(2z+\tfrac{1}{2}+1) + \dots$$
$$= \chi(z) - t\,\chi(z-\tfrac{1}{2}) + (t^2/1.2)\,\chi(z-1) - \dots \qquad \text{(33)}$$

Let $R(2z+\frac{1}{2}) > 0$, and take the limits of the two sides of (33) for $t=1$. The series of term by term limits on the right converges, if $R(2z+\frac{1}{2}) > 0$. In this case, then

$$\chi(z) + \chi(z-1)/1\,.\,2 + \chi(z-2)/4! + \dots$$
$$= \chi(z-\tfrac{1}{2})/1 + \chi(z-\tfrac{3}{2})/3! + \chi(z-\tfrac{5}{2})/5! + \dots \qquad \text{(34)}$$

The function of z, which in the region $R(z) < -\frac{1}{4}$, is given by either side of (34) will be denoted by $\psi(z)$; it plays a fundamental part in the analysis which follows.

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By putting $t=1$ in (32), we find

$$2\,\psi(z)=2^{2z+\frac{1}{2}}/\Pi(2z+\tfrac{1}{2})+B_1\,2^{2z+\frac{1}{2}+1}/\Pi(2z+\tfrac{1}{2}+1)+\dots\ \dots\ (35)$$

It is easy to prove, as in Art. 6, that this series represents $2\,\psi(z)$ asymptotically (Art. 7) as well as analytically in the region $R(z)>-\frac{1}{4}$, when $|z|\rightarrow\infty$.

10. We can now, for the case when $\cos 2\mu\pi=0$, find a series for B_n from which its asymptotic form as $n\rightarrow\infty$ will be apparent. In (33), let $2z+\frac{1}{2}+n=0$, n being a positive integer.

Then $B_n+B_{n+1}(1-t)/1+B_{n+2}(1-t)^2/1\,.\,2+\dots$
$$=\chi(-\tfrac{1}{2}n-\tfrac{1}{4})-t\,\chi(-\tfrac{1}{2}n-\tfrac{3}{4})+(t^2/1\,.\,2)\,\chi(-\tfrac{1}{2}n-\tfrac{5}{4})-\dots$$

The limit of the function on the left for $t=1$ is B_n. The function on the right has therefore the same limit; and this limit can be found term by term, by a well known theorem of Abel's, provided the series converges for $t=1$. Now there is one and only one case in which *both* of the series

$$\chi(z)+\chi(z-1)/1\,.\,2+\dots$$

and
$$\chi(z-\tfrac{1}{2})/1+\chi(z-\tfrac{3}{2})/3\,!+\dots$$

converge, outside the region $R(z)>-\frac{1}{4}$.

The first series converges if $z=N\pm\mu$, the second if $z-\frac{1}{2}=N'\pm\mu$, where N and N' are integers. If both converge, we must have, by subtraction, $\frac{1}{2}=\pm 2\mu+$ an integer, so that 2μ is half an odd integer, and $\cos 2\mu\pi=0$.

In this case, therefore, we have

$$B_n=\chi(-\tfrac{1}{2}n-\tfrac{1}{4})-\chi(-\tfrac{1}{2}n-\tfrac{3}{4})/1+\chi(-\tfrac{1}{2}n-\tfrac{5}{4})/1\,.\,2-\dots\quad\dots(35\text{A})$$

Using (28), and writing A for $\phi(-\frac{1}{4})/\phi(\frac{1}{4})$, we easily deduce from this, *when n is even*,

$$B_n=A\,\{k^{2n+1}\,\chi(\tfrac{1}{2}n+\tfrac{1}{4})+k^{2n+5}\,\chi(\tfrac{1}{2}n+\tfrac{5}{4})/1\,.\,2+\dots\}$$
$$-(1/A)\{k^{2n+3}\,\chi(\tfrac{1}{2}n+\tfrac{3}{4})/1+k^{2n+7}\,\chi(\tfrac{1}{2}n+\tfrac{7}{4})/1\ \ 2\,.\,3+\dots\}\dots(35\text{B})$$

When n is *odd*, B_n is given by the same series, with A changed into $1/A$.

Hence, for the asymptotic values, as $n\rightarrow\infty$,

$$B_n\approx A\,k^{2n+1}/\Pi(n+1),\ (n\text{ even});$$
$$B_n\approx(1/A)\,k^{2n+1}/\Pi(n+1),\ (n\text{ odd}).\qquad\dots\dots\dots(35\text{C})$$

For this case ($\cos 2\mu\pi=0$), it should be noted that the series (35) for $2\psi(z)$ converges for every value of z.

11. The function $\psi(z)$ of (34) is a solution of the difference equation

$$(4z^2 - s^2)\,\psi(z) = k^4\,\psi(z+1) + (4z-1)\,\psi(z-\tfrac{1}{2}). \quad \text{.........(36)}$$

This may easily be verified from either of the defining expansions of (34), with the help of the difference equation (11) for $\chi(z)$; or from the expansion in (35), with the help of the recurrence equation (15) for the coefficients B. The equation (36) is only proved in this way for values of z with real part greater than a certain number; from the principle of continuation it is true for every z, and may be regarded as giving a definition of the function ψ for values of z beyond the region for which (34) and (35) are valid.

In (36) write $\quad z = -\tfrac{1}{2}\xi - \tfrac{1}{4}$, or $\xi = -(2z+\tfrac{1}{2})$.

Thus
$$\{(\xi+\tfrac{1}{2})^2 - s^2\}\,\psi(-\tfrac{1}{2}\xi - \tfrac{1}{4})$$
$$= k^4\,\psi\{-\tfrac{1}{2}(\xi-2) - \tfrac{1}{4}\} - 2(\xi+1)\,\psi\{-\tfrac{1}{2}(\xi+1) - \tfrac{1}{4}\}.$$

If we bring in a function H_ξ, where

$$\psi(-\tfrac{1}{2}\xi - \tfrac{1}{4}) = H_\xi / \sin \xi\pi,$$

this becomes

$$\{(\xi+\tfrac{1}{2})^2 - s^2\} H_\xi = k^4 H_{\xi-2} + 2(\xi+1)\,H_{\xi+1}, \quad \text{.........(36A)}$$

which is the same as the equation (15) for B_n, with H for B, and ξ for n.

The function H_z is $\sin \pi z\,\psi(-\tfrac{1}{2}z - \tfrac{1}{4})$, and vanishes if $z = -1$, $-2, \ldots$; and we shall now show that $H_0 = -\tfrac{1}{2}\cos 2\mu\pi$.

We have, so long as $R(t) > 0$,

$$-2H_{-t}/\sin t\pi = 2\,\psi(\tfrac{1}{2}t - \tfrac{1}{4})$$
$$= 2^t/\Pi(t) + B_1\,2^{t+1}/\Pi(t+1) + \ldots, \quad \text{......(37)}$$

by (35).

The series on the right of (37) may be expressed as the sum of two series, one of which converges for $t = 0$, while the other can be easily summed for any t with $R(t) > 0$. This is shown as follows.

If we multiply the series in (37) by $\pi/\cos 2\mu\pi$, and write b_n for $B_n\,\pi/\cos 2\mu\pi$, we can write the remainder after n terms, which is
$b_n\,2^{t+n}/\Pi(t+n) + b_{n+1}\,2^{t+n+1}/\Pi(t+n+1) + \ldots,$
in the form

$$\{b_n - 2^{-n}\Pi(n-1)\}\,2^{t+n}/\Pi(t+n)$$
$$+ \{b_{n+1} - 2^{-n-1}\Pi(n)\}\,2^{t+n+1}/\Pi(t+n+1) + \ldots$$
$$+ 2^t\{\Pi(n-1)/\Pi(t+n) + \Pi(n)/\Pi(t+n+1) + \ldots\} \ldots (38)$$

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Now, from (18) and similar equations, by addition, it is seen at once that $D_n = D + O(1/n^3)$, and (16) then gives

$$B_n = 2^{-n}\,\Pi(n-1)\,D\,\{1 + O(1/n)\},$$

or

$$b_n = 2^{-n}\,\Pi(n-1)\,\{1 + O(1/n)\}.$$

The general term in the first series of (38) is therefore

$$O(1/p)\,.\,2^t\,\Pi(p-1)\,/\,\Pi(p+t),$$

so that the series converges if $R(t) > -1$, and is, therefore, finite for $t=0$.

The second series of (38) has the sum

$$(1/t)\,\Pi(n-1)\,/\,\Pi(n+t-1),$$

i.e., $1/t +$ a value which is finite for $t = 0$.

From (37),

$$\begin{aligned} H_0 &= \operatorname*{Lt}_{t=0} H_{-t} \\ &= -\tfrac{1}{2}\operatorname*{Lt}_{t=0+}\left[\sin t\pi\,\{2^t/\Pi t + B_1\,2^{t+1}/\Pi(t+1)+\ldots\}\right] \\ &= -\tfrac{1}{2}(\cos 2\mu\pi/\pi)\,\mathrm{Lt}\,(\sin t\pi/t) = -\tfrac{1}{2}\cos 2\mu\pi, \end{aligned}$$

as was to be proved.

If, then, we introduce a function B_z with the definition

$$B_z = -(2/\cos 2\mu\pi)\,H_z$$

or

$$B_z = -(2/\cos 2\mu\pi)\sin\pi z\;\psi(-\tfrac{1}{2}z - \tfrac{1}{4}),\;\ldots\ldots\ldots\ldots \quad (39)$$

so that

$$\psi(\xi) = \tfrac{1}{2}(\cos 2\mu\pi/\cos 2\xi\pi)\,B_{-2\xi-\frac{1}{2}},\;\ldots\ldots\ldots\ldots \quad (40)$$

we shall have $B_0 = 1$, $B_{-1} = 0$, $B_{-2} = 0$, and, by (36A),

$$\{(z+\tfrac{1}{2})^2 - s^2\}\,B_z - 2(z+1)\,B_{z+1} - k^4\,B_{z-2} = 0\;\ldots\ldots\ldots \quad (41)$$

Hence, by (15), the coefficients 1, B_1, B_2, ... of (14) are the values for $z = 0, 1, 2, \ldots$ of the function B_z defined by (39).

From (37) and (39),

$$\begin{aligned} B_z = -(\sin\pi z)/\cos 2\mu\pi)\,\{&2^{-z}/\Pi(-z) \\ &+ B_1\,2^{-z+1}/\Pi(-z+1)+\ldots\},\;\ldots\ldots\ldots\ldots \quad (42) \end{aligned}$$

if $R(z) < 0$.

If in (41) we write C_z for $\Pi(z)\,B_z$, we find

$$\{(z+\tfrac{1}{2})^2 - s^2\}\,C_z - 2C_{z+1} - k^4 z(z-1)\,C_{z-2} = 0.\;\ldots\ldots\ldots \quad (43)$$

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From (39), C_z is finite for every z with $R(z)<0$; it follows then, by successive applications of (43), that C_z is finite for every z.

B_z is therefore a holomorphic function of z. The function does not exist, however, in the exceptional case when $\cos 2\mu\pi = 0$.

Since $\psi(z) = \frac{1}{2}\cos 2\mu\pi\, B_{-2z-\frac{1}{2}} / \sin(2z+\frac{1}{2})\pi$, the singularities of $\psi(z)$ at a finite distance are simple poles at $z = -\frac{1}{2}n - \frac{1}{4}$, where n is zero or a positive integer; and the residue of $\psi(z)$ at the pole $z = -\frac{1}{2}n - \frac{1}{4}$ is $(-1)^n B_n \cos 2\mu\pi / 4\pi$. Note that when $\cos 2\mu\pi = 0$, $\psi(z)$ is holomorphic, in agreement with the last sentence of Art. 10.

12. The function $\psi(z)$ has been defined explicitly for $R(z) > -\frac{1}{4}$, and its finite singularities have been found. For other values of z, it is virtually defined by the difference equation (36). There is an obvious advantage, however, in having an expression for $\psi(z)$— and consequently for B_z—which is valid all over the z-plane. Such an expression will now be obtained.

Consider the integral

$$I_z = \int \frac{1}{2i} \frac{\chi(z-t)\, dt}{\Pi(2t) \sin 2t\pi \sin(t-z-\mu)\pi \sin(t-z+\mu)\pi}, \quad \ldots\ldots\ldots\ldots(44)$$

the path (P, say) being any straight line drawn downwards parallel to the imaginary axis, passing to the left of $t=0$, and avoiding any zero of $\sin(t-z-\mu)\pi$ or $\sin(t-z+\mu)\pi$. For any t, on or to the right of the path P, and having a large modulus, we have, by (23),

$$\chi(z-t) \approx (2/\pi)\sin(z-t-\mu)\pi \sin(z-t+\mu)\pi / \Pi(2t-2z-\tfrac{3}{2}),$$

so that the integrand in I_z

$$\approx (1/i\pi)\,\Pi(2t-2z-\tfrac{3}{2}) / \Pi(2t)\sin 2t\pi$$

$$\approx (1/i\pi)\,(2t)^{-2z-\frac{3}{2}} / \sin 2t\pi.$$

Hence, if $R(2z+\frac{3}{2})>1$, *i.e.* if $R(z) > -\frac{1}{4}$,

$I_z = 2i\pi$ (sum of residues at poles to the right of path P); or, writing down in order the residues at the zeros of $\sin t\pi$, $\cos t\pi$, $\sin(t-z-\mu)\pi$, $\sin(t-z+\mu)\pi$, we have

$$I_z = \frac{\chi(z) + \chi(z-1)/1.2}{2\sin(z+\mu)\pi \sin(z-\mu)\pi} + \frac{\chi(z-\frac{1}{2})/1 + \chi(z-\frac{3}{2})/3!}{(-)\,2\cos(z+\mu)\pi\cos(z-\mu)\pi}$$

$$+ \sum_n{}' \frac{\chi(-n-\mu)}{\Pi(2n+2z+2\mu)\sin 2(z+\mu)\pi \sin 2\mu\pi}$$

$$- \sum_n{}'' \frac{\chi(-n+\mu)}{\Pi(2n+2z-2\mu)\sin 2(z-\mu)\pi \sin 2\mu\pi}.$$

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Here Σ'_n and Σ''_n include all values of n, in ascending order, such that $n+z+\mu$ and $n+z-\mu$, respectively, are to the right of the path P; and the sum of the other two series, by the definition at (34), is $\psi(z) \,.\, 2\cos 2z\pi / \sin 2(z+\mu)\pi \sin 2(z-\mu)\pi$. Hence

$$\left.\begin{aligned} 2\cos 2z\pi\ \psi(z) &= I_z \sin 2(z+\mu)\pi \sin 2(z-\mu)\pi \\ &- \{\sin 2(z-\mu)\pi/\sin 2\mu\pi\}\Sigma'_n\ \chi(-n-\mu)/\Pi(2n+2z+2\mu) \\ &+ \{\sin 2(z+\mu)\pi/\sin 2\mu\pi\}\Sigma''_n\ \chi(-n+\mu)/\Pi(2n+2z-2\mu) \end{aligned}\right\} \quad \text{..(45)}$$

It is easy to prove, with the help of the principle of continuation, that the expression on the right of (45) gives a representation of the function $2\cos 2z\pi\ \psi(z)$, or $\cos 2\mu\pi\ B_{-2z-\frac{1}{2}}$, valid for every z.

13. If, in the integral I_z of (44) we substitute for $\chi(z-t)$ the expression derived from (27) by changing z into $z-t$, we can derive an asymptotic series, for $\psi(z)$, valid when z is real and $\rightarrow -\infty$. The question whether the result is valid, on the more general supposition that $(z)\rightarrow\infty$ in any direction, subject to $R(z)<a$, is left open for the present.

Stopping, in the meantime, at the first term of (27), *i.e.* the term in (23), we have

$$\chi(z) = (2/\pi)\sin(z+\mu)\pi \sin(z-\mu)\pi\Pi(-2z-\tfrac{3}{2})f(z),$$

where $f(z)\rightarrow 1$, when $|z|\rightarrow\infty$, with $R(z)<a$.

Thus $$I_z = (1/i\pi)\int \frac{\Pi(2t-2z-\frac{3}{2})f(z-t)\,dt}{\Pi(2t)\sin 2t\pi}, \quad \text{......................(46)}$$

path P as at (44);

and $f(z-t)$ is nearly equal to 1 along the whole path, when z is large and negative. We may write

$$I_z = (i/\pi^2)\int \Pi(2t-2z-\tfrac{3}{2})\,\Pi(-2t-1)f(z-t)\,dt,$$

or, on putting $2t = z+\frac{1}{4}+u$,

$$I_z = (i/2\pi^2)\int \Pi(-z-\tfrac{5}{4}+u)\Pi(-z-\tfrac{5}{4}-u)f\tfrac{1}{2}(z-\tfrac{1}{4}-u)\,du. \quad \text{......(47)}$$

If the path P is taken through the point $2t = z+\frac{1}{4}$, *i.e.* the point $u=0$, and we put $u = iv$, we find

$$I_z = (1/2\pi^2)\int_{-\infty}^{\infty} \Pi(-z-\tfrac{5}{4}+iv)\ \Pi(-z-\tfrac{5}{4}-iv)f\tfrac{1}{2}(z-\tfrac{1}{4}-iv)\,dv.$$

We take z to be real; so that the product of the Π functions here is real and positive; it follows easily, as in the similar case at (29), that

$$I_z = (1/2\pi^2) f_0 \int_{-\infty}^{\infty} \Pi\left(-z - \tfrac{5}{4} + iv\right) \Pi\left(-z - \tfrac{5}{4} - iv\right) dv,$$

where $f_0 \to 1$, when $z \to -\infty$.

In other words, the asymptotic value of I_z is found by using the asymptotic value of $\chi(z-t)$ under the integral sign at (44); and obviously a similar result follows by the same method, if we stop at any term of the asymptotic series for $\chi(z-t)$ instead of the first.

The coefficient of B_p in the asymptotic series thus obtained for I_z is

$$(1/i\pi)(-1)^p \int \frac{\Pi(2t - 2z - \frac{3}{2} - p)dt}{\Pi(2t) \sin 2\pi t}, \text{ path } P. \qquad \ldots\ldots\ldots\ldots(48)$$

Consider the integral

$$(1/2i) \int \Pi(u+c)\, du / \Pi u \sin \pi u; \qquad \ldots\ldots\ldots\ldots\ldots(49)$$

suppose $R(c) < -1$, and for convenience let c be complex; the path being downwards and parallel to the imaginary axis, except that it bends so as to keep the poles of $\Pi(u+c)$ to one side and those of $1/\Pi u \sin \pi u$ to the other; then the integral $(49) = 2\pi i$ (sum of residues to right of path).

$$= \Pi c \{1 - (c+1)/1 + (c+1)(c+2)/1 \cdot 2 - \ldots\}$$
$$= \Pi c (1+1)^{-c-1} = \Pi c \cdot 2^{-c-1}. \qquad \ldots\ldots\ldots(50)$$

By the continuation theorem, this result holds when $R(c)$ is positive, in which case the integral can be taken along the path P of (44).

We thus find the value of (48) to be

$$(-1)^p (1/\pi) \Pi(-2z - \tfrac{3}{2} - p)\, 2^{2z + \frac{1}{2} + p}. \qquad \ldots\ldots\ldots\ldots(51)$$

Hence

$$I_z \approx (1/\pi)\{2^{2z+\frac{1}{2}} \Pi(-2z - \tfrac{3}{2}) - B_1 2^{2z+\frac{1}{2}+1} \Pi(-2z - \tfrac{3}{2} - 1) + \ldots\} \ldots(52)$$

In (45), when z is large and negative, the series Σ' and Σ'' are represented asymptotically by their first terms, and are negligible, unless $2(z \pm \mu)$ is an integer, a case considered in next article; thus, when z is real and $\to -\infty$,

$$\psi(z) \approx \{\sin 2(z+\mu)\pi \sin 2(z-\mu)\pi / 2\pi \cos 2z\pi\}. \{2^{2z+\frac{1}{2}}\Pi(-2z-\tfrac{3}{2}) - B_1 2^{2z+\frac{1}{2}+1}\Pi(-2z-\tfrac{3}{2}-1)+\dots\}; \quad \dots(53)$$

also, by (39), when z is real and $\to +\infty$,

$$B_z \approx \{\cos(z+2\mu)\pi \cos(z-2\mu)\pi / \pi \cos 2\mu\pi\}. \{2^{-z}\Pi(z-1) - B_1 2^{-z+1}\Pi(z-2)+\dots\}; \quad \dots\dots\dots\dots(54)$$

and, in particular, taking $z=n$, a large positive integer,

$$B_n \approx (\cos 2\mu\pi/\pi) \{2^{-n}\Pi(n-1) - 2^{-n+1}\Pi(n-2)B_1 + 2^{-n+2}\Pi(n-3)B_2 - \dots\}, \quad \dots(55)$$

an extension of the result already found at (31).

We may write (53) in the form

$$\psi(z) \approx \{-\sin 2(z+\mu)\pi \sin 2(z-\mu)\pi / 2\cos^2 2z\pi\}. \{2^{2z+\frac{1}{2}}/\Pi(2z+\tfrac{1}{2}) + B_1 2^{2z+\frac{1}{2}+1}/\Pi(2z+\tfrac{1}{2}+1)+\dots\} \quad \dots(56)$$

The relationship of (56) and (35) is interesting; compare also (27) and (14).

14. Returning now to (45), we have to consider the exceptional case when $2(z \pm \mu)$ is an integer. The results will lead us immediately to the asymptotic expansions of the functions of (2) and (3). In (45), take $z-\mu = \frac{1}{2}N$, where N is a positive or negative integer.

We find $\psi(\frac{1}{2}N+\mu) = \sum_n'' \chi(-n+\mu)/\Pi(2n+N)$: $\dots\dots\dots\dots(57)$

(a) N even; $2n+N=0, 2, 4, \dots$

$$\psi(\tfrac{1}{2}N+\mu) = \chi(\tfrac{1}{2}N+\mu) + \chi(\tfrac{1}{2}N+\mu-1)/1.2+\dots \quad \dots(58)$$

(b) N odd; $2n+N=1, 3, 5, \dots$

$$\psi(\tfrac{1}{2}N+\mu) = \chi(\tfrac{1}{2}N+\mu-\tfrac{1}{2})/1 + \chi(\tfrac{1}{2}N+\mu-\tfrac{3}{2})/1.2.3+\dots(59)$$

In (58), put $N=2p$, and in (59) put $N=2p+1$; then

$$\psi(p+\mu) = \chi(p+\mu) + \chi(p+\mu-1)/1.2+\dots \quad \dots(60)$$

$$\psi(p+\mu+\tfrac{1}{2}) = \chi(p+\mu)/1 + \chi(p+\mu-1)/1.2.3+\dots, \quad \dots(61)$$

where p is any positive *or negative* integer. In (60) and (61) μ is any root of the equation (8); μ may therefore be changed into $-\mu$.

It is interesting to note that although the series (34), which define $\psi(z)$ in the region $R(z) > -\frac{1}{4}$, fail to converge in general outside this region, yet when one of them does converge, it gives the correct value of the function—a result which of course could not have been anticipated. The formulae (60) and (61) could, however, be deduced by mathematical induction from (36), since they are known, from (34) to be true when $R(p+\mu) > -\frac{1}{4}$.

15. If we change from the variable α of Art. 1 to a variable ρ, such that $\rho = \frac{1}{2} c e^{\alpha}$, ...(62)
and if we use the function χ as defined in (10), the defining series (2) for $J(\nu, s, \kappa c, \alpha)$ takes the form

$$\sum_{n=-\infty}^{\infty} (-1)^n \sqrt{2/\pi}\, \chi(n+\mu)(\kappa\rho)^{2n+2\mu}; \quad \text{............(63)}$$

or, if $\kappa\rho = i\sigma$, so that $\sigma = \kappa\rho/i = \kappa c e^{\alpha}/2i$, ...(64)
the series for $J(\nu, s, \kappa c, \alpha)$ is

$$i^{2\mu} \sqrt{2/\pi}\, \Sigma \chi(n+\mu)\, \sigma^{2n+2\mu}; \quad \text{..................(65)}$$

write this for the moment as J_ν simply.

Then, from (3),

$$G(\nu, s, \kappa c, \alpha) = (\pi/2 \sin 2\mu\pi)\,(J_{-\nu} - e^{-i\nu\pi} J_\nu)$$
$$= \sqrt{\pi/2}\,(e^{-i\mu\pi}/\sin 2\mu\pi)\,\{\Sigma\chi(n-\mu)\,\sigma^{2n-2\mu}$$
$$- \Sigma\chi(n+\mu)\,\sigma^{2n+2\mu}\} \text{......(66)}$$

The asymptotic expansion of this function as $|\sigma| \to \infty$ contains the factor $e^{-\sigma}$; and we shall deduce it from the ordinary power expansion of the product $e^{\sigma} G(\nu, s, \kappa c, \alpha)$.

16. In the product of e^{σ} by the series in 65, *i.e.* the product of the two absolutely convergent series

$$1 + \sigma + \sigma^2/1.2 + \dots, \text{ and } \Sigma\chi(n+\mu)\,\sigma^{2n+2\mu},$$

the coefficient of $\sigma^{2n+2\mu}$ is

$$\chi(n+\mu) + \chi(n+\mu-1)/1.2 + \dots, \text{ or } \psi(n+\mu), \text{ by (60); ...(67)}$$

and the coefficient of $\sigma^{2n+2\mu+1}$ is

$$\chi(n+\mu)/1 + \chi(n+\mu-1)/1.2.3 + \dots, \text{ or } \psi(n+\mu+\tfrac{1}{2}), \text{ by (61). ...(68)}$$

Hence $e^{\sigma}\,\Sigma\,\chi\,(n+\mu)\,\sigma^{2n+2\mu}$

$$= \sum_{n=-\infty}^{\infty} \psi\,(\tfrac{1}{2}n+\mu)\,\sigma^{n+2\mu}. \qquad \text{.....................(69)}$$

Also $\quad e^{\sigma}\,\{\Sigma\chi\,(n-\mu)\,\sigma^{2n-2\mu} - \Sigma\chi\,(n+\mu)\,\sigma^{2n+2\mu}\}$

$$= \Sigma\psi\,(\tfrac{1}{2}n-\mu)\,\sigma^{n-2\mu} - \Sigma\psi\,(\tfrac{1}{2}n+\mu)\,\sigma^{n+2\mu}. \qquad \text{...............(70)}$$

17. The case when $\cos\,2\mu\pi = 0$ being exceptional in some respects, it is convenient to treat this case by a special method. We may put $\mu = \tfrac{1}{4}$. We have here

$$e^{\sigma}\,\Sigma\chi\,(n-\tfrac{1}{4})\,\sigma^{2n-\frac{1}{2}} = \Sigma\sigma^{2n-\frac{1}{2}}\{\chi\,(n-\tfrac{1}{4}) + \chi\,(n-\tfrac{5}{4})/1\,.\,2 + \ldots\}$$
$$+ \Sigma\sigma^{2n+\frac{1}{2}}\{\chi(n-\tfrac{1}{4})/1 + \chi\,(n-\tfrac{5}{4})/1\,.\,2\,.\,3 + \ldots\};$$
$$e^{\sigma}\,\chi\,(n+\tfrac{1}{4})\,\sigma^{2n+\frac{1}{2}} = \Sigma\sigma^{2n+\frac{1}{2}}\{\chi\,(n+\tfrac{1}{4}) + \chi\,(n-\tfrac{3}{4})/1\,.\,2 + \ldots\}$$
$$+ \Sigma\sigma^{2n+\frac{3}{2}}\{\chi\,(n+\tfrac{1}{4})/1 + \chi\,(n-\tfrac{3}{4})/1\,.\,2\,.\,3 + \ldots\}$$
$$= \Sigma\sigma^{2n+\frac{1}{2}}\{\chi\,(n+\tfrac{1}{4}) + \chi\,(n-\tfrac{3}{4})/1\,.\,2 + \ldots\}$$
$$+ \Sigma\sigma^{2n-\frac{1}{2}}\{\chi\,(n-\tfrac{3}{4})/1 + \chi(n-\tfrac{7}{4})/1\,.\,2\,.\,3 + \ldots\},$$

the last line being obtained by changing n into $n-1$.

Hence $\quad e^{\sigma}\,\{\Sigma\,\chi\,(n-\tfrac{1}{4})\,\sigma^{2n-\frac{1}{2}} - \Sigma\,\chi\,(n+\tfrac{1}{4})\,\sigma^{2n+\frac{1}{2}}\}$

$$= \Sigma\sigma^{2n-\frac{1}{2}}\{\chi\,(n-\tfrac{1}{4}) - \chi\,(n-\tfrac{3}{4})\,/\,1 + \chi\,(n-\tfrac{5}{4})\,/\,1\,.\,2 - \ldots\}$$
$$- \Sigma\sigma^{2n+\frac{1}{2}}\{\chi\,(n+\tfrac{1}{4}) - \chi\,(n-\tfrac{1}{4})\,/\,1 + \chi\,(n-\tfrac{3}{4})\,/\,1\,.\,2 - \ldots\}$$
$$= \Sigma\,(-1)^p \quad \sigma^{p-\frac{1}{2}}\,\{\chi\,(\tfrac{1}{2}p-\tfrac{1}{4}) - \chi\,(\tfrac{1}{2}p-\tfrac{3}{4})\,/\,1 + \chi\,(\tfrac{1}{2}p-\tfrac{5}{4})\,/\,1\,.\,2 - \ldots\}\,.$$

But if $p>0$, the coefficient of $\sigma^{p-\frac{1}{2}}$ here vanishes, by (34); and if p is 0 or negative, that coefficient is $(-1)^p\,B_{-p}$, by (35A).

Hence, when $\cos 2\mu\pi = 0$,

$$e^{\sigma}\,\{\Sigma\,\chi\,(n-\tfrac{1}{4})\,\sigma^{2n-\frac{1}{2}} - \Sigma\,\chi\,(n+\tfrac{1}{4})\,\sigma^{2n+\frac{1}{2}}\}$$
$$= \sigma^{-\frac{1}{2}} - B_1\,\sigma^{-\frac{3}{2}} + B_2\,\sigma^{-\frac{5}{2}} - \ldots + (-1)^p\,B_p\,\sigma^{-p-\frac{1}{2}} + \ldots \quad (71)$$

Thus, when $\mu = \tfrac{1}{2}N + \tfrac{1}{4}$, when N is any integer, we have from (66)

$$e^{\sigma}\,.\,G\,(\nu,\,s,\,\kappa c,\,\alpha) = \sqrt{(\pi/2)}\,(e^{-i\mu\pi}\,/\sin 2\mu\pi)$$
$$\{\sigma^{-\frac{1}{2}} - B_1\,\sigma^{-\frac{3}{2}} + B_2\,\sigma^{-\frac{5}{2}} - \ldots\}. \qquad \text{...........(72)}$$

It will now be proved that, in the general case when μ is unrestricted, the series in (72) represents the function $e^{\sigma}\,G$ *asymptotically.*

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18. It was pointed out at the end of Art. 9 that if $|z| \to \infty$ in any direction within the region $R(z) > -\frac{1}{4}$, then

$$\psi(z) \approx 2^{2z-\frac{1}{2}} / \Pi(2z+\tfrac{1}{2})\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(73)$$

this holds in particular if $z \to \infty$ in a direction parallel to the imaginary axis The latter result is true even when $R(z) \leq -\frac{1}{4}$, and is needed for what follows. To prove it, in the difference equation (36) for $\psi(z)$ put

$$\psi(z) = 2^{2z-\frac{1}{2}}\,\Pi(2z-\tfrac{1}{2})\,E_z / \Pi(2z+s)\,\Pi(2z-s). \quad \ldots\ldots(74)$$

The difference equation becomes

$$E_z - E_{z-\frac{1}{2}} = E_{z+1} \,.\, 4k^4(2z+\tfrac{1}{2})(2z+\tfrac{3}{2}) \div$$
$$\{(2z)^2 - s^2\}\{(2z+1)^2 - s^2\}\{2z+2)^2 - s^2\}. \quad \ldots\ldots(75)$$

From (73) and (74) $E_z \to 1$ when $I(z) \to \infty$, with $R(z)$ constant and $> -\frac{1}{4}$. It follows from (75) that $E_{z-\frac{1}{2}N} \approx E_z \approx 1$ (*cf.* Art. 4). Hence the required result follows.

19. Consider now the integral

$$I_t = \int \frac{\sigma^{2t}\,\psi(t)\,dt}{\sin 2(t+\mu)\pi \sin 2(t-\mu)\pi}, \quad \ldots\ldots\ldots\ldots(76)$$

the path, Q say, being downwards parallel to the imaginary axis along the line $t = -q$, where q is real and positive; we take q so that the path avoids any pole of the integrand.

Towards infinity on this path $\psi(t) \approx 1/\Pi(2t+\frac{1}{2})$, (Art. 18); and, if we write $2t + \frac{1}{2} = z = x + iy$, where x and y are real, then the integrand $\approx \sigma^{z-\frac{1}{2}} / \Pi(z) \,.\, \frac{1}{2}\cos 2\pi z$

$$\approx 2\sigma^{x-\frac{1}{2}}\,\sigma^{iy} / (iy)^x\,\Pi(iy)\cosh 2\pi y$$

If $|\sigma| = r$, and arg. $\sigma = \theta$, then $\sigma^{iy} = r^{iy}e^{-\theta y}$, and $|\sigma^{iy}| = e^{-\theta y}$; also $\Pi(iy)\,\Pi(-iy) = \pi y / \sinh \pi y$, and $|\Pi iy| = \sqrt{\pi y} / \sqrt{\sinh \pi y}$.
Hence $|\text{integrand}| \approx (2/\sqrt{\pi})\,|\sigma|^{x-\frac{1}{2}}\,y^{-x-\frac{1}{2}}\,e^{-\theta y - \frac{3}{2}\pi|y|}$.

Hence the integral (76) converges absolutely at both ends if $|\theta| < \frac{3}{2}\pi$, *i.e.* if arg. σ lies between $\frac{3}{2}\pi - \delta$ and $-\frac{3}{2}\pi + \delta$, where δ is a small positive number.

If arg. σ lies between these limits, it is easy to show that the integral (76) $= 2\pi i$ (sum of residues of integrand to right of path

Q). To prove this, consider the integral taken round the rectangle the equations of whose sides are $t = -q$, $2t + \frac{1}{2} = X$, $2t + \frac{1}{2} = \pm Y$, where X and Y will be made to tend to infinity.

Take $z = x + iy = 2t + \frac{1}{2}$, as before.

When x is finite, say between q and $-q$, the value of the integrand on $y = \pm Y$ has already been shown to tend to zero. When $x > q$, say, $| x + iy |$ is large over the sides of the rectangle other than $t = -q$, so that

$$\sigma^z / \Pi(z) \approx (1/\sqrt{2\pi z})(\sigma e / z)^z;$$

and $$z^z = e^{z \log z} = e^{(x+iy)(\log \sqrt{x^2+y^2} + i \arg. z)},$$

so that $$| z^z | > e^{x \log x - \frac{1}{2}\pi |y|}.$$

Hence the integral over $y = Y$ or $y = -Y$ is a convergent integral multiplied by a factor in Y which tends to zero; and similarily for $x = X$.

20. The integrand in (76) has simple poles at $t = \frac{1}{2}n \pm \mu$; and at the poles of $\psi(t)$, viz. (Art. 11) when $n \geqq 0$, at $t = -\frac{1}{2}n - \frac{1}{4}$; n being an integer.

Residue at $t = \frac{1}{2}n - \mu$ is $\sigma^{n-2\mu} \psi(\frac{1}{2}n - \mu) / (-2\pi \sin 4\mu\pi)$;

residue at $t = \frac{1}{2}n + \mu$ is $\sigma^{n+2\mu} \psi(\frac{1}{2}n + \mu) / 2\pi \sin 4\mu\pi$;

residue at $t = -\frac{1}{2}n - \frac{1}{4}$ is $\sigma^{-n-\frac{1}{2}}(-1)^n B_n \cos 2\mu\pi / 4\pi \cos^2 2\mu\pi$.

Hence, summing only for terms in which the index of the power of σ is greater than $(-2q)$,

$$\{\Sigma\sigma^{n-2\mu}\psi(\tfrac{1}{2}n - \mu) - \Sigma\sigma^{n+2\mu}\psi(\tfrac{1}{2}n + \mu)\} / \sin 2\mu\pi$$
$$= \sum_{n=0} (-1)^n B_n \sigma^{-n-\frac{1}{2}} + 2i \cos 2\mu\pi \, I_t. \quad \text{............(77)}$$

The difference between the series on the left of (77) and the complete series on the right of (70) is of order higher than $(1/\sigma)^{2q}$; and, in I_t, σ^{2t} can be written $\sigma^{-2q}\sigma^{2iy}$, so that I_t is of order $(1/\sigma)^{2q}$. Hence, by (66) and (70) the series

$$\sqrt{(\tfrac{1}{2}\pi)}\, e^{-i\mu\pi} \sum_{n=0}^{\infty} (-1)^n B_n \sigma^{-n-\frac{1}{2}} \quad \text{..................(78)}$$

gives the asymptotic expansion of the product of e^σ and a solution of Mathieu's equation, viz., of

$$\sqrt{(\tfrac{1}{2}\pi)}\,(e^{-i\mu\pi}/\sin 2\mu\pi) e^\sigma \{\Sigma\chi(n-\mu)\sigma^{2n-2\mu} - \Sigma\chi(n+\mu)\sigma^{2n+2\mu}\}, \quad (79)$$

i.e. of $e^{\sigma} G(\nu, s, \kappa c, \alpha)$, where, as at (64),

$\sigma = \kappa c\ e^{\alpha}/2i$; the argument of σ must be less than $\frac{3}{2}\pi$ and greater than $-\frac{3}{2}\pi$.

21. From the asymptotic expansion of $G(\nu, s, \kappa c, \alpha)$ in (78), (79), with the help of the definition (3) we can find at once the corresponding expansions for $J(\nu, s, \kappa c, \alpha)$. Referring to Art. 15 (64), (65), write for brevity $J(\nu, i\sigma)$ instead of $J(\nu, s, \kappa c, \alpha)$; so that

$$J(\nu, i\sigma) = \sqrt{(2/\pi)}\, e^{i\mu\pi} \Sigma\chi(n+\mu)\ \sigma^{2n+2\mu}.$$

This gives obviously

$$J(\nu, i\sigma e^{i\pi}) = J(\nu, i\sigma)\, e^{i\nu\pi}, \quad \text{...............(80)}$$

and

$$J(-\nu, i\sigma e^{i\pi}) = J(-\nu, i\sigma)\, e^{-i\nu\pi}. \quad \text{...............(81)}$$

In this notation the definition of G in (3) becomes

$$G(\nu, i\sigma) = (\pi/2\sin\nu\pi)\{J(-\nu, i\sigma) - e^{-i\nu\pi} J(\nu, i\sigma)\} \quad \text{......(82)}$$

We therefore have

$$G(\nu, i\sigma e^{i\pi}) = (\pi/2\sin\nu\pi)\{e^{-i\nu\pi} J(-\nu, i\sigma) - J(\nu, i\sigma)\} \quad \text{......(83)}$$

and $G(\nu, i\sigma e^{-i\pi}) = (\pi/2\sin\nu\pi)\{e^{i\nu\pi} J(-\nu, i\sigma) - e^{-2i\nu\pi} J(\nu, i\sigma)\}$ (84)

From (82), (83), $\pi i J(\nu, i\sigma) = G(\nu, i\sigma) - e^{i\nu\pi} G(\nu, i\sigma e^{i\pi})$; ......(85)

and from (82), (84), $\pi i J(\nu, i\sigma) = e^{i\nu\pi} G(\nu, i\sigma e^{-i\pi}) - e^{2i\nu\pi} G(\nu, i\sigma)$...(86)

If arg. σ lies between $-\frac{3}{2}\pi$ and $\frac{1}{2}\pi$, we may use (85); and if arg. σ lies between $-\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, we may use (86); the extreme values in these limits not being included.

The asymptotic expansion of $G(\nu, i\sigma)$ is, from (78), (79)

$$G(\nu, i\sigma) = \sqrt{\tfrac{1}{2}\pi}\ e^{-i\mu\pi} e^{-\sigma} \sum_{n=0} (-1)^n B_n \sigma^{-n-\frac{1}{2}}, \quad \text{.........(87)}$$

this being valid if arg. σ lies between $-\frac{3}{2}\pi$ and $\frac{3}{2}\pi$, these values themselves excluded.

22. The corresponding asymptotic expansions for the case when $R(\alpha) \to -\infty$, or $|\sigma| \to 0$, are now easily found.

The part played in the above analysis by the function $G(\nu, s, \kappa c, \alpha)$ is now taken by $G(\nu, s, \kappa c, -\alpha)$. Its asymptotic expansion is found from (87) by simply changing α into $-\alpha$, σ being $\kappa c e^{\alpha}/2i$. The expression for $G(\nu, s, \kappa c, -\alpha)$ in terms of the two functions. $J(\pm\nu, s, \kappa c, \alpha)$ is given at once by I (115), (119).

23. In terms of the variable σ, *i.e.* $\kappa c e^{a} / 2i$, the equation (1) becomes

$$\frac{d^2u}{d\sigma^2} + \frac{1}{\sigma}\frac{du}{d\sigma} - \left(1 + \frac{s^2}{\sigma^2} + \frac{k^4}{\sigma^4}\right) u = 0. \qquad \text{.................(88)}$$

If we put $u = ve^{-\sigma}$, then

$$\frac{d^2v}{d\sigma^2} + \left(\frac{1}{\sigma} - 2\right)\frac{dv}{d\sigma} - \left(\frac{1}{\sigma} + \frac{s^2}{\sigma^2} + \frac{k^4}{\sigma^4}\right) v = 0. \qquad \text{...........(89)}$$

If we try a solution

$$v = \sum_{n=0} (-1)^n B_n \sigma^{-n-\frac{1}{2}},$$

we find the equation (15) for the coefficients B; if we take $B_0 = 1$, the cofficients are found to be the same as the coefficients B already obtained as in Art. 3.

24. Equation (88), for the special case of $k = 0$, is the Bessel's equation whose solutions are $J_s(i\sigma)$, $G_s(i\sigma)$.

The equation for ν is now $\cos \nu\pi = \cos s\pi$, or $\nu = 2N \pm s$.

We find $\quad B_n = (1/2^n n!)\,(\frac{1}{4} - s^2)\,(\frac{9}{4} - s^2) \,..\, \{(n - \frac{1}{2})^2 - s^2\}$;

$$\chi(z) = \sqrt{(\tfrac{1}{2}\pi)}/2^{2z}\, \Pi(z + r)\, \Pi(z - r).$$

The expansion (14) for $\chi(z)$ becomes

$$\{1/\Pi(2z + \tfrac{1}{2})\}\; F(\tfrac{1}{2} - s,\ \tfrac{1}{2} + s,\ 2z + \tfrac{1}{2} + 1\ ;\ \tfrac{1}{2}),$$

F being the hypergeometric function. The identity of these two expressions for $\chi(z)$ is a well-known theorem of Kummer's, useful in Spherical Harmonics. The theorem (32) degenerates into another fundamental result in Spherical Harmonics. It will be found interesting to trace out the degenerate forms of the other formulae of this paper in a similar manner.