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SOME OBSERVATIONS ON THE DIOPHANTINE EQUATION $y^2 = x! + A$ AND RELATED RESULTS

MACIEJ ULAS

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Dedicated to Professor Andrzej Schinzel on the occasion of his 75th birthday

Abstract

We consider the Brocard–Ramanujan type Diophantine equation $y^2 = x! + A$ and ask about values of $A \in \mathbb{Z}$ for which there are at least three solutions in the positive integers. In particular, we prove that the set $\mathcal{A}$ consisting of integers with this property is infinite. In fact we construct a two-parameter family of integers contained in $\mathcal{A}$. We also give some computational results related to this equation.

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1. Introduction

One among many classical problems in Diophantine equations is a question posed by Brocard in [4, 5]. He asked about the characterisation of all integer solutions of the Diophantine equation

$$y^2 = n! + 1.$$

The same question was posed by Ramanujan in [13]. It is well known that $(n, y) = (4, 5), (5, 11), (7, 71)$ are solutions of this equation and that there are no additional solutions with $n \le 10^9$; this was proved by Berndt and Galway [2]. Under the assumption of the weak Szpiro conjecture, Overholt [12] proved that there are only finitely many solutions. Let us recall that the Szpiro conjecture says that there exists a constant $s > 0$ such that, for any triple of positive integers a, b, c with $a + b = c$,

$$c \le N(abc)^s,$$

where for a given integer m we have $N(m) = \prod_{p|m} p$, p prime. The number $N(m)$ is called the *radical of the integer* m and is just the product of primes dividing m

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taken without repetition. Overholt's result was generalised by Dąbrowski [6], who considered the equation

$$y^2 = x! + A, \quad A \in \mathbb{N}. \tag{1.1}$$

He proved unconditionally that if A is not a square then (1.1) has only finitely many solutions. The same result holds for values of A which are squares provided that the Szpiro conjecture is true. Some variants on (1.1) were considered by Dąbrowski [7], Kihel and Luca [9] and Kihel *et al.* [10].

One can also consult the paper of Berend and Harmse [1] and the section D25 in Guy [8].

In the papers of Togbé [14, 15] one can also find an interesting discussion related to (1.1). He proved that the system of equations

$$y^2 = x_1! + 4k + 7, \quad y^2 = x_2! + 4k + 6, \quad y^2 = x_3! + 4k + 2,$$

has exactly one solution in integers x_1, x_2, x_3, y provided that $k + 2$ is a square, say u^2, and then $x_1 = 0$ or 1, $x_2 = 2, x_3 = 3$ and $y = 2u$. He also noted that, for $A = 1$ and $A = 505$, (1.1) has at least three solutions in positive integers. As an open question he stated the following: *Can we find another value of A such that* (1.1) *has at least three solutions?*

The aim of this note is to give some computational and theoretical results devoted to (1.1). In particular, in Section 2 we prove that there are infinitely many positive integers A such that (1.1) has at least three solutions in the integers. We also state some open questions and conjectures which arose during our research. In Section 3 we give some results on variants of (1.1).

2. Results

In [14] it is shown that for $A = 505$ the Diophantine equation (1.1) has at least three solutions in integers. The question whether there exist other integers with this property is raised. From the work of Dąbrowski we know that in this case we necessarily have $A \equiv 0, 1 \pmod 4$ [6, Remark 1]. Because it is unclear what we can expect we decided to run a computer search for solutions of (1.1). To do this, a simple script was written in Magma [3]. Let us define the set

$$\mathcal{A} := \{A \in \mathbb{N} : \text{there are at least three solutions of the equation } y^2 = x! + A\}.$$

In Table 1 we tabulate values of $A \le 10^{10}$ for which $A \in \mathcal{A}$ and the solutions for x satisfying the condition $1 \le x \le 40$.

In the ranges considered we found 11 elements of the set $\mathcal{A}$. The large number of required As convinced us that we can expect that the set $\mathcal{A}$ is infinite. Let us also note that for any given A from Table 1 which is not a square, that is, $A \ne 1, 11664$, the presented set of solutions in x of (1.1) is the complete set. Indeed, from the work of Dąbrowski, as was noted in [2], we know that in this case each solution in x of (1.1)

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TABLE 1. Solutions of $y^2 = x! + A$ with $A \le 10^{10}$ and $x \le 40$.

A	x	A	x	A	x
1	4, 5, 7	29905	4, 6, 8	10806084	9, 11, 14
505	4, 5, 6	172201	4, 7, 8	119145744	9, 11, 15
2529	6, 7, 8	399249	8, 9, 10	226621476	9, 10, 11
11664	8, 9, 10	1608336	8, 9, 11		

is less than the smallest prime p for which the Legendre symbol $(A/p) = -1$. In each case of A from Table 1 we get that $x \le 19$.

Although we know that for any given nonsquare A there are only finitely many solutions of (1.1) and for relatively small A we can find them very quickly, it is a nontrivial task to find more elements of $\mathcal{A}$. To show that $\mathcal{A}$ is infinite we start with a simple preparatory observation.

LEMMA 2.1. *Let $a, b, u \in \mathbb{N}$ be given and suppose that $4u \mid a!$ and $a < b$. Then there exist positive integers y, A such that*

$$y^2 = a! + A, \quad (y + 2u)^2 = b! + A. \tag{2.1}$$

PROOF. To prove the lemma it is enough to note that (2.1) is equivalent to the following system:

$$4uy + 4u^2 = b! - a!, \quad y^2 = a! + A.$$

We have replaced the first equation in (2.1) by its difference with respect to the second equation. In this form the above system can easily be solved with respect to y and A. We thus get

$$y = \frac{1}{4u}(b! - a!) - u, \quad A = \frac{1}{16u^2}(b! - a!)^2 - \frac{1}{2}(b! + a!) + u^2.$$

From the assumption $4u \mid a!$ and $a < b$ we deduce that y, A are integers and the proof is complete. □

REMARK 2.2. We note that for given $u \in \mathbb{N}_+$ it is enough to assume that $a \ge 4u$ and then we clearly have $4u \mid a!$.

Lemma 2.1 shows that there are infinitely many positive integers A such that $y^2 = x! + A$ has at least two solutions. A slightly different result of this type was also noted in [11]. Now we are ready to prove the following result.

THEOREM 2.3. *Let $a, u \in \mathbb{N}$ be given and suppose that $a \ge 4u + 2$. Then there exist $A, b, c \in \mathbb{N}$ with $a < b < c$ such that*

$$a! + A = \square, \quad b! + A = \square, \quad c! + A = \square.$$

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Proof. Let us fix a positive integer u and let us suppose that $a \geq 4u$. Then it is enough to take

$$b = \frac{1}{4u^2}a! - 1, \quad c = b + 1 = \frac{1}{4u^2}a!, \quad A = \frac{1}{16u^2}(b! - a!)^2 - \frac{1}{2}(b! + a!) + u^2. \quad (2.2)$$

We then get

$$a! + A = \left(\frac{1}{4u}(b! - a!) - u\right)^2,$$
$$b! + A = \left(\frac{1}{4u}(b! - a!) + u\right)^2,$$
$$c! + A = \left(\frac{1}{4u}(b! - a!) + (2b + 1)u\right)^2.$$

From Lemma 2.1 we know that the first and second equations are satisfied for all $u, a, b \in \mathbb{N}$ with $4u \leq a < b$. We thus see that in order to complete the proof we need to check the last equality for $a \geq 4u + 2$ and b, c given by (2.2). Let us put

$$D = (b + 1)! + A - \left(\frac{1}{4u}(b! - a!) + (2b + 1)u\right)^2.$$

Expanding the expression for D and simplifying, $D = b(a! - 4u^2(b + 1))$. From our assumption we know that $b = (1/4u^2)a! - 1$. This equality immediately implies that $a! - 4u^2(b + 1) = 0$ and so $D = 0$. This equality implies the third equality above. □

Remark 2.4. Note that for $u = 1$ it is enough to assume that $a \geq 4$. If $u = 1$ then we take $a = 4$ and get $b = 5$. This gives $A = 505$ with solutions at $x = 4, 5, 6$. For the next value of a, that is, $a = 5$, we get $b = 29$ and the value of A is

$$A = 4886047197121241831575642137408516548543712516920573952000841,$$

which has 61 digits with solutions at $x = 5, 29, 30$.

Corollary 2.5. *The system of Diophantine equations*

$$y_1^2 - x_1! = y_2^2 - x_2! = y_3^2 - x_3!$$

has infinitely many solutions in the integers x_i, y_i, $i = 1, 2, 3$.

Because we were unable to find a value of $A \in \mathbb{Z}$ for which $y^2 = x! + A$ has at least four solutions we state a natural question.

Question 2.6. Does there exist an integer A such that $y^2 = x! + A$ has at least four solutions in positive integers?

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REMARK 2.7. It would also be interesting to study the Diophantine equation

$$y^2 = A - x!,$$

which clearly has only finitely many solutions in integers. One can easily prove that there exist infinitely many As such that the above equation has at least two integer solutions. We also found some instances of $A \leq 10^{10}$ for which there exist at least three solutions:

$$\begin{array}{llll} A = 145, & x = 1, 4, 5, & A = 46249, & x = 4, 7, 8, \\ A = 63121, & x = 5, 7, 8, & A = 4291662, & x = 8, 10, 11. \end{array}$$

We believe that the following conjecture is true.

CONJECTURE 2.8. The set

$$\mathcal{A}' = \{A \in \mathbb{Z} : \text{the equation } y^2 = A - x! \text{ has at least three solutions}\}$$

is infinite.

The next interesting case of (1.1) is the case when the value of A is a square. From the work of Dąbrowski we know that under the assumption of the weak Szpiro conjecture (1.1) has only finitely many solutions. A natural question arises whether it is possible to construct integers A such that A is square and (1.1) has ‘many’ solutions in integers. We used a computer search to find such values of A. In Table 2 we tabulate all values of $a \in \mathbb{N}$ for which there exist at least two positive integer solutions of (1.1), where $A = a^2$, and at least one solution satisfies the condition $1 \leq x \leq 35$.

To find entries in Table 2 we proceeded as follows. For given $x \leq 35$ we computed the set

$$D(x) := \{d \in \mathbb{N} : d \mid x!\},$$

which is just the set of divisors of $x!$. We thus know that for any given $d \in D(x)$ such that $a = \frac{1}{2}(d - (x!/d))$ is an integer (we can assume that this number is positive) there exists $y \in \mathbb{N}$ such that $y^2 = x! + A$, where $A = a^2$. This follows from the fact that if $y^2 = x! + a^2$ then $y - a$ and $y + a$ are divisors of $x!$, so for some $d \in D(x)$ we have $y - a = d$, $y + a = x!/d$ and we obtain

$$y = \frac{1}{2}\left(d + \frac{x!}{d}\right), \quad a = \frac{1}{2}\left(d - \frac{x!}{d}\right).$$

We denote by $P(x)$ the set of positive integers of the form $\frac{1}{2}(d - (x!/d))$, where $d \in D(x)$. We thus see that if $a \in P(x)$ then (1.1) with $A = a^2$ has at least one solution in integers. Moreover, the set $P(x)^2$, that is, the set of square values of elements of $P(x)$, contains all possible values of A such $y^2 = x! + A$ has a solution (here x is fixed).

To find two solutions for given $A = a^2$ we looked for further solutions up to $x \leq 50$. As the result of our search we found the values of a presented in Table 2 together with solutions in x. All the computations were performed by Magma. From the table

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TABLE 2. Values of a such that $y^2 = x! + a^2$ has at least two solutions and the smaller solution for x is at most 35.

a	x	a	x	a	x
1	4, 5, 7	35280	14, 15	271106640	17, 19
108	8, 9, 10	36576	12, 13	4409475840	22, 23
179	6, 8	48024	12, 13	7935621120	19, 21
204	7, 9	81720	12, 14	18641145600	22, 23
288	12, 13	267120	14, 15	561589459200	24, 27
508	9, 12	851040	16, 18	1143137318400	25, 27
828	8, 11	1060920	14, 15	2094434496000	24, 25
996	9, 11	1068480	16, 17	4693095288000	21, 24
1140	10, 11	2152080	14, 16	11579564304000	25, 27
2934	11, 13	4800160	16, 17	52250931532800	27, 29
3060	11, 15	25744320	16, 17	148245349824000	27, 30
11040	11, 13	29306592	16, 17	868006971127296000	32, 33
22640	10, 15	88361280	18, 19	144169160495044608000	35, 37
27360	11, 13	239499435	15, 18	507404900179457280000	35, 38

we can also deduce that if we want to find a such that $y^2 = x! + a^2$ has at least two solutions and one among them is relatively big then we should expect that a will be big too.

It is clear that this method allows us to find all $A = a^2$ such that (1.1) has at least one solution with $x \leq C$ where C is a given constant. However, the computation of the sets $D(x)$ and $P(x)$ is time- and memory-consuming and the question arises whether it can be done faster. We encourage readers with more computational skills to improve the method presented or find a new one.

For all $A = a^2$, where a is given in Table 2, we looked for a third (or a fourth when $a = 1$ or 108) solution of $y^2 = x! + A$ in the range $x \leq 10^4$ using the approach in Berndt and Galway [2] in the search for solutions of $y^2 = x! + 1$ with $x \leq 10^9$. For any given value of x and $A = a^2$ we computed the value of the Legendre symbol

$$\left(\frac{x! + A}{p}\right)$$

for each $p \in S$, where S is the set of 15 first primes which are greater than the square of the last entry for a in Table 2. For a given prime p, the computations were performed modulo p. If x was found such that, for each $p \in S$, $(x! + A)/p) \in \{0, 1\}$ then we tested such x for the next 15 primes. Unfortunately, in each case of A from Table 2 we did not find any additional solution with $x \leq 10^4$.

A very interesting question arises whether it can be proved that there are infinitely

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many $a \in \mathbb{N}$ such that $y^2 = x! + A$, where $A = a^2$, has at least two solutions in integers. We believe that this can be done and we decided to state the following conjecture.

CONJECTURE 2.9. The set

$$\mathcal{B} = \{A \in \mathbb{N}^2 : \text{the equation } y^2 = x! + A \text{ has at least two solutions}\}$$

is infinite.

One can also ask the following question.

QUESTION 2.10. Does there exist an integer $a > 108$ such that (1.1) with $A = a^2$ has at least three solutions?

We could not find any value of A such that A is negative with $-A \leq 10^{10}$ and $y^2 = x! + A$ has at least three solutions with $x \leq 50$. In fact, in the ranges considered there are only 204 values of A for which there exist at least two solutions of $y^2 = x! + A$. Moreover, for any given A we looked for a third solution in the range $x \leq 10^4$. We found no additional solution. In light of these computations it is natural to ask the following question.

QUESTION 2.11. Does there exist a negative integer A such that $y^2 = x! + A$ has at least three solutions in integers?

3. A few remarks on related Diophantine equations

In this section we give some remarks on two Diophantine equations which are similar to $y^2 = x! + A$. First we are interested in the following equation

$$y^2 = x!! + A, \tag{3.1}$$

where $x!!$ is called the double factorial, defined by

$$(2x)!! = 2 \cdot 4 \cdot \dots \cdot (2x-2) \cdot (2x), \quad (2x+1)!! = 1 \cdot 3 \cdot \dots \cdot (2x-1) \cdot (2x+1).$$

One can easily verify that

$$(2x)!! = 2^x x!, \quad (2x+1)!! = \frac{(2x+1)!}{2^x x!}.$$

Let us recall that the Hall conjecture asserts that for every $\epsilon > 0$, there exists a constant $C(\epsilon)$, depending only on ϵ, such that if X, Y and k are nonzero integers satisfying $Y^2 = X^3 + k$, then

$$\max\{|X|^3, Y^2\} \leq C(\epsilon)|k|^{6+\epsilon}.$$

We are now ready to prove the following result.

THEOREM 3.1. *If $A \in \mathbb{Z}$ is not a square then (3.1) has only finitely many solutions in integers. If A is a square then, provided the Hall conjecture is true, (3.1) has only finitely many solutions in integers.*

PROOF. If A is not a square then there exists a smallest prime p such that $(A/p) = -1$. This immediately implies that if $x > 2p$ then (3.1) is insolvable modulo p.

To prove the second statement of the theorem we consider two cases: x even and x odd. Because the proof is essentially the same in both cases we present the details only for the case of x even. We thus assume that x is even, say $x = 2n$, and let d and z be the two integers with d cube-free such that $x!! = (2n)!! = 2^n n! = dz^3$. From the well-known Chebyshev bound for the product of primes less than n,

$$d \leq \Big(\prod_{p<n, p \text{ prime}} p\Big)^2 < 4^{2n}.$$

Equation (3.1) gives

$$y^2 = dz^3 + A,$$

and thus

$$Y^2 = X^3 + d^2 A,$$

where $X = dz$ and $Y = dy$. Taking $\epsilon = 1$ in the Hall conjecture,

$$d^2 2^n n! = d^3 z^3 = X^3 \leq C(1)|d^2 A|^7.$$

To complete the proof, we note that $(n/2)^{n/2} < n!$ and thus

$$\left(\frac{n}{2}\right)^{n/2} 2^n < 2^n n! \leq C(1)|d|^{12}|A|^7 \leq C(1) 4^{24n}|A|^7.$$

Finally, we obtain

$$\left(\frac{n}{2^{95}}\right)^{n/2} \leq C(1)|A|^7,$$

which immediately implies that n is bounded. □

We performed a numerical search to find As such that (3.1) has at least three solutions in integers. The results of our computations in the range $0 < A \leq 10^{10}$ and $x \leq 50$ are presented in Table 3.

We now prove that the set

$$\mathcal{A}' := \{A \in \mathbb{N} : \text{there are at least three solutions of the equation } y^2 = x!! + A\}$$

is infinite. More precisely, we have the following result.

THEOREM 3.2. *Let $a \in \mathbb{N}_{\geq 3}$. Then there exist $A, b, c \in \mathbb{N}$ with $a < b < c$ such that*

$$(2a)!! + A = \square, \quad (2b)!! + A = \square, \quad (2c)!! + A = \square.$$

TABLE 3. Solutions of $y^2 = x!! + A$ with $A \leq 10^{10}$ and $x \leq 50$.

A	x	A	x	A	x
1	3, 4, 5, 6	72256	7, 12, 16	47195136	16, 18, 20
16	6, 7, 8, 9	896761	6, 10, 12	84584704	7, 14, 20
241	5, 6, 8	1738816	9, 12, 14	290492416	16, 18, 20
736	6, 7, 9	2968681	6, 11, 14	313102336	12, 16, 18
9616	8, 10, 12	3102976	14, 16, 18	450553104	16, 17, 20
20736	14, 15, 16	3322944	14, 15, 18	604265104	15, 17, 18
53776	6, 12, 14	6194176	9, 16, 18	2120510016	14, 15, 18
55224	1, 9, 15	25049641	8, 11, 14		

PROOF. It is clear that we consider $y^2 = 2^x x! + A$. We start with the easy observation that for $3 \leq a < b$ and

$$A = \frac{1}{16}(2^b b! - 2^a a!)^2 - \frac{1}{2}(2^b b! + 2^a a!) + 1,$$

we have $y^2 = 2^a a! + A$, $(y+2)^2 = 2^b b! + A$ with $y = \frac{1}{4}(2^b b! - 2^a a!)$. To complete the proof we take $b = 2^{a-3} a! - 1$ and $c = b + 1 = 2^{a-3} a!$. We show that

$$2^{b+1}(b+1)! + A = (y + 4b + 3)^2.$$

Let us put

$$D = 2^{b+1}(b+1)! + A - (\tfrac{1}{4}(2^b b! - 2^a a!) + 4b + 3)^2.$$

Expanding the expression for D and simplifying,

$$D = 8(2b+1)(2^{a-3} a! - b - 1).$$

From our assumption we know that $b = 2^{a-3} a! - 1$. This equality immediately implies that $2^{a-3} a! - b - 1 = 0$ and so $D = 0$. Our theorem is proved. □

REMARK 3.3. We can also ask whether similar results can be proved for the more general Diophantine equation $y^2 = Bt^x x! + A$, where $A \in \mathbb{Z}$ and $B, t \in \mathbb{N}$ are given. In a forthcoming paper ('Variations on the Brocard–Ramanujan equation') we investigate this and even more general equations and get some new results.

We also performed some numerical calculations related to (3.1) with A square, say $A = a^2$. Using the same approach as in the case of $y^2 = x! + A$, we obtained the results in Table 4. We looked for solutions with $x \leq 58$.

In the light of the results of our computations we believe that the following conjecture is true.

CONJECTURE 3.4. The set

$$\mathcal{B}' = \{A \in \mathbb{N}^2 : \text{the equation } y^2 = x!! + A \text{ has at least two solutions}\}$$

is infinite.

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TABLE 4. Values of a such that $y^2 = x!! + a^2$ has at least two solutions and the smaller solution for x is at most 58.

a	x	a	x	a	x
4	6, 7, 8, 9	22500	15, 25	882155520	30, 32
16	7, 10	25968	15, 18	905354240	28, 32
24	9, 12	34560	17, 22	977577984	30, 33
29	8, 11	40316	14, 20	1130250240	34, 36
64	9, 12	75648	20, 24	1659248640	32, 34
88	12, 14	92132	15, 16	8072904960	31, 36
144	14, 15, 16	114432	22, 26	32799621120	36, 40
192	12, 18	238080	20, 24	46800875520	34, 36
2592	16, 17	910080	24, 26	70258089825	35, 37
3264	15, 18	1363200	23, 24	2355255705600	40, 42
5704	15, 18	1898496	24, 28	4609934622720	42, 44
5724	16, 17	3048000	25, 26	2453155268788224000	56, 60
9600	15, 22	4113216	25, 26	4779351557552128000	58, 60
13248	16, 18	82122240	28, 30		

Moreover, one can ask the following question.

QUESTION 3.5. Does there exist an integer $a > 4$ such that (3.1) has at least four solutions in integers? Does there exist an integer $a > 144$ such that (3.1) has at least three solutions in integers?

The next equation, which is more or less a natural variant of (1.1), is

$$y^2 = P_x + A \quad \text{where } P_x = \prod_{i=1}^{x} p_i, \tag{3.2}$$

where p_i is the ith prime. The number P_x is called the *xth primordial number*. Primordial numbers were used originally by Euclid in the proof of the infinitude of the set of prime numbers. We prove the following easy result.

THEOREM 3.6. *If $A \in \mathbb{Z}$ is not a square then the Diophantine equation (3.2) has only finitely many solutions in integers. If A is a square then (3.2) has no solutions in integers.*

PROOF. If A is not a square then there exists a smallest prime p such that $(A/p) = -1$. This immediately implies that if $p_x > p$ then (3.2) cannot be solved modulo p.

In the case where A is square, say $A = a^2$, equation (3.2) can be rewritten as $(y - a)(y + a) = P_x$. We know that $2 \,\|\, P_x$ for each $x \in \mathbb{N}_+$, which implies that the numbers $y - a$ and $y + a$ are even (because they are of the same parity) which immediately implies that $4 \mid P_x$. This is clearly impossible. □

TABLE 5. Primitive solutions of $y^2 = L_x + A$ with $-10^9 \le A \le 10^9$ and $x \le 100$.

A	x	A	x
109	4, 5, 7	300609	11, 13, 17
184	4, 8, 9	5539284	16, 17, 19
841	8, 11, 13	14850196	16, 17, 23
3256	8, 9, 11	126044689	8, 13, 19
42424	4, 8, 9	639937204	17, 19, 23

We performed a small numerical search for the solutions of (3.2) with $-10^9 < A \le 10^9$ and $x \le 50$. We found only one positive A such that (3.2) has at least three solutions. For $A = 190$ we found solutions for $x = 2, 4, 5$. For A negative we found $A = -2141$ with solutions for $x = 5, 6, 7$. However, we can easily show that there are infinitely many integers A (positive and negative) such that (3.2) has at least two solutions. This leads us to the following question.

QUESTION 3.7. Does there exist an integer $A > 190$ such that (3.2) has at lest three solutions in integers? Does there exist an integer $A < -2141$ such that (3.2) has at least three solutions in integers?

Finally, we state the following provocative conjecture.

CONJECTURE 3.8. There exists a constant C independent of A such that $y^2 = P_x + A$ has no more than C solutions in integers.

REMARK 3.9. Just for fun, we also looked for instances of A such that

$$y^2 = \binom{2x}{x} + A, \tag{3.3}$$

and

$$y^2 = L_x + A, \tag{3.4}$$

where $L_x = \operatorname{lcm}(1, 2, \ldots, x-1, x)$ has at least three solutions in positive integers.

In the range $-10^9 < A \le 10^9$ we found only three values of A such that (3.3) has at least three solutions with $x \le 50$. These values and solutions for x are as follows:

$$A = 1192,\ x = 5, 6, 7; \quad A = 50605,\ x = 3, 6, 9; \quad A = 3456168,\ x = 7, 11, 12.$$

It would be interesting to construct infinitely many As such that (3.3) has at least three solutions in integers. We believe that this should be relatively easy.

In the case of (3.4) we are interested only in *primitive solutions*, that is, solutions $x_1, \ldots, x_k$ such that $L_{x_i} \ne L_{x_j}$ for $i \ne j$. Indeed, there exist different positive integers a, b such that $L_a = L_b$. For example $L_5 = L_6 = 60$. In Table 5 we present the results of our search for primitive solutions of (3.4) with $-10^9 \le A \le 10^9$ and $x \le 100$.

We found only one negative A in this range with the required property, namely, $A = -164$ with solutions at $x = 7, 8, 11$. We note that in the range $x \leq 100$ there are 36 essentially different values of the function L_x.

We expect the existence of infinitely many As such that (3.4) has at least three primitive solutions. However, we think that the proof of this fact may be difficult.

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MACIEJ ULAS, Jagiellonian University,
Faculty of Mathematics and Computer Science,
Institute of Mathematics, Łojasiewicza 6, 30-348 Kraków, Poland
e-mail: maciej.ulas@uj.edu.pl