

Semi-Affine Coxeter-Dynkin Graphs and $G \subseteq \text{SU}_2(\mathbb{C})$

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Abstract. The semi-affine Coxeter-Dynkin graph is introduced, generalizing both the affine and the finite types.

Semi-Affine Graphs

It is profitable to treat the so-called Coxeter-Dynkin diagrams as graphs. A classification of finite graphs with an adjacency matrix having 2 as the largest eigenvalue is made in a paper of John Smith [JHS]. It is in a combinatorial context and no reference is made to Coxeter-Dynkin diagrams there. This maximal eigenvalue property is a defining property of the affine diagrams. What is introduced in this note is a more weakly constrained graph in which we interpret the rational functions which arise in terms of a q -deformation of my correspondence [M1], [K]. Since these semi-affine graphs do not have symmetrizable matrices, this appears to imply a connection with singularities rather than Lie algebras.

Here we shall deal only with those of type **A**, **D**, and **E**. Undirected edges are treated as a pair of edges directed in opposing directions as in [FM], [M1], [M2]. By so doing, we can introduce the semi-affine graph which may be defined in terms of a graph of finite type with an additional edge (two for **A**-type) directed toward the affine node; equivalently it may be defined as an affine graph with any undirected edge connecting the affine node replaced by a directed edge directed toward the affine node. This is done by removing one of the two opposed directed edges.

Effectively the semi-affine graph generalizes both the affine and finite type graph since the affine node acts as a sink, and when weighted at the nodes, it satisfies the same constraints as the affine graph except the constraint imposed by the additional directed edge(s) in the affine graph. Note that weighting the extra node with zero yields the same constraints as the finite type graph. In some sense the semi-affine graph lies intermediate between the finite and affine graphs yet generalizes both.

We recall the original correspondence between subgroups, G , of $\text{SU}_2(\mathbb{C})$ and affine graphs of type **A**, **D**, and **E**. The nodes are the irreducible representations of G , and R_j is a successor of R_i if $R_j \subseteq R \otimes R_i$.

It is helpful to have a simple example at hand: for this we choose the type **D**₄. The finite,

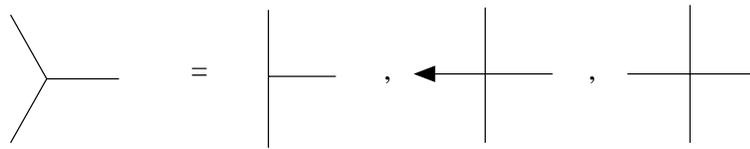
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semiaffine, and affine graphs of this type are:



To each of the $r + 1$ nodes, i , we attach a weight function, n_i , satisfying

$$tn_i = \sum_{j \leftarrow i} n_j,$$

summed over the successor nodes, j , of i . Consistent with the geometrical interpretation of summing over successor nodes, see [M1], we note that the affine weight, n_0 , occurs on the right side only. We initially normalize to $n_0 = 1$. For our example, this yields weights of $t/(t^2 - 3)$ for the central node, and $1/(t^2 - 3)$ for the other nodes. We now renormalize to make the weights polynomials in t by multiplying by the denominator, $t^2 - 3$.

Now $t^2 - 3$ is the minimal polynomial of $2 \cos(2\pi/2h)$ where h is the Coxeter number ($= 6$ here). This suggests writing $t = q + 1/q$ and clearing denominators. This yields respective weights of $1 - q^2 + q^4$, $q + q^3$, and q^2 for the affine, central, and other nodes. These are the numerators of generalized Molien series (described below) when written as a rational function of q with no common factor.

Normalizing the affine node to $1 + q^h = 1 + q^6$ by multiplying by $1 + q^2$, we find that the weights: $1 + q^6$, $q + 2q^3 + q^5$, and three of $q^2 + q^4$, are the numerators of the generalized Molien series [NJAS] but now written with standard denominator for finite $G \subset SU_2(\mathbb{C})$.

More generally, we find that each n_i is a rational function of t . For a finite graph we have a common denominator polynomial, $\text{Cox}(t)$, which is the minimal polynomial of $2 \cos(2\pi/2h)$, for h , the Coxeter number of the Weyl group associated with the graph. We put $t = q + q^{-1}$ and clear denominators to obtain polynomials, $n_i(q)$, for weights normalized so that $n_0(q) = 1 + q^h$.

For a matrix group, G , and irreducible character χ_i , we have the Molien series

$$m_i(G) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi_i(g)}{\det(I - gq)},$$

which, see [NJAS], for finite $G \subset SU_2(\mathbb{C})$, can be expressed in standard form as

$$m_i(G) = \frac{N_i(q)}{(1 - q^a)(1 - q^b)}$$

with $ab = 2|G|$, $a + b = h + 2$, h the Coxeter number assigned to G by the McKay correspondence, and $\{\chi_i\}$ ($\chi_0(g) = 1$), being the set of irreducible characters of G . We find that

$$(*) \quad (q + q^{-1})m_i(G) = \sum_{j \leftarrow i} m_j(G),$$

and $N_i(q) = n_i(q)$.

The semi-affine graph has characteristic polynomial $t^d \text{Cox}(t)$ of total degree = rank + 1, where degree $\text{Cox}(t) = \varphi(2h)/2$ and φ is Euler's function.

Specialization

The condition $N_0(q) = 1 + q^h = 0$ yields numeric weights for the finite type graphs.

We may instead impose the extra condition obtained by making the semi-affine graph into an affine one—this gives numeric weights for the affine graph, and these values are those for which the denominator of the standard form of the Molien series vanishes.

$$\mathbf{A}: (q + q^{-1})(1 + q^h) = 2(q + q^{h-1}) \quad \text{implying} \quad (1 - q^2)(1 - q^h) = 0;$$

$$\mathbf{D}: (q + q^{-1})(1 + q^h) = q + q^3 + q^{h-3} + q^{h-1} \quad \text{implying} \quad (1 - q^4)(1 - q^{h-2}) = 0;$$

$$\mathbf{E}: (q + q^{-1})(1 + q^h) = q + q^{a-1} + q^{b-1} + q^{h-1} \quad \text{implying} \quad (1 - q^a)(1 - q^b) = 0.$$

It is useful to note:

- (1) Chains starting at the affine node have the smallest exponent of q increasing by 1 at each successive node—similarly the largest exponent decreases by 1.
- (2) For even h , the exponents of q are alternately all even and all odd at adjacent nodes. For odd h ($= \mathbf{A}_m, m$ even), each odd exponent pairs with an even one.
- (3) The number of powers of q at a node is half the number of powers of q summed over adjacent nodes.

For brevity, I give the exponents of q along the longest chain starting at the affine node. Other nodes are either given last, or are determined by a graph symmetry fixing the affine node.

$$\mathbf{A}_m: n_k = q^k + q^{h-k}, k = 0, \dots, m, h = m + 1.$$

$$\mathbf{D}_m: \text{At the tips, } n_0 = 1 + q^h, n_1 = q^2 + q^{h-2} \text{ and } n_m = n_{m+1} = q^{m-2} + q^m, \text{ while the } m - 3 \text{ central nodes are weighed } q^k + q^{h-k} + q^{k+2} + q^{h-k-2}, k = 1, \dots, m - 3, h = 2m - 2.$$

$$\mathbf{E}_6: (0 + 12), (1 + 5 + 7 + 11), (2 + 4 + 2 \times 6 + 8 + 10), (3 + 5 + 7 + 9), (4 + 8) + \text{symmetry},$$

$$\mathbf{E}_7: (0 + 18), (1 + 7 + 11 + 17), (2 + 6 + 8 + 10 + 12 + 16), (3 + 5 + 7 + 2 \times 9 + 11 + 13 + 15), (4 + 6 + 8 + 10 + 12 + 14), (5 + 7 + 11 + 13), (6 + 12) + (4 + 8 + 10 + 14),$$

$$\mathbf{E}_8: (0 + 30), (1 + 11 + 19 + 29), (2 + 10 + 12 + 18 + 20 + 28), (3 + 9 + 11 + 13 + 17 + 19 + 21 + 27), (4 + 8 + 10 + 12 + 14 + 16 + 18 + 20 + 22 + 26), (5 + 7 + 9 + 11 + 13 + 2 \times 15 + 17 + 19 + 21 + 23 + 25), (6 + 8 + 12 + 14 + 16 + 18 + 22 + 24), (7 + 13 + 17 + 23) + (6 + 10 + 14 + 16 + 20 + 24).$$

Interpretation of q

The $n_i(q)$ are self-reciprocal since we start with rational functions of $t = q + q^{-1}$. This may be interpreted in terms of Poincaré duality.

The coefficients of q^k in the numerators N_i of $m_i(G)$ count the dimensions of certain fixed spaces under the group action [NJAS]; this exhibits q as a dimension-shifter. One may also interpret the main equation (*) as a trace equation in which we see $q + q^{-1}$ as the trace of an element in $\text{SU}_2(\mathbf{C})|_G$.

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