PART IV

PERIODIC ORBITS

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THE MECHANISM OF BRANCHING OF THREE-DIMENSIONAL PERIODIC ORBITS FROM THE PLANE

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ABSTRACT

A linearised treatment is presented of vertical bifurcations of symmetric periodic orbits (bifurcations of plane with three-dimensional orbits) in the circular restricted problem. Recent work on bifurcations from vertical-critical orbits ($a_v = \pm 1$) is extended to deal with the more general situation of bifurcations from vertical self-resonant orbits ($a_v = \cos(2\pi n/m)$ for integer m,n) and it is shown that in this more general case bifurcating families of three-dimensional orbits always occur in pairs, the orbital symmetry properties being governed by the evenness or oddness of the integer m. The applicability of the theory to the elliptic restricted problem is discussed.

1. INTRODUCTION

The occurrence of intersections of planar with three-dimensional periodic orbits of the circular restricted problem ("vertical" bifurcations) at planar orbits for which the vertical stability index $a_v = \pm 1$ ("vertical-critical" orbits) was first proposed by Hénon (1973). Markellos et al (1981) have discussed the mechanism of such bifurcations and calculated entire series of the vertical bifurcations of the basic "Strömgren families" of periodic orbits of the problem. Zagouras and Markellos (1977) and Zagouras and Kalogeropoulou (1978) have presented numerical results on the continuation of vertical-critical orbits into three dimensions, and showed that vertical bifurcations also occur at planar orbits for which

$$a_v = \cos(2\pi n/m), \qquad (1.1)$$

for integer values of m and n. Robin and Markellos (1980) gave examples of families of three-dimensional periodic orbits generated from such "multiple" vertical bifurcations for values of m up to 8, and found that, as had been anticipated by the second author, the bifurcating families always occur in pairs, each pair arising from the same self-

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resonant orbit, and that the three-dimensional symmetry properties of these families depend solely on whether the integer m, the "multiplicity" of the bifurcation, is even or odd.

The object of the present paper is to provide an analytical background to the above-mentioned numerical results on "multiple" vertical bifurcations of symmetric periodic orbits, and to show that the observed pattern of occurence of the bifurcating families of three-dimensional orbits is of general validity.

2. BIFURCATION CONDITION

With respect to the usual dimensionless barycentric rotating coordinate system ($x_1 = x$, $x_2 = y$, $x_3 = z$) (see e.g. Robin and Markellos, 1980, hereafter referred to as "Paper I"), the initial conditions for a symmetric planar periodic orbit may be written

$$\underset{\sim}{x}_o = (x_{01},\ 0,0,0,\ x_{05},\ 0). \tag{2.1}$$

This initial state of the massless third body of the system corresponds to a mirror configuration (Roy and Ovenden, 1955) in the horizontal plane, or plane of the primaries. The periodicity conditions for this "unperturbed" orbit can be expressed in the form

$$\begin{aligned} x_2\ (x_{01},\ x_{05},\ T/2) &= 0 \\ x_4\ (x_{01},\ x_{05},\ T/2) &= 0 \end{aligned} \tag{2.2}$$

where T is the orbital period. (Note that the components x_3 and x_6 of the state vector vanish for all values of the epoch t in this unperturbed orbit).

Let us now consider the orbit resulting from small "vertical perturbations" δx_{03} and δx_{06} in the initial conditions (2.1). The initial conditions of this perturbed orbit are

$$\underset{\sim}{x}_0 = (x_{01},\ 0,\ \delta x_{03},\ 0,\ x_{05},\ \delta x_{06}). \tag{2.3}$$

As Hénon (1973) has pointed out, the horizontal components (x_1, x_2, x_4, x_5) of the state vector are, in the linear approximation, unaffected by purely vertical perturbations (see also, Markellos et al, 1981). Denoting the "vertical" components of the state vector in the perturbed orbit by (δx_3, δx_6), we may express these in terms of the initial perturbations (δx_{03}, δx_{06}) as follows:

$$\begin{pmatrix} \delta x_3 \\ \delta x_6 \end{pmatrix} = \begin{pmatrix} v_{33} & v_{36} \\ v_{63} & v_{66} \end{pmatrix} \begin{pmatrix} \delta x_{03} \\ \delta x_{06} \end{pmatrix}, \tag{2.4}$$

where the $v_{k\ell}$'s are the first-order "variations" $\partial x_k/\partial x_{0\ell}$ for the

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unperturbed orbit.

We now seek to establish periodicity conditions for the perturbed orbit in terms of the Periodicity Theorem of Roy and Ovenden (1955). Referring to Equations (2.1) and (2.2) of Paper I, we see that the initial conditions of the perturbed orbit correspond to a mirror configuration if and only if

$$\delta x_{06} = 0 \tag{2.5}$$

or

$$\delta x_{03} = 0 \tag{2.6}$$

according as the mirror configuration is of type (P) (in-plane) or type (A) (on-axis). (Note that this distinction arises because we are now considering a "three-dimensional" rather than a planar mirror configuration, as was the case for the unperturbed orbit).

By the Periodicity Theorem, the perturbed orbit resulting from initial perturbations (δx_{03}, δx_{06}) satisfying either of the above conditions will be periodic if, at some epoch $t \neq 0$, another mirror configuration occurs. Since, as we have just seen, the horizontal part of the perturbed motion is (to first order) unaffected, it is clear that a mirror configuration can only take place at those epochs corresponding to the occurrence of a mirror configuration in the unperturbed planar periodic orbit: that is, for values of t given by

$$t = N(\frac{T}{2}), \tag{2.7}$$

where N is some positive integer. The condition for a mirror configuration in the perturbed orbit at epoch t satisfying Equation (2.7) is then either

$$\delta x_6 = 0 \tag{2.8}$$

or

$$\delta x_3 = 0, \tag{2.9}$$

again depending on the type of configuration. Combining Equations (2.4)-(2.9), we see that the periodicity conditions for the perturbed orbit can be written

$$\delta x_j = v_{ji}(NT/2) \ \ \delta x_{0i} = 0 , \tag{2.10}$$

where $i = 3$ for a type (P) and $i = 6$ for a type (A) mirror configuration at the initial epoch, while $j = 6$ for a type (P) and $j = 3$ for a type (A) mirror configuration at the final epoch (as in Table I of Paper I), and $v_{ji}(NT/2)$ denotes the variation $\partial x_j / \partial x_{0i}$ evaluated at $t = NT/2$ on the unperturbed orbit. Thus, δx_{0i} is always the non-zero initial perturbation; $\delta x_{0i} = 0$ is the trivial solution of Equation (2.10) corresponding to the original unperturbed orbit.

Equation (2.10) expresses the condition that in the linear approximation, there exists a family of three-dimensional symmetric periodic orbits parametrised by the perturbation δs_{0i}, bifurcating from the planar periodic orbit. The condition for the occurrence of a vertical bifurcation from a symmetric planar periodic orbit is therefore that one (at least) of the four elements (v_{33}, v_{36}, v_{63}, v_{66}) of the vertical submatrix V_v of the full variational matrix $V = [v_{k\ell}]_{6x6}$ vanishes at an epoch t equal to an integer number of half-periods of the orbit.

3. PROPERTIES OF BIFURCATING FAMILIES

The symmetry properties of the bifurcating three-dimensional orbits depend on the mirror configuration types occurring at the initial (t = 0) and final (t = NT/2) epochs, and hence on the values of the subscripts i and j in Equation (2.10). This means that we can predict the symmetry class (plane symmetric, axisymmetric or doubly-symmetric) of the bifurcating family by identifying which of the four elements of the matrix V_v vanishes at the appropriate epoch. For example, a family of plane symmetric three-dimensional orbits would be expected to bifurcate from a planar periodic orbit for which $v_{63}(NT/2) = 0$ for some value of N. The four possible cases are listed in Table I.

The interval between successive mirror configurations for the three-dimensional periodic orbits in the neighbourhood of the bifurcation, as we have seen, is equal to NT/2 for some integer N. This interval is equal to half of the orbital period for a three-dimensional orbit of simple symmetry (plane symmetric or axisymmetric), and equal to a quarter of the period for a doubly-symmetric orbit. Thus, in the linear approximation, the period of the three-dimensional orbits arising from a vertical bifurcation is equal to NT or 2NT according to whether the orbits are of simple or double symmetry, respectively (T being the period of the planar orbit at which the bifurcation takes place). The final column of Table I gives the values of the periods in each case.

Table I

Case	Type of Mirror Congiguration at: Initial Epoch	Final Epoch	Symmetry Class	i	j	Orbital Period
1	P	P	Plane Symmetric	3	6	NT
2	A	A	Axisymmetric	6	3	NT
3	A	P	Doubly-	6	6	2NT
4	P	A	Symmetric	3	3	2NT

The elements of the "vertical variational matrix"

$$V_v = \begin{pmatrix} v_{33} & v_{36} \\ v_{63} & v_{66} \end{pmatrix} \tag{3.1}$$

satisfy the well-known area-preserving property

$$\det V_v = v_{33}v_{66} - v_{36}v_{63} = 1 \tag{3.2}$$

(see, e.g. Hénon, 1973). The terms $v_{33}v_{66}$ and $v_{36}v_{63}$ cannot both vanish, and so zero elements of V_v, indicating a vertical bifurcation, can only occur either singly, or as one of the diagonal pairs (v_{33},v_{66}) or (v_{36},v_{63}). Table I shows that in consequence of this fact, a vertical bifurcation orbit can only give rise either to a single family of three-dimensional orbits, corresponding to one and only of Cases 1-4 of the table, or else to two families of three-dimensional orbits, corresponding to Cases 1 and 2 or to Cases 3 and 4 (that is, one family of plane symmetric and one of axisymmetric orbits, or a pair of families of doubly-symmetric orbits). As we shall see presently, the former situation applies in general to vertical-critical orbits, and the latter to vertical self-resonant (non-critical) orbits.

4. BIFURCATION FROM VERTICAL-CRITICAL ORBITS

Let us first of all consider the case of a planar orbit which has a zero element appearing in the matrix $V_v(NT/2)$ for $N = 1$. The full set of vertical stability indices a_v, b_v, c_v, d_v is defined by

$$V_v(T) = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} , \tag{4.1}$$

and the elements of $V_v(T/2)$ are denoted

$$V_v(T/2) = \begin{pmatrix} A_v & B_v \\ C_v & D_v \end{pmatrix} . \tag{4.2}$$

It can easily be shown that for a symmetric orbit, these two sets of quantities are related by

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$$\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} = \begin{pmatrix} A_vD_v + B_vC_v & 2B_vD_v \\ 2A_vC_v & A_vD_v + B_vC_v \end{pmatrix} \quad (4.3)$$

(Hénon, 1973). This incidentally shows the important property $a_v = d_v$ for a symmetric planar periodic orbit.

By Equations (2.10) and (4.2), with $N = 1$, Cases 1-4 of Table I correspond respectively to

$$C_v = 0; \ B_v = 0; \ D_v = 0; \ A_v = 0. \quad (4.4)$$

Now from Equations (3.2) and (4.3), the vertical stability index a_v is given by

$$a_v = 2A_vD_v - 1 = 2B_vC_v + 1. \quad (4.5)$$

In each of the cases (4.4), $|a_v| = 1$: we are therefore dealing with bifurcation from vertical-critical orbits, which has been discussed by Hénon (1973). In the first two cases ($C_v = 0$ and $B_v = 0$), a_v has the value ± 1, corresponding to the values $m = n = 1$ in Equation (1.1). This can be described as a "simple bifurcation", since in the neighbourhood of the bifurcation, the period of the three-dimensional orbits is (to first order) equal to T, the period of the vertical-critical orbit; the orbital multiplicities are of course also equal. In the second two cases of (4.4), $A_v = 0$ and $D_v = 0$, Equation (4.5) shows that $a_v = -1$, corresponding to the values $m = 2$, $n = 1$ in Equation (1.1): this can be described as a "double bifurcation", since it involves a doubling of the period and orbital multiplicity of the vertical-critical orbit (the "multiplicity" of an orbit being defined as half the number of crossings of the (x_1,x_3)-plane occurring in one period).

Since the parameters A_v, B_v, C_v and D_v are all independent, zero elements of the matrix $V_v(T/2)$ will in general occur singly: thus, a vertical-critical orbit will as a rule give rise to only one family of three-dimensional periodic orbits. The summetry properties of the bifurcating family depend on which of the four elements of $V_v(T/2)$ vanishes, as indicated in Table I; this has been clearly illustrated by Hénon (op. cit.).

5. BIFURCATION FROM VERTICAL SELF-RESONANT ORBITS

Let us now consider the case of a planar orbit for which $V_v(NT/2)$ $(N > 1)$ has at least one zero element, such that all of the elements A_v, B_v, C_v and D_v of $V_v(T/2)$ are non-zero. It can be seen from Equation (4.5) that this latter constraint excludes from consideration the special case of vertical-critical orbits ($a_v = \pm 1$), which were dealt with separately in the previous section. As we shall see later, we are

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now dealing with vertical self-resonant orbits, for which a_v is given by Equation (1.1) with values of the integer m greater than 2.

In order to relate the occurrence of a zero element of $V_v(NT/2)$ for some $N > 1$ to the value of the vertical stability index a_v, we make use of the well-known property of the variational matrix

$$V(t + T) = V(t)V(T) \tag{5.1}$$

(e.g. Wintner,1946); this general property can be applied in particular to the "vertical submatrix" V_v of V. It is convenient to consider separately the cases of even and of odd values of the integer N. Using Equation (5.1), we may express $V_v(NT/2)$ for odd values of $N = 2r+1$ as

$$V_v(NT/2) = V_v(T/2 + rT) = V_v(T/2)\,[V_v(T)]^r \tag{5.2}$$

$$(r = 0,1,2,\ldots).$$

Similarly, for even values of $N = 2r$ we have

$$V_v(NT/2) = V_v(rT) = [V_v(T)]^r \tag{5.3}$$

$$(r = 0,1,2,\ldots).$$

Note that although we are restricting our attention to values of $N > 1$, the above formulae are valid for all non-negative values of N.

It is easily shown by induction that the vertical variational matrix V_v computed at $t = NT/2$ $(N = 0,1,2,\ldots)$ satisfies the following two equations:

$$V_v(T/2 + rT) = \begin{pmatrix} \alpha_r A_v & \beta_r B_v \\ \beta_r C_v & \alpha_r D_v \end{pmatrix} \quad (r \geq 0) \tag{5.4}$$

where α_r and β_r are functions of A_v, B_v, C_v and D_v only;

$$V_v(rT) = \begin{pmatrix} \gamma_r & 2\delta_r B_v D_v \\ 2\delta_r A_v C_v & \gamma_r \end{pmatrix} \quad (r \geq 0), \tag{5.5}$$

where γ_r and δ_r are functions of A_v, B_v, C_v and D_v only. We may therefore state the following:

If the elements A_v, B_v, C_v and D_v of $V_v(T/2)$ are all non-zero, then for all values of $N > 1$, $V_v(NT/2)$ has either no zero elements, or exactly two zero elements on the same diagonal.

This follows from Equation (3.2), together with Equation (5.4) for odd values of N ($=2r+1$, $r = 1,2,3,\ldots$), and Equation (5.5) for even values of N ($=2r$, $r = 1,2,3,\ldots$), the important point being the appearance of the common factors (α_r, β_r), (γ_r, δ_r) in the diagonal pairs of elements: since A_v, B_v, C_v and D_v are assumed to be all non-zero, an element of $V_v(NT/2)$ can vanish only if one of the functions α_r, β_r, γ_r or δ_r is zero.

We therefore have the important result that families of three-dimensional periodic orbits bifurcating from a vertical self-resonant periodic orbit for which $N > 1$ (that is, excluding vertical-critical orbits) always occur in pairs, and as we have already seen, both families must consist either of simply-symmetric or of doubly-symmetric orbits.

Using Equations (5.1) (as applied to V_v), (4.1) and (4.3), together with Equations (5.4) and (5.5), pairs of simultaneous recurrence relations can be established for the functions α_r, β_r, γ_r and δ_r:

$$\begin{aligned} \alpha_r &= a_v\alpha_{r-1} + (a_v-1)\beta_{r-1}, \\ \beta_r &= a_v\beta_{r-1} + (a_v+1)\alpha_{r-1}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \gamma_r &= a_v\gamma_{r-1} + (a_v^2-1)\delta_{r-1}, \\ \delta_r &= a_v\delta_{r-1} + \gamma_{r-1}. \end{aligned} \tag{5.7}$$

Since $|a_v| \neq 1$, the β's can be eliminated from Equations (5.6), and the δ's from Equations (5.7), giving

$$\begin{aligned} \alpha_{r+1} - 2a_v\alpha_r + \alpha_{r-1} &= 0, \\ \gamma_{r+1} - 2a_v\gamma_r + \gamma_{r-1} &= 0. \end{aligned} \tag{5.8}$$

The general solutions of these two identical second-order recurrence relations are

$$\begin{aligned} \alpha_r &= Ae^{ir\varphi} + Be^{-ir\varphi}, \\ \gamma_r &= Ce^{ir\varphi} + De^{-ir\varphi}, \end{aligned} \tag{5.9}$$

where

$$\cos\varphi = a_v, \tag{5.10}$$

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and A, B, C, D are constants to be determined from the initial conditions

$$\alpha_0 = 1, \qquad \alpha_1 = 2a_v - 1; \qquad \gamma_0 = 1, \qquad \gamma_1 = a_v . \tag{5.11}$$

Calculation of the four constants yields

$$\alpha_r = \frac{\sin (r+1)\varphi - \sin r\varphi}{\sin \varphi}, \tag{5.12}$$

$$\gamma_r = \cos r\varphi, \tag{5.13}$$

and the associated solutions for β_r, δ_r are found to be

$$\beta_r = \frac{\sin (r+1)\varphi + \sin r\varphi}{\sin \varphi}, \tag{5.14}$$

$$\delta_r = \frac{\sin r\varphi}{\sin \varphi} . \tag{5.15}$$

Note that since $|\cos \varphi| = |a_v| \neq 1$ for vertical self-resonant orbits, $\sin\varphi \neq 0$.

Let us consider the conditions for the occurrence of a pair of zero elements of the matrix $V_v(NT/2)$, for odd values of $N = 2r+1$ ($r = 1,2,3, \ldots$). It is clear from Equation (5.4) that this requires either α_r or β_r to vanish, for some $r > 0$. By Equation (5.12), the function α_r is equal to zero if and only if

$$\sin (r+1)\varphi = \sin r\varphi , \tag{5.16}$$

with solutions

$$\varphi = \left(\frac{2k+1}{2r+1}\right)\pi, \tag{5.17}$$

where k is an arbitrary integer, such that $(2k+1)/(2r+1)$ is not an integer. Substitution of the solutions (5.17) into Equation (5.10) gives

$$a_v = \cos\left(\frac{2k+1}{2r+1}\right)\pi, \tag{5.18}$$

and with $0 \leq k < r$, the complete set of r roots of α_r, a polynomial in a_v of degree r, is obtained. (A duplicate set of r solutions is obtained for $r < k \leq 2r$).

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In a similar way, the roots of β_r, also an r^{th} degree polynomial in a_v, are found to be given by

$$a_v = \cos\left(\frac{2k}{2r+1}\right)\pi \ , \tag{5.19}$$

which with $0 < k \leq r$ gives the complete set of r roots of β_r. Note that the two sets of solutions (5.18) and (5.19) are mutually exclusive, as would be expected, since α_r and β_r cannot vanish simultaneously.

We now consider the conditions for the occurrence of a bifurcation for even values of $N = 2r$ $(r = 1,2,3,\ldots)$, that is, for the appearance of a pair of zero elements of $V_v(rT)$. This requires that either γ_r or δ_r vanishes, for some $r > 0$. From Equation (5.13), the roots of γ_r are found to be

$$a_v = \cos\left(\frac{2k+1}{2r}\right)\pi, \tag{5.20}$$

the complete set of r solutions being given by $0 \leq k < r$. Similarly, from Equation (5.15), δ_r has roots

$$a_v = \cos\left(\frac{k}{r}\right)\pi \ , \tag{5.21}$$

the complete set of roots of the polynomial δ_r, of degree r-1 in a_v, being given by $0 < k < r$.

Let us now relate these results to the vertical self-resonance condition (1.1),

$$a_v = \cos\left(\frac{2\pi n}{m}\right), \tag{5.22}$$

m and n being mutually prime integers with $0 < n \leq m$. The vertical-critical cases $m = n = 1(a_v = \pm 1)$ and $m = 2$, $n = 1(a_v = -1)$ are excluded, having been dealt with already. From Equations (5.18)-(5.21), the condition for the occurrence of a bifurcation associated with the vanishing of one of the functions α_r, β_r, γ_r, δ_r (and therefore of one of the diagonal pairs of elements (v_{33}, v_{66}), (v_{36}, v_{63}) of the matrix $V_v(NT/2)$) can be expressed in the form (5.22), with the values of the integers m and n in each case as given in Table II, with r any positive integer.

The final entry of Table II, corresponding to $\delta_r = 0$, is essentially redundant, since all the possible combinations of values of m and of n can be constructed from the entries corresponding to the cases $\alpha_r = 0$ and $\gamma_r = 0$. This redundancy of solutions reflects the fact that a doubly-symmetric periodic orbit can be regarded as simply-symmetric if one of its symmetries is ignored; the bifurcation of doubly-symmetric orbits corresponding to $v_{33}(N_1T/2) = v_{66}(N_1T/2) = 0$, for some $N_1 > 1$ (α_r or γ_r equal to zero), automatically gives $v_{36}(N_2T/2) = v_{63}(N_2T/2) = 0$, where $N_2 = 2N_1$ is even (that is, $\delta_{N_1} = 0$). The occurrence of a bifurca-

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Table II

Function	m	n	
α_r	$2(2r+1)$	$2k+1$	: $k = 0,1,2,\dots,r-1$
β_r	$2r+1$	k	: $k = 1,2,\dots,r$
γ_r	$4r$	$2k+1$	: $k = 0,1,2,\dots,r-1$
δ_r	$2r$	k	: $k = 1,2,\dots,r-1$

tion of genuinely simply-symmetric orbits is associated with the vanishing of the function β_r, for some $r > 1$.

The possible values of the integer m in Equation (5.22) in each case of Table II (with $r = 1,2,3,\dots$) are

$$\alpha_r = 0 : \quad m = 6,10,14,\dots$$

$$\beta_r = 0 : \quad m = 3,5,7,\dots$$

$$\gamma_r = 0 : \quad m = 4,8,12,\dots$$

which together account for all integer values greater than 2 ; the values $m = 1$ and $m = 2$ applying to the special case of vertical-critical orbits. It is evident that an even value of m corresponds to the case of doubly-symmetric three-dimensional bifurcating orbits ($v_{33} = v_{66} = 0$), while an odd value of m corresponds to a bifurcation with a family of simply-symmetric orbits ($v_{36} = v_{63} = 0$). The following conclusion may therefore be stated:

> A vertical self-resonant orbit, with vertical stability index given by Equation (5.22), gives rise to one family of axisymmetric and one of plane symmetric three-dimensional orbits if m is odd, or to two families of doubly-symmetric orbits if m is even.

6. REMARK

The foregoing discussion, which is of general validity in the circular restricted problem, can easily be extended to the elliptic restricted problem, the only difference being that the orbital period of the three-dimensional orbits arising from a vertical bifurcation in the elliptic case have fixed period (an integer multiple of the period of the primaries), the eccentricity of the orbit of the primaries varying along the bifurcating family instead of the period (Robin,1981). The pattern of vertical bifurcations in the two versions of the problem would therefore appear to be identical, in terms of the occurrence of pairs of vertical branches whose symmetry properties are governed by the evenness or oddness of the "multiplicity" m of the bifurcation and the special nature of bifurcation from vertical-critical orbits.

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