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MAXIMAL PRE-PRIMAL CLUSTERS

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A number of unsolved problems of primal algebra theory concern the existence of certain collections of dependent primal algebras. In [**3**] E. S. O'Keefe showed that any collection of pairwise non-isomorphic primal algebras of type $\{n\}$ with $n > 1$ forms a primal cluster. Recently the author has discovered that if τ is any type containing at least two elements, one of which is > 1, then there are at least two non-isomorphic dependent primal algebras of type τ, except possibly in the case $\tau = \{2, 0\}$; this result will appear later. (In [**1**] it is stated that F. M. Sioson proved in [**5**] that any collection of pairwise non-isomorphic primal algebras of type $\{n, 0\}$ with $n > 1$ also forms a primal cluster; an examination of Sioson's proof, however, reveals that each of the primals considered is required to satisfy a certain permutation condition which need not hold for an arbitrary primal algebra of that type.)

The exact number of distinct maximal primal clusters of a given type is unknown, except for the case $\{n\}$ mentioned above when there is only one. It is not even known whether the number must be finite for a type containing only finitely many finite elements.

By definition the class of polynomial functions of a primal algebra is complete in the sense that every finitary function defined on the carrier of the algebra is representable by a polynomial in the primitive operations of the algebra. A set $\mathscr{U}$ of finitary functions defined on a finite set A is said to be *pre-complete* provided (i) $\mathscr{U}$ is closed under composition, (ii) $\mathscr{U}$ is not complete in the sense that there is a finitary function defined on A which is not contained in $\mathscr{U}$, and (iii) the set $\mathscr{V}$ is complete in the sense of (ii), where $\mathscr{V}$ is the set of finitary functions on A generated under composition by $\mathscr{U}$ and any finitary function on A which is not in $\mathscr{U}$. Pre-complete sets of functions have been studied and classified by S. V. Jablonskii in [**2**]. We define a *pre-primal algebra* $\mathfrak{A}$ to be an algebra of finite or countably infinite, finitary type whose carrier is a finite set containing more than one element and whose set of polynomial functions is pre-complete. By a *pre-primal cluster* we mean a set of similar pre-primal algebras which is also a *cluster* in the sense that any finite collection of pairwise non-isomorphic algebras from the set is independent; by a *maximal* pre-primal cluster we mean a pre-primal cluster which is not properly contained in any other pre-primal cluster. We call two maximal pre-primal clusters of the same type *distinct* provided each contains an algebra which is isomorphic to none of the algebras in the other. We will show, assuming the

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Axiom of Choice, that there are infinitely many maximal pre-primal clusters of most types.

In the course of the proof we will need a special case of a result of Jablonskii's. Let A be a finite set containing more than one element and let θ be an equivalence relation defined on A. Let $f : A^n \to A$ be any finitary function defined on A. Then f is said to conserve θ provided $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ with $a_i \theta b_i$ for $i = 1, \ldots, n$ implies $f(a_1, \ldots, a_n) \theta f(b_1, \ldots, b_n)$. Denote by $\mathscr{U}(\theta, A)$ (or simply by $\mathscr{U}(\theta)$ if no confusion can arise) the totality of finitary functions defined on A which conserve θ.

THEOREM A (Jablonskii [**2**]). *If the set A has finite, non-prime cardinality, then $\mathscr{U}(\theta, A)$ is pre-complete.*

Let $A_n = \{0, 1, \ldots, 2^n - 1\}$ denote the first 2^n non-negative integers, where $n > 1$. Define the functions F_1, F_2, F_3, and G, of ranks 2, 2, 2, and 1, respectively, as follows:

$$F_1(x, y) = x \cdot y \quad (\text{modulo } 2^n),$$
$$F_2(x, y) = x + y \quad (\text{modulo } 2^n),$$
$$F_3(x, y) = \begin{cases} 2 & \text{if } x = y = 0 \\ 0 & \text{otherwise,} \end{cases}$$
$$G(x) = x + 1 \quad (\text{modulo } 2^n).$$

THEOREM 1. *The set of functions generated by F_1, F_2, F_3, and G is pre-complete.*

Proof. Define the equivalence relation θ on A_n by the partition

$$\theta : \{0, 2, 4, \ldots, 2^n - 2\}, \quad \{1, 3, 5, \ldots, 2^n - 1\}.$$

Then each of the functions F_1, F_2, F_3, and G belongs to $\mathscr{U}(\theta)$ and thus so does the set of functions they generate. Moreover, by Theorem A, this latter set is not complete.

The following are easily seen to be polynomials of A_n:

(i) $0(x) = x \cdot G(x) \ldots G^{(2^n-1)}(x) = 0$ for all $x \in A_n$;

(ii) if $r \in A_n$, then $R_r(x) = G^{(r)}(0(x)) = r$;

(iii) $\delta_0(x) = x^{2^n} = \begin{cases} 0 \text{ if } x \text{ is even,} \\ 1 \text{ if } x \text{ is odd;} \end{cases}$

(iv) $\delta_1(x) = \delta_0(G(x)) = \begin{cases} 1 \text{ if } x \text{ is even,} \\ 0 \text{ if } x \text{ is odd.} \end{cases}$

Let $a_1, \ldots, a_t \in A_n$, with repetitions allowed, and define the function $\Delta[a_1, \ldots, a_t] : A^t \to A$ by

$$\Delta[a_1, \ldots, a_t]\ (x_1, \ldots, x_t) = \begin{cases} 2, \text{ if } x_1 = a_1, \ldots, x_t = a_t \\ 0, \text{ otherwise.} \end{cases}$$

Then we have

(v) $\Delta[a](x) = F_3(G^{(2^n-a)}(x), 0(x))$.

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Suppose inductively that for any positive integer $k \leqq t$, $(t \geqq 1)$, we can represent $\Delta[a_1, \ldots, a_k]$ $(x_1, \ldots, x_k)$ as a composition of F_1, F_2, F_3, and G for any $a_1, \ldots, a_k \in A_n$. Let $a_1, \ldots a_t, b \in A_n$. Then

$$\Delta[a_1, \ldots a_t, b](x_1, \ldots, x_t, y) = F_3(G^{(2^n-2)}(\Delta[a_1, \ldots, a_t](x_1, \ldots, x_t)), G^{(2^n-b)}(y)).$$

Thus by induction we can represent any $\Delta[a_1, \ldots, a_m](x_1, \ldots, x_m)$ as a composition of F_1, F_2, F_3, and G.

Suppose now that $f(x_1, \ldots, x_m) \in \mathscr{U}(\theta)$. Define

$$f'(x_1, \ldots, x_m) = \begin{cases} f(x_1, \ldots, x_m), & \text{if } f(x_1, \ldots, x_m) \text{ is even,} \\ f(x_1, \ldots, x_m) - 1, & \text{otherwise.} \end{cases}$$

Then $f'(x_1, \ldots, x_m) \in \mathscr{U}(\theta)$, as is

$$f''(x_1, \ldots, x_m) = f(x_1, \ldots, x_m) - f'(x_1, \ldots, x_m).$$

If we can show that f' and f'' can both be obtained as compositions of F_1, F_2, F_3, and G it will follow that f can also be so obtained.

We observe that the range of f' is a subset of $\{0, 2, \ldots, 2^n - 2\}$. Consequently

$$f'(x_1, \ldots, x_m) = \sum R[\tfrac{1}{2}f'(i_1, \ldots, i_m)](x) \cdot \Delta[i_1, \ldots, i_m](x_1, \ldots, x_m)$$

where the sum runs independently over all $(i_1, \ldots, i_m) \in (A_n)^m$.

The range of f'' is a subset of $\{0, 1\}$. Moreover, f'' is completely determined by its restriction to $\{0, 1\}^m$. This is so since $f''(x_1, \ldots, x_m) = f''(y_1, \ldots, y_m)$ if $x_i \equiv y_i$ modulo 2 for all $i = 1, \ldots, m$, and thus, in particular, $f''(x_1, \ldots, x_m) = f''(j_1, \ldots, j_m)$ where $x_i \equiv j_i$ modulo 2 and $j_i \in \{0, 1\}$ for all $i = 1, \ldots, m$. Also, if $(j_1, \ldots, j_m) \in \{0, 1\}^m$ we have

$$\delta_{j_1}(x_1) \ldots \delta_{j_m}(x_m) = \begin{cases} 1, & \text{if } x_i \equiv j_i \text{ modulo 2 for all } i = 1, \ldots, m, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently

$$f''(x_1, \ldots, x_m) = R[f''(j_1, \ldots, j_m)](x_1) \cdot \delta_{j_1}(x_1) \ldots \delta_{j_m}(x_m),$$

where the sum runs independently over all $j_1, \ldots, j_m \in \{0, 1\}^m$.

Then $P(x_1, \ldots, x_m) = f(x_1, \ldots, x_m)$ and thus each element of $\mathscr{U}(\theta)$ is representable as a composition of F_1, F_2, F_3, and G. Hence the set of functions generated by these functions is pre-complete.

Because of Theorem 1 we can show that an algebra with carrier A_n is pre-primal by showing that each of its primitive operations belongs to $\mathscr{U}(\theta)$ and that F_1, F_2, F_3, and G are all representable as polynomials modulo the algebra.

THEOREM 2. *If* $\tau = \{n_i | i \in I\}$ *is any finite or countably infinite finitary type satisfying at least one of the following three conditions* (A), (B), *or* (C), *then* *there exists at least a countable infinity of maximal pre-primal clusters of type* τ.

(A) *The type* τ *contains at least two elements, one of which is greater than or equal to* 5.

(B) *The type* τ *contains at least two elements, one of which is greater than or equal to* 3, *while the other is greater than or equal to* 2.

(C) *The type* τ *contains at least three elements, each greater than or equal to* 2.

Proof. Case (A). Suppose A holds for τ and suppose for definiteness that $n_j \geqq 5$. Define the algebra $\mathfrak{A}_n = \langle A_n; o_i | i \in I \rangle$ by letting $o_k(x_1, \ldots, x_{nk}) = 0(\mathfrak{A}_n)$ for $k \in I$, $k \neq j$, while

$$o_j(x_1, \ldots, x_{n_j}) = x_1 + x_1 \cdot x_2 + x_2 + x_3 + x_2 \cdot F_3(x_4, x_5) + 1(\mathfrak{A}_n).$$

Then obviously $o_i(x_1, \ldots, x_{n_i}) \in \mathscr{U}(\theta)$ for all $i \in I$.

The following are seen to be polynomials of $\mathfrak{A}_n$:

(i) $0(x) = 0$;

(ii) $G(x) = o_j(x, 0(x), \ldots, 0(x))$;

(iii) $R_r(x) = G^{(r)}(0(x)) = r$;

(iv) $p(x_1, \ldots, x_5) = G^{(2^n-1)}(o_j(x_1, \ldots, x_{n_j})) = x_1 + x_1 \cdot x_2 + x_2 + x_3 + x_2 \cdot F_3(x_4, x_5)$;

(v) $F_3(x, y) = p(0(x), R_1(x), R_2n_{-1}(x), x, y)$;

(vi) $F_2(x, y) = p(0(x), x, y, x, x) = x + y$;

(vii) $q(x, y) = p(x, y, 0(x), y, y) = x + x \cdot y + y$;

(viii) $s(x) = G^{(2)}(q(R_2n_{-2}(x), x)) = 2^n - 2 + (2^n - 2)y + y + 2 = (2^n - 1)y = -y$;

(ix) $F_1(x, y) = F_2(F_2(s(x), s(y)), q(x, y)) = -x - y + x + x \cdot y + y = x \cdot y$.

Thus by (ix), (vi), (v), and (ii) we may represent F_1, F_2, F_3, and G as polynomials modulo $\mathfrak{A}_n$; this implies that $\mathfrak{A}_n$ is pre-primal for each n.

Now let p and q be distinct positive integers greater than one. We will show that $\mathfrak{A}_p$ and $\mathfrak{A}_q$ are dependent. Let $\sigma(x)$ be any unary polynomial symbol. We claim that $\sigma(0)$ modulo $(\mathfrak{A}_p)$ has the same parity as $\sigma(0)$ modulo $\mathfrak{A}_q$. This is certainly true if σ contains no primitive operation symbol or if σ contains exactly one primitive operation symbol. Suppose inductively that it is true for all polynomial symbols containing fewer than $t (t > 1)$ occurrences of primitive operation symbols and let $\sigma(x)$ be any polynomial symbol containing exactly t occurrences of primitive operation symbols. If $k \in I$, $k \neq j$, and $\sigma(x) = o_k(\sigma_1(x), \ldots, \sigma_{nk}(x))$, our claim is obviously valid, while if

$$\sigma(x) = o_j(\sigma_1(x), \ldots, \sigma_{nk}(x)) = \sigma_1(x) + \sigma_1(x) \cdot \sigma_2(x) + \sigma_2(x) + \sigma_3(x) + \sigma_2(x) \cdot F_3(\sigma_4(x), \sigma_5(x)) + 1$$

where $\sigma_1(x), \ldots, \sigma_5(x)$ each satisfies our induction hypothesis, it is easy to check that $\sigma(0)$ modulo $\mathfrak{A}_p$ and $\sigma(0)$ modulo $\mathfrak{A}_q$ have the same parity. Thus our claim is true by induction. Because of this there can exist no polynomial symbol $\Gamma(x)$ satisfying both $\Gamma(x) = 0$ modulo $\mathfrak{A}_p$ and $\Gamma(x) = 1$ modulo $\mathfrak{A}_q$, since this would imply that $\Gamma(0) = 0$ modulo $\mathfrak{A}_p$ while $\Gamma(0) = 1$ modulo $\mathfrak{A}_q$, a contradiction. Consequently $\mathfrak{A}_p$ and $\mathfrak{A}_q$ are dependent.

By the Axiom of Choice we may imbed each $\mathfrak{A}_n$ in at least one maximal pre-primal cluster. By our previous work no such cluster can contain two distinct $\mathfrak{A}_n$. Thus there must be at least countably infinitely many such clusters.

Case (B). Suppose B holds for τ and suppose for definiteness that $n_j \geqq 3$, $n_k \geqq 2, j \neq k$. Define the algebra $\mathfrak{B}_n = \langle A_n; o_i | i \in I \rangle$ by letting $o_t(x_1, \ldots, x_{nt}) = 0$ for $t \in I$, $t \neq j, k$ (if any such t exist), while

$$o_j(x_1, \ldots, x_{n_j}) = x_1 + x_1 \cdot x_2 + x_3 + 1 \ (\mathfrak{B}_n),\ o_k(x_1, \ldots, x_{nk}) = F_3(x_1, x_2) \ (\mathfrak{B}_n).$$

Again each $o_t \in \mathscr{U}(\theta)$. Furthermore, the following are polynomials of $\mathfrak{B}_n$:

(i) $F_3(x, y) = o_k(x, y, \ldots, y)$;
(ii) $0(x) = 0 = F_3(x, F_3(x, x))$;
(iii) $G(x) = o_j(x, 0(x), \ldots, 0(x))$;
(iv) $R_r(x) = G^{(r)}(0(x)) = r$;
(v) $p(x, y) = o_j(x, 0(x), y, \ldots, y) = x + y + 1$;
(vi) $F_2(x, y) = G^{(2^n-1)}(p(x, y)) = x + y$;
(vii) $s(x) = o_j(R_2n_{-1}(x), x, R_2n_{-1}(x), \ldots, R_2n_{-1}(x)) = -x - 1$;
(viii) $F_1(x, y) = o_j(x, y, s(x), \ldots, s(x)) = x + x \cdot y + (-x - 1) + 1 = x \cdot y$.

Then by (i), (iii), (vi), and (viii), $\mathfrak{B}_n$ is pre-primal. As in Case A we can establish the pairwise dependence of the $\mathfrak{B}_n$'s and thus obtain at least a countable infinity of maximal pre-primal clusters.

Case (C). Suppose (C) holds for τ and suppose for definiteness that $n_j \geqq 2$, $n_k \geqq 2$, and $n_r \geqq 2$ with j, k, r pairwise unequal. Define the algebra $\mathfrak{C}_n = \langle A_n; o_i | i \in I \rangle$ by letting $o_t(x_1, \ldots, x_{nt}) = 0 \ (\mathfrak{C}_n)$ for $t \in I$, $t \neq j, k, r$ (if any such t exist), while

$$o_j(x_1, \ldots, x_{n_j}) = x_1 \cdot x_2 \ (\mathfrak{C}_n),$$
$$o_k(x_1, \ldots, x_{nk}) = x_1 + x_2 + 1 \ (\mathfrak{C}_n),$$
$$o_r(x_1, \ldots, x_{n_r}) = F_3(x_1, x_2) \ (\mathfrak{C}_n).$$

As before, we observe that $0(x) = 0 = F_3(x, F_3(x, x))$ is a polynomial of $\mathfrak{C}_n$; it is now easy to proceed as in the previous cases and show that each $\mathfrak{C}_n$ is pre-primal. We can now show as before that the $\mathfrak{C}_n$'s are pairwise dependent and hence that there exists at least a countable infinity of maximal pre-primal clusters of type τ.

COROLLARY. *Let τ be any finite or countably infinite finitary type satisfying at least one of the conditions* (A), (B), *or* (C). *Then there exists at least a countable infinity of pairwise dependent pre-primal algebras of type τ.*

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