J. Austral. Math. Soc. (Series A) 33 (1982), 76–85

GROUPS FIXING GRAPHS IN SWITCHING CLASSES

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(Received 1 December 1980)

Communicated by D. E. Taylor

Abstract

A permutation group G on a finite set Ω is *always exposable* if whenever G stabilises a switching class of graphs on Ω, G fixes a graph in the switching class. Here we consider the problem: given a finite group G, which permutation representations of G are always exposable? We present solutions to the problem for (i) 2-generator abelian groups, (ii) all abelian groups in semiregular representations, (iii) generalised quaternion groups and (iv) some representations of the symmetric group S_n.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 C 25.

Introduction

According to Harries and Liebeck (1978), a permutation group G on a finite set Ω is *always exposable* if whenever G stabilises a switching class of graphs on Ω, G fixes a graph in the switching class. Equivalently, in the notation of Cameron (1977), G is always exposable if the first invariant γ of G and τ is zero for every 2-graph τ on Ω on which G acts. Here we consider the following problem:

PROBLEM. *Given a finite group G, which permutation representations of G are always exposable?*

The problem has been solved when G is cyclic by Mallows and Sloane (1975) and when G is dihedral by Harries and Liebeck (1978). Here we present solutions for the following groups: (i) 2-generator abelian groups, (ii) all abelian groups in semiregular representations, (iii) generalised quaternion groups, (iv) some representations of the symmetric group S_n. The methods used for (i), (ii) and (iii) are

https://doi.org/10.1017/S1446788700017638 Published online by Cambridge University Press

based on those introduced by Harries and Liebeck (1978); it will readily be seen that they suffice to solve the problem for any group G, given a presentation for G and a very large supply of patience. For (iv) we use a different technique.

One application of the results is given by the observation that if X is a 2-transitive automorphism group of a nontrivial 2-graph and P is a Sylow 2-subgroup of X then P is *not* always exposable (see Corollary 3.7 of Harries and Liebeck (1978) or Proposition 2.5 of Cameron (1977)). Hence solutions to the above problem give restrictions on possible permutation representations of Sylow 2-subgroups of 2-transitive automorphism groups of nontrivial 2-graphs (the fact that such 2-graphs must be regular gives further restrictions).

For notation and an introduction to switching classes see Harries and Liebeck (1978); for connections with 2-graphs (and other things) see Sections 2, 3 of Cameron (1977).

1. Abelian groups in semiregular representations

If Γ is a graph on a finite set Ω we denote the switching class of Γ by $\mathbb{S}(\Gamma)$; if s is the switch with respect to the subset Φ of Ω then $s\Gamma$ is the graph obtained from Γ by switching with respect to Φ. For a permutation α of Ω, $\alpha(\Phi)$ is the image of Φ under α, and ${}_{\alpha}s$ is the switch with respect to $\alpha(\Phi)$. We say that Φ is *compatible* with α if each cycle of α involves an even number of elements of Φ. Write Φ_{α} for the symmetric difference $\Phi\Delta\alpha(\Phi)$; note that if s is the switch with respect to Φ then $s_{\alpha}s$ is the switch with respect to Φ_{α}. As observed in Section 3 of Harries and Liebeck (1978), the graphs on Ω are permuted by switches s, by permutations α and by compositions $s\alpha$ of these operations, which are called *switch-permutations*. These satisfy the rule $\alpha s = {}_{\alpha}s\alpha$. If Ψ is a fixed set of $\langle\alpha\rangle$ then α^{Ψ} denotes the action of α on Ψ. Let Z_n denote a cyclic group of order n.

THEOREM 1.1. *Let G be a finite, abelian, semiregular permutation group. Then G is always exposable if and only if G has no subgroup isomorphic to $Z_2 \times Z_4$.*

PROOF. First suppose that $Z_2 \times Z_4 \leqslant G$ and write $G = Z_{r_1} \times Z_{r_2} \times \cdots \times Z_{r_n}$ where $4 \mid r_1$ and $2 \mid r_2$. We show that G is not always exposable. Let $\alpha_1, \ldots, \alpha_n$ be generators for the cyclic factors $Z_{r_1}, \ldots, Z_{r_n}$ respectively. For simplicity we suppose that G is regular (the proof extends readily to the semiregular case). We may identify Ω with $\{1, 2, \ldots, r_1 r_2 \cdots r_n\}$ and take

$$\alpha_1 = (1\ 2\ \cdots r_1)(r_1 + 1\ \cdots 2r_1)\cdots,$$

$$\alpha_i = (1\ r_1 \cdots r_{i-1} + 1\ \cdots\ (r_i - 1)r_1 \cdots r_{i-1} + 1)(2\ r_1 \cdots r_{i-1} + 2\ \cdots)\cdots$$

for $i \geqslant 2$. Let $\Phi = \{r \in \Omega \mid r \text{ odd}\}$ and let s be the switch with respect to Φ. For $i = 3,\ldots,n$ define

$$\beta_i = \begin{cases} \alpha_i & \text{if } r_i \text{ is even,} \\ \alpha_2\alpha_i & \text{if } r_i \text{ is odd,} \end{cases}$$

so that each β_i has even order and Φ is compatible with $\alpha_1, \alpha_2, \beta_3,\ldots,\beta_n$ (since $4 \mid r_1$). Note that $\alpha_1(\Phi) = \Omega \setminus \Phi$, $\alpha_2(\Phi) = \Phi$ and $\beta_i(\Phi) = \Phi$ for $i \geqslant 3$. Let Q be the group $\langle s\alpha_1, s\alpha_2, s\beta_3,\ldots,s\beta_n\rangle$ of switch-permutations. It is easy to verify, in the notation of Harries and Liebeck (1978), (i) that $Q \cong \langle \alpha_1, \alpha_2, \beta_3,\ldots,\beta_n\rangle$ $(= G)$, hence that no element Q involves switch 1-cycles $(\bar{a})$ (b), and (ii) that no involution of Q involves a switch-transposition $(\bar{a}b)$. Hence by Theorem 3.8 of Harries and Liebeck (1978), Q fixes a graph Γ on Ω. Then G stabilises $\mathbb{S}(\Gamma)$; however if G fixes the graph $s'\Gamma \in \mathbb{S}(\Gamma)$ then $s = s'_{\alpha_1}s' = s'_{\alpha_2}s'$. It is not hard to see that no such switch s' can exist, so G is not always exposable.

Conversely, suppose that G has no subgroup $Z_2 \times Z_4$ and let P be a Sylow 2-subgroup of G; then P is cyclic or elementary abelian. If P is cyclic then G is always exposable (Theorem 4.6 of Harries and Liebeck (1978) or the remark after Theorem 3.4 of Cameron (1977)). So suppose that $P = \langle \alpha_1,\ldots,\alpha_n\rangle \cong (Z_2)^n$. We show that P, and hence G, is always exposable by induction on n. Let P stabilise a switching class $\mathbb{S}(\Gamma)$. By induction, the subgroup $Q = \langle \alpha_1\alpha_2, \alpha_1\alpha_3,\ldots,\alpha_1\alpha_n\rangle$ fixes a graph $\Gamma_1 \in \mathbb{S}(\Gamma)$, so $\alpha_1\Gamma_1 = \alpha_2\Gamma_1 = \cdots = \alpha_n\Gamma_1 = s\Gamma_1$, say, and $\langle s\alpha_1,\ldots,s\alpha_n\rangle$ fixes Γ_1. If Φ is the subset switched by s then Φ is compatible with each α_i (Lemma 4.4 of Harries and Liebeck (1978)). Hence from the action of P on an orbit we see that Φ must be a union of P-orbits. For each P-orbit $\Psi \subseteq \Phi$, let Ψ' be a Q-orbit contained in Ψ. Then since $P = Q\langle \alpha_i\rangle$, we have $\Psi'_{\alpha_i} = \Psi$ for $i = 1,\ldots,n$ (recall that $\Psi'_{\alpha_i} = \Psi'\Delta\alpha_i(\Psi')$). Hence if s' is the switch with respect to a union of Q-orbits, one for each P-orbit in Φ, then $s = s'_{\alpha_i}s'$ for all i and so P fixes the graph $s'\Gamma_1$. By induction then, P, and so G, is always exposable.

2. Two-generator abelian groups

In this section we obtain a necessary and sufficient condition for a 2-generator abelian permutation group X to be always exposable. We shall see that X is always exposable if and only if a Sylow 2-subgroup of X is always exposable (Theorem 2.7), so we first restrict our attention to 2-generator abelian 2-groups. Throughout this section the group G is defined by

$$G = \left\langle \alpha, \beta \mid \alpha^{2^m} = \beta^{2^n} = [\alpha, \beta] = 1 \right\rangle \qquad (m, n \geqslant 1).$$

Let G act on a finite set Ω. Then Ω is a union of G-orbits and since G is abelian, each orbit is determined by the kernel of the action of G on it. We shall see that

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the exposability of G depends closely on the nature of the intersections of the kernels of the actions of G on its various orbits. First we list the proper subgroups of G in four classes:

(*A*) 1, $\langle\alpha^{2^i}\rangle$, $\langle\beta^{2^j}\rangle$, $\langle\alpha^{2^i},\beta^{2^j}\rangle$, $\langle\alpha^{2^i}\beta^{2r}\rangle$, $\langle\alpha^{2^k},\alpha^{2^l}\beta^{2r}\rangle$ ($i\geqslant 1, j\geqslant 1, k>l\geqslant 1$, any r);

(*B*) $\langle\alpha\rangle$, $\langle\alpha^{2^i},\beta\rangle$, $\langle\alpha^{2^l}\beta^{2r+1}\rangle$, $\langle\alpha^{2^k},\alpha^{2^l}\beta^{2r+1}\rangle$ ($i\geqslant 1, k>l\geqslant 1$);

(*C*) subgroups in (B) with α, β interchanged;

(*D*) $\langle\alpha\beta^{2r+1}\rangle$, $\langle\alpha^{2^i},\alpha\beta^{2r+1}\rangle$ ($i\geqslant 1$).

Write $\Psi_{\gamma_1,\ldots,\gamma_r}$ or Ψ_H for an orbit of G on Ω with kernel $H=\langle\gamma_1,\ldots,\gamma_r\rangle$ and $n_{\gamma_1,\ldots,\gamma_r}$ or n_H for the number of such orbits. Denote by $\Psi_{(A)}(\Psi_{(B)},\Psi_{(C)},\Psi_{(D)})$ the union of the orbits with kernels in $(A)((B),(C),(D))$ and by $n_{(A)}(n_{(B)},n_{(C)},n_{(D)})$ the number of these orbits.

Our strategy in considering the exposability of G is as follows: if G stabilises the switching class $\mathbb{S}(\Gamma)$ then there is a switch s such that $\langle s\alpha, s\beta\rangle$ fixes a graph $\Gamma_1\in\mathbb{S}(\Gamma)$ (Lemma 5.1 of Harries and Liebeck (1978)). The group G fixes the graph $s'\Gamma_1\in\mathbb{S}(\Gamma)$ if and only if

$$(*)\qquad s=s'_\alpha s'=s'_\beta s'.$$

Thus we seek all switches s such that $\langle s\alpha, s\beta\rangle$ fixes a graph and determine whether or not there is a switch s' satisfying $(*)$. The following definition is useful in this strategy.

DEFINITION 2.1. Let Ψ be an orbit of G on Ω and let $\Sigma\subseteq\Psi$.

(i) The subset Σ' of Σ is an (α,β)-*subset* of Σ (respectively $(\alpha,\bar\beta)$-*subset*; $(\bar\alpha,\beta)$-*subset*) if Σ'_α $(=\Sigma'\Delta\alpha(\Sigma'))=\Sigma$ and $\Sigma'_\beta=\Sigma$ (respectively $\Sigma'_\alpha=\Sigma$ and $\Sigma'_\beta=\Psi\setminus\Sigma$; $\Sigma'_\alpha=\Psi\setminus\Sigma$ and $\Sigma'_\beta=\Sigma$). We say that Σ is *amenable* if it has (α,β)-, $(\alpha,\bar\beta)$- and $(\bar\alpha,\beta)$-subsets.

(ii) The orbit Ψ is *strictly* (α,β) if it has an (α,β)-subset but no $(\alpha,\bar\beta)$- or $(\bar\alpha,\beta)$-subset (similar definitions apply for a *strictly* $(\alpha,\bar\beta)$ and a *strictly* $(\bar\alpha,\beta)$ orbit).

EXAMPLE 2.2. Let $m=3$, $n=2$ and let Δ_1, Δ_2 be the orbits of G with kernels $K_1=\langle\alpha^2\beta^2\rangle$, $K_2=\langle\alpha^2\beta^3\rangle$ respectively. We may write

$$\alpha^{\Delta_1\cup\Delta_2}=(1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11\,12),$$

$$\beta^{\Delta_1\cup\Delta_2}=(1\,5\,3\,7)(2\,6\,4\,8)(9\,11)(10\,12).$$

It is easy to see that Δ_1 is amenable, whereas Δ_2 is strictly $(\alpha,\bar\beta)$.

LEMMA 2.3. *The orbits with kernels in class* (A), (B), (C), (D) *are, respectively, amenable, strictly* $(\alpha,\bar\beta)$, *strictly* $(\bar\alpha,\beta)$, *strictly* (α,β) *orbits.*

PROOF. We consider only an orbit $\Psi = \Psi_{\alpha^{2^i}\beta^{2^j}}$ (r odd); other cases are similar. Write $H = \langle \alpha^{2^i}\beta^{2^j r}\rangle$. There exists an odd integer k such that $\beta^{2^j} \equiv \alpha^{2^i k} \pmod H$ and assuming that $m + j \leqslant n + i$, we may write

$$\alpha^{\Psi} = (1\ 2\ \cdots 2^m)\cdots\big((2^j - 1)2^m + 1\ \cdots 2^{m+j}\big)\cdots,$$
$$\beta^{\Psi} = \big(1\ 2^m + 1\ 2.2^m + 1\ \cdots\ (2^j - 1)2^m + 1\ 2^i k + 1\ 2^m + 2^i k + 1\ \cdots\big)\cdots.$$

Let $\delta_1,\ldots,\delta_{2^j}$ be the orbits of $\langle \alpha^{\Psi}\rangle$ in the order written above. If $i \geqslant 1, j \geqslant 1$ then $\{r \text{ odd} \mid 1 \leqslant r \leqslant 2^{m+j}\}$, $\delta_1 \cup \delta_3 \cup \cdots \cup \delta_{2^j-1}$, $\{r \text{ odd} \mid r \in \delta_{2l+1}, l = 0, 1, \ldots\} \cup \{r \text{ even} \mid r \in \delta_{2l}, l = 1, 2, \ldots\}$ are, respectively, $(\alpha, \bar\beta)$-, $(\bar\alpha, \beta)$-, (α, β)-subsets of Ψ, so Ψ is amenable. If $j = 0$, $i \geqslant 1$ then Ψ is strictly $(\alpha, \bar\beta)$ and if $i = 0, j \geqslant 1$ then Ψ is strictly $(\bar\alpha, \beta)$. Finally if $i = j = 0$ then $\beta^{\Psi} = (\alpha^{\Psi})^k$ so Ψ is strictly (α, β).

LEMMA 2.4. *Let H_λ ($\lambda \in \Lambda$) be subgroups of G and let s be the switch with respect to a union $\Psi = \bigcup_{\lambda\in\Lambda}\Psi_{H_\lambda}$ of G-orbits. Then $\langle s\alpha, s\beta\rangle$ does not fix a graph on Ω if and only if there is a subgroup K of G such that* (i) *$\Omega \setminus \Phi$ contains an orbit Ψ_K, and* (ii) *for some λ, $H_\lambda \cap K$ contains an element of odd length in α, β (that is, an element $\alpha^t\beta^u$ where $t + u$ is odd).*

PROOF. We use Theorem 3.8 of Harries and Liebeck (1978). Since s is a switch with respect to a union of G-orbits, $\langle s\alpha, s\beta\rangle$ contains no elements which involve a switch-transposition $(\bar a\, b)$. Also an element $(s\alpha)^x(s\beta)^y$ involves switch 1-cycles $(\bar a)(b)$ if and only if for some λ, $a \in \Psi_{H_\lambda} \subseteq \Phi$ and $b \in \Psi_K \subseteq \Omega\setminus\Phi$ where $\alpha^x\beta^y \in H_\lambda \cap K$ and $x + y$ is odd.

In view of Lemma 2.4 we make the following definition.

DEFINITION 2.5. (i) The *B-graph* of G is defined as follows: its vertices are those subgroups H in (B) with $n_H > 0$, and H is joined to K if and only if $H \cap K$ contains an element of odd length in α, β. The *C-graph* is similarly defined.

(ii) The *D-graph* of G has vertex set $\{H \text{ in } (D) \mid n_H > 0\}$ and H is joined to K if and only if $H \cap K$ contains $\alpha^t\beta^u$ for some odd t, u.

By Lemma 2.4, the subsets $\Phi \subseteq \Psi_{(B)}$ which are unions of G-orbits such that $\langle s\alpha, s\beta\rangle$ fixes a graph (where s is the switch with respect to Φ), are in 1-1 correspondence with unions of components of the B-graph; if $\mathcal{C}$ is the set of vertices in a union of components, the corresponding subset Φ is $\bigcup_{H\in\mathcal{C}}\Psi_H$.

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THEOREM 2.6. *The 2-generator abelian 2-group G acting on Ω is not always exposable if and only if* fix $G = \varnothing$ *and one of the following holds*:

(i) *all orbits of G have their kernels in class (A) and* $n_{\alpha^2,\beta^2} = 0$;

(ii) $n_{(B)}n_{(C)}n_{(D)} > 0$;

(iii) *the B-graph is disconnected*;

(iv) *the C-graph is disconnected*;

(v) *the D-graph is disconnected*.

PROOF. If fix $G \neq \varnothing$ then G is always exposable (every switching class contains a unique graph in which a given vertex is isolated), so suppose that fix $G = \varnothing$. Let G stabilise $\mathbb{S}(\Gamma)$; then there is a switch s with respect to Φ such that $\langle s\alpha, s\beta\rangle$ fixes a graph $\Gamma_1 \in \mathbb{S}(\Gamma)$ (Lemma 5.1 of Harries and Liebeck (1978)). As explained before, we seek all such switches s and determine whether or not there is a switch s' with $s = s'_\alpha s = s'_\beta s$. Certainly $s\alpha s\beta = s\beta s\alpha$, so $s_\alpha s = s_\beta s$. Consequently $\alpha(\Phi)$ is either $\beta(\Phi)$ or $\Omega \setminus \beta(\Phi)$.

Case 1. Φ *is a union of G-orbits*. Since orbits with kernels in (A) are amenable (Lemma 2.3) we may assume that $\Phi \subseteq \Psi_{(B)} \cup \Psi_{(C)} \cup \Psi_{(D)}$. As noted after Definition 2.5 the sets $\Phi \subseteq \Psi_{(B)}$ for which $\langle s\alpha, s\beta\rangle$ fixes a graph are in 1-1 correspondence with unions of components of the B-graph. Since $\Psi_{(B)}$ consists of strictly $(\alpha, \bar{\beta})$ orbits, the required switch s' will not exist if and only if $\Omega \setminus \Phi$ contains either a strictly $(\alpha, \bar{\beta})$ orbit or both a strictly $(\bar{\alpha}, \beta)$ and a strictly (α, β) orbit. Hence G is not always exposable in cases (ii), (iii) of the theorem. The case $\Phi \subseteq \Psi_{(C)}$ yields (iv) of the theorem; other possibilities for Φ give no further cases where s' does not exist.

Case 2. Φ *is not a union of G-orbits*. Recall that $\alpha(\Phi)$ is either $\beta(\Phi)$ or $\Omega \setminus \beta(\Phi)$. Suppose first that $\alpha(\Phi) = \beta(\Phi)$. By considering the relevant permutation representations we find that if $\varnothing \neq \Phi \cap \Psi_H \neq \Psi_H$ for some $H \in (A) \cup (B) \cup (C)$ then $\Phi \cap \Psi_H$ is amenable. Now consider an orbit $\Psi = \Psi_{\alpha^{2^i},\alpha^{-r}\beta}$ (r odd) in (D). We have (assuming $i \leqslant n$)

$$\alpha^\Psi = (1\,2\,\cdots\,2^i), \qquad \beta^\Psi = (\alpha^\Psi)^r.$$

If $\alpha^{-1}\beta \notin \langle \alpha^{2^i}, \alpha^{-r}\beta\rangle$ (so that $r \neq 1$) and $r - 1 = 2^j k$ where k is odd, then $\Phi_\Psi = \{1, 2^j + 1, 2.2^j + 1, \ldots, 2^i - 2^j + 1\}$ is an orbit of $\langle \alpha^{-1}\beta\rangle$ and Φ_Ψ has $(\alpha, \bar{\beta})$- and $(\bar{\alpha}, \beta)$-subsets but no (α, β)-subset. Further, if s_1 is the switch with respect to Φ_Ψ then $(s_1\alpha)^r(s_1\beta)^{-1} = s_\Psi \alpha^r\beta^{-1}$ where s_Ψ is the switch with respect to Ψ. Hence if

$$\Phi = \bigcup \{\Phi_\Psi \mid \Psi = \Psi_H, H \text{ ranges over a component of the } D\text{-graph}\}$$

then $\langle s\alpha, s\beta\rangle$ fixes a graph (Theorem 3.8 of Harries and Liebeck (1978)). Consequently s' does not exist in case (v) of the theorem; this is the only case where $\alpha(\Phi) = \beta(\Phi)$ and no switch s' exists.

Finally, if $\alpha(\Phi) = \Omega \setminus \beta(\Phi)$ then, using Theorem 3.8 of Harries and Liebeck (1978) we see that $n_{(B)} = n_{(C)} = n_{(D)} = n_{\alpha^2,\beta^2} = 0$, since $\langle s\alpha, s\beta\rangle$ fixes a graph; and if this is the case it is easy to construct a switch s for which there is no s'. For instance, for an orbit $\Psi = \Psi_{\alpha^{2^i},\beta^{2^j}}$ $(i \geqslant 2, j \geqslant 1)$ we have

$$\alpha^\Psi = \left(1\ 2\ \cdots 2^i\right) \cdots \left((2^j-1)2^i+1\ \cdots 2^{i+j}\right),$$
$$\beta^\Psi = \left(1\ 2^i+1\ 2.2^i+1\ \cdots\ (2^j-1)2^i+1\right) \cdots$$

and we take $\Phi \cap \Psi$ to be $\{r \text{ odd} \mid 1 \leqslant r \leqslant 2^{i+j}\}$.

THEOREM 2.7. *Let X be a 2-generator abelian permutation group on a finite set Ω and let G be a Sylow 2-subgroup of X. Then X is always exposable if and only if G is always exposable.*

PROOF. Write $X = \langle \alpha, \beta \mid \alpha^m = \beta^n = [\alpha, \beta] = 1\rangle$. The result is clearly true if m or n is odd (for then G is cyclic), so suppose that m, n are even. It is not difficult to see that if H is a subgroup of X containing elements $\alpha^{t_1}\beta^{u_1}$, $\alpha^{t_2}\beta^{u_2}$ where t_1, u_2 are odd and t_2, u_1 are even then $G \leqslant H$. Hence we may partition the subgroups $K = \langle \alpha^{v_1}\beta^{w_1}, \alpha^{v_2}\beta^{w_2}\rangle$ of X into five classes: (A) subgroups K with v_i, w_i even $(i = 1,2)$; (B) subgroups with v_1, v_2 even, w_1 odd; (C) subgroups with w_1, w_2 even, v_1 odd; (D) subgroups with v_i, w_i odd $(i = 1,2)$; (E) subgroups K containing G. Note that this agrees with the previous use of (A), (B), (C), (D).

As before we write $n_{(A)}$ for the number of orbits of X with kernel in class (A), and so on. An orbit of X with kernel K breaks up into isomorphic orbits of G, each having kernel $G \cap K$. And if K belongs to class (A) ((B), (C), (D)) then, as a subgroup of G, $G \cap K$ belongs to class (A) ((B), (C), (D) respectively). If X has an orbit Ψ with kernel in class (E) then Ψ has odd size and so X is always exposable by Corollary 3.6 of Harries and Liebeck (1978); also $\Psi \subseteq \operatorname{fix} G$. The B-, C- and D-graphs of X are defined in the same way as in Definition 2.5.

The method of proof of Theorem 2.6 shows that X is not always exposable if and only if $n_{(E)} = 0$ and one of the following holds: (i) all orbits of X have kernel in (A) and $n_{\alpha^{2q},\beta^{2r}} = 0$ for any odd q, r; (ii) $n_{(B)}n_{(C)}n_{(D)} > 0$; (iii) the B-, C- or D-graph is disconnected. It is easy to see that connectedness of the B-graph (C-graph, D-graph) of X is equivalent to connectedness of the B-graph (C-graph, D-graph) of G. Hence by Theorem 2.6, X is always exposable if and only if G is.

3. Generalised quaternion groups

Let G be the generalised quaternion group of order $2^{a+1} \geqslant 8$ defined by

$$G = \left\langle \alpha, \beta \mid \alpha^{2^a} = 1, \beta^2 = \alpha^{2^{a-1}}, \beta^{-1}\alpha\beta = \alpha^{-1} \right\rangle.$$

For $j = 1, \ldots, a-1$ denote by Ψ_j the set of right cosets of $\langle \alpha^{2^{a-j}}, \alpha\beta \rangle$ in G. The methods of the proof of Theorem 2.6 yield

THEOREM 3.1. *The generalised quaternion group G acting on a finite set Ω is not always exposable if and only if one of the following holds*:

(i) *G is semiregular*;

(ii) fix $G = \varnothing$, fix $\alpha \neq \varnothing$, fix $\beta \neq \varnothing$ *and for some j, G has an orbit isomorphic to Ψ_j*.

4. Some representations of S_n

The methods used in the previous two sections to solve the problem of the Introduction are only really efficient when the group G is easily presented on few generators. We now introduce a different technique which applies to any permutation group; we apply it only to certain representations of the symmetric group S_n.

THEOREM 4.1. *Let S_n act naturally on $\Sigma = \{1, 2, \ldots, n\}$, let k be a positive integer and denote by $\Sigma^{\{k\}}$ the set of k-subsets of Σ. If $n \geqslant 4k - 2$ then the action of S_n on $\Sigma^{\{k\}}$ is always exposable.*

PROOF. Write $\Omega = \Sigma^{\{k\}}$. Our strategy is as follows: firstly, by looking at the orbits of S_n on $\Omega^{\{2\}}$ we classify all the switching classes on Ω in which S_n fixes a graph; then by considering the orbits of S_n on $\Omega^{\{3\}}$ we show that any 2-graph on Ω on which S_n acts corresponds to one of these switching classes.

Step 1. *Orbits on* $\Omega^{\{3\}}$. An orbit of S_n on $\Omega^{\{3\}}$ is uniquely determined by a 4-tuple (r, s, t, u) of nonnegative integers with $u \leqslant r \leqslant s \leqslant t$, where $\{A, B, C\}$ is in the orbit (r, s, t, u) $(A, B, C \in \Omega)$ if and only if $|A \cap B| = r$, $|A \cap C| = s$, $|B \cap C| = t$ and $|A \cap B \cap C| = u$. Clearly (i) $u \leqslant r \leqslant s \leqslant t < k$ and (ii) $s + t \leqslant k + u$; and if (r, s, t, u) satisfies (i) and (ii) then it corresponds to an orbit of S_n on $\Omega^{\{3\}}$.

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Step 2. *Orbits on* $\Omega^{\{2\}}$ *and corresponding* 2-*graphs*. Denote the orbits of S_n on $\Omega^{\{2\}}$ by $(0), (1), \ldots, (k-1)$ where $\{A, B\} \in (r)$ if and only if $|A \cap B| = r$. The graphs on Ω on which S_n acts are $\Gamma_{\mathcal{C}}$ ($\mathcal{C}$ any subset of $\{0, 1, \ldots, k-1\}$), where the edge-set of $\Gamma_{\mathcal{C}}$ is $\bigcup_{c \in \mathcal{C}}(c)$. Recall (see Section 2 of Cameron (1977)) that the 2-graph $\Delta_{\mathcal{C}}$ corresponding to the graph $\Gamma_{\mathcal{C}}$ is the set of triples of vertices containing an odd number of edges of $\Gamma_{\mathcal{C}}$. Thus $\Delta_{\mathcal{C}}$ is the union of all orbits (r, s, t, u) of S_n on $\Omega^{\{3\}}$ such that $|\mathcal{C} \cap \{r, s, t\}|$ is odd. Clearly any switching class on Ω in which S_n fixes a graph corresponds to some $\Delta_{\mathcal{C}}$. So to complete the proof we must show that the $\Delta_{\mathcal{C}}$ are the only 2-graphs on Ω on which S_n acts.

Thus let S_n act on a 2-graph $\Delta \subseteq \Omega^{\{3\}}$. Put $\mathcal{C} = \{c \mid (c, 0, 0, 0) \subseteq \Delta\}$.

Step 3. *We have* $\Delta = \Delta_{\mathcal{C}}$. Let $\{A, B, C\} \in (r, s, t, u)$ where $rs \neq 0$. Then

$$|A \cup B \cup C| = 3k - (r + s + t) + u \leqslant 3k - (r + s) \leqslant 3k - 2.$$

Since $n \geqslant 4k - 2$ we may pick $D \in \Omega$ with D disjoint from $A \cup B \cup C$. Consider the 4-set $\{A, B, C, D\}$. We have

$$\{A, B, C\} \in (r, s, t, u), \qquad \{A, C, D\} \in (s, 0, 0, 0),$$
$$\{A, B, D\} \in (r, 0, 0, 0), \qquad \{B, C, D\} \in (t, 0, 0, 0).$$

Since Δ is a 2-graph, an even number of these triples lies in Δ. Hence $(r, s, t, u) \subseteq \Delta$ if and only if an odd number of r, s, t lies in $\mathcal{C}$. Thus $\Delta = \Delta_{\mathcal{C}}$.

COROLLARY 4.2. *If* $n \geqslant 4k - 2$ *and* $n > 10$ *then the action of* A_n *on* $\Sigma^{\{k\}}$ *is always exposable.*

PROOF. This follows from the fact that A_n is $3k$-transitive and hence has the same orbits on $\Omega^{\{3\}}$ (where $\Omega = \Sigma^{\{k\}}$) as S_n (see Remark 2 below).

REMARKS. 1. The restriction $n \geqslant 4k - 2$ was made entirely for convenience in the proof of Theorem 4.1 and can probably be relaxed considerably.

2. The following observations are elementary: let G and H be permutation groups on Σ and suppose that H is always exposable. Then G is always exposable if either (i) $H \leqslant G$ and G has the same orbits as H on $\Sigma^{\{2\}}$, or (ii) $G \leqslant H$ and G has the same orbits as H on $\Omega^{\{3\}}$.

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