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Almost sure convergence of cover times for ψ-mixing systems

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Abstract. Given a topologically transitive system on the unit interval, one can investigate the *cover time*, that is, the time for an orbit to reach a certain level of resolution in the repeller. We introduce a new notion of dimension, namely the *stretched Minkowski dimension*, and show that under mixing conditions, the asymptotics of typical cover times are determined by Minkowski dimensions when they are finite, or by stretched Minkowski dimensions otherwise. For application, we show that for countably full-branched affine maps, results using the usual Minkowski dimensions fail to give a finite limit of cover times, whilst the stretched version gives a finite limit. In addition, cover times for irrational rotations are calculated as counterexamples due to the absence of mixing.

Key words: cover time, exponentially ψ-mixing, irrational rotations, Minkowski dimension
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1. *Introduction*

Consider $\mathcal{X} \subset [0, 1]$ and $f : \mathcal{X} \to [0, 1]$ a topologically transitive piecewise expanding Markov map equipped with an ergodic invariant probability measure μ. We want to study the *cover times* for points in the repeller Λ, that is, given $x \in \Lambda$, let

$$\tau_r(x) := \inf\{k : \text{for all } y \in \Lambda, \text{ there exists } j \leq k : d(f^j(x), y) < r\}.$$

The first quantitative result of expected cover times $\mathbb{E}[\tau_r]$ was obtained for Brownian motions in [**M**]. It is generalised in recent works [**BJK**, **JT**] for *chaos games* associated to iterative function systems and one-dimensional dynamical systems. In [**BJK**], an almost sure convergence for $-\log \tau_r / \log r$ was also demonstrated, assuming the invariant measure μ supported on the attractor of the iterated function systems (IFS) satisfies rapid mixing conditions. All results suggest that the asymptotic behaviour of τ_r is crucially linked to the *Minkowski dimensions*: for each $r > 0$, let $M_\mu(r) := \min_{x\in \mathrm{supp}(\mu)} \mu(B(x,r))$. The *upper and lower Minkowski dimensions* of μ are defined respectively by

$$\overline{\dim}_M(\mu) := \limsup_{r\to 0} \frac{\log M_\mu(r)}{\log r}, \quad \underline{\dim}_M(\mu) := \liminf_{r\to 0} \frac{\log M_\mu(r)}{\log r}.$$

We write $\dim_M(\mu)$ when the two quantities coincide. In other words, these dimension-like quantities reflect the decay rate of the minimal μ-measure ball at scale r and they are closely related to the box-counting dimension of the ambient space (see [FFK] for more details). In addition, the Minkowski dimensions of μ govern the asymptotic behaviour of hitting times associated to the balls which are most 'unlikely' to be visited at small scales. Our first result below gives an almost sure asymptotic growth rate of cover times in terms of $\overline{\dim}_M(\mu)$ and $\underline{\dim}_M(\mu)$.

THEOREM 1.1. *Let (f, μ) be a probability-preserving system where f is topologically transitive, Markov and piecewise expanding. If $\overline{\dim}_M(\mu) < \infty$, then for μ-almost every (a.e.) x in the repeller,*

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \ge \overline{\dim}_M(\mu), \quad \liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \ge \underline{\dim}_M(\mu).$$

If (f, μ) is exponentially ψ-mixing, then for μ-a.e. $x \in \Lambda$, the inequalities above are improved to

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \overline{\dim}_M(\mu), \quad \liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \underline{\dim}_M(\mu).$$

In particular, it is true if the invariant measure in question is *doubling*.

Remark 1.2. We remark that systems with finite Minkowski dimensions, or at least $\overline{\dim}_M(\mu) < \infty$, are fairly common. In particular, if μ is *doubling*, that is, there exists a constant $D > 0$ such that for all $x \in \mathrm{supp}(\mu)$ and $r > 0$, $D\mu(B(x,r)) \ge \mu(B(x,2r)) > 0$, then $\overline{\dim}_M(\mu) < \infty$. This can be seen from the following: for each $n \in \mathbb{N}$, let $x_n \in \mathrm{supp}(\mu)$ be such that $\mu(B(x_n, 2^{-n})) = M_\mu(2^{-n})$. By the doubling property,

$$\begin{aligned} M_\mu(2^{-n}) = \mu(B(x_n, 2^{-n})) &\ge D^{-1}\mu(B(x_n, 2^{-n+1})) \\ &\ge D^{-1}M_\mu(2^{-n+1}) = D^{-1}\mu(B(x_{n-1}2^{-n+1})). \end{aligned}$$

Reiterating this, one gets $M_\mu(2^{-n}) \ge D^{-n+1}M_\mu(1/2)$. In other words,

$$\frac{\log M_\mu(2^{-n})}{-n\log 2} \le \frac{-(n-1)\log D + \log M_\mu(1/2)}{-n\log 2}.$$

As for all $r > 0$, there is unique $n \in \mathbb{N}$ such that $2^{-n} < r \le 2^{-n+1}$ and $(\log 2^{-n}/\log 2^{-n+1}) = 1$,

$$\limsup_{r\to 0} \frac{\log M_\mu(r)}{\log r} = \limsup_{n\to\infty} \frac{\log M_\mu(2^{-n})}{-n\log 2} \le \frac{\log D}{\log 2} < \infty.$$

However, the Minkowski dimensions are not always finite due to non-doubling behaviours or more extreme decay of $M_\mu(r)$ (see Example 3.2). Hence, we need a new

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notion of dimension, invariant under scalar multiplication (replacing $M_\mu(r)$ by $M_\mu(cr)$ for any $c > 0$, the limit does not change), to capture such decay rate in r.

Definition 1.3. Define the upper and lower *stretched Minkowski dimensions* by

$$\overline{\dim}_M^s(\mu) := \limsup_{r\to 0} \frac{\log|\log M_\mu(r)|}{-\log r}, \quad \underline{\dim}_M^s(\mu) := \liminf_{r\to 0} \frac{\log\log|M_\mu(r)|}{-\log r}.$$

Those quantities should be of independent interest. Our second theorem below deals with almost sure cover times for systems in which $M_\mu(r)$ decays at stretched-exponential rates.

THEOREM 1.4. *Let (f, μ) be an ergodic probability preserving system where f is topologically transitive, Markov and piecewise expanding. If $\overline{\dim}_M(\mu) = \infty$, but $0 < \underline{\dim}_M^s(\mu), \overline{\dim}_M^s(\mu) < \infty$, then for μ-a.e. $x \in \Lambda$,*

$$\liminf_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} \geq \underline{\dim}_M^s(\mu), \quad \limsup_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} \geq \overline{\dim}_M^s(\mu). \tag{1.1}$$

If (f, μ) is exponentially ψ-mixing, then for μ-a.e. $x \in \Lambda$,

$$\liminf_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} = \underline{\dim}_M^s(\mu), \quad \limsup_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} = \overline{\dim}_M^s(\mu). \tag{1.2}$$

1.1. *Layout of the paper.* Basic definitions are introduced in §2 and we delay the proofs of the main theorems to §4. Several examples that satisfy Theorems 1.1 and 1.4 will be discussed in §3. In §5, we will also prove that for irrational rotations, which are known to have no mixing behaviour, Theorem 1.1 fails for almost every point when the rotations are of type η (see Definition 5.1) for some $\eta > 1$. Lastly, in §6, we show that similar results hold for flows under some natural conditions.

2. *Setup*

Let $\mathcal{A}$ be a finite or countable index set and $\mathcal{P} = \{P_a\}_{a\in\mathcal{A}}$ a collection of subintervals in $[0, 1]$ with disjoint interiors covering $\mathcal{X}$. We say $f : \mathcal{X} \to [0, 1]$ is a *piecewise expanding Markov map* if:

(1) for any $a \in \mathcal{A}$, $f_a := f|_{P_a}$ is continuous, injective and $f(P_a)$ a union of elements in $\mathcal{P}$;

(2) there is a uniform constant $\gamma > 1$ such that for all $a \in \mathcal{A}$, $|Df_a| \geq \gamma$.

The *repeller* of *f*, denoted by Λ, is the collection of points with all their forward iterates contained in $\mathcal{P}$, namely

$$\Lambda := \left\{x \in \mathcal{X} : f^k(x) \in \bigcup_{a\in\mathcal{A}} P_a \text{ for all } k \geq 0\right\}.$$

We study the dynamics of $f : \Lambda \to \Lambda$ together with an ergodic invariant probability measure μ supported on Λ. There is a shift system associated to *f*: let M be an $\mathcal{A} \times \mathcal{A}$ matrix such that $M_{ab} = 1$ if $f(P_a) \cap P_b \neq \emptyset$ and 0 otherwise. Here, f is *topologically transitive* if for all $a, b \in \mathcal{A}$, there exists k such that $M_{ab}^k > 0$. Let Σ denote the space of all *infinite admissible words*, that is,

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$$\Sigma := \{x = (x_0, x_1, \ldots) \in \mathcal{A}^{\mathbb{N}_0} : M_{x_k, x_{k+1}} = 1 \text{ for all } k \geq 0\}.$$

A natural choice of metric on Σ is $d_s(x, y) := 2^{-\inf\{j \geq 0 : x_j \neq y_j\}}$, and we define the projection map $\pi : \Sigma \to \Lambda$ by

$$x = \pi(x_0, x_1, \ldots) \quad \text{if and only if } x \in \bigcap_{i=0}^{\infty} f^{-i} P_{x_i}.$$

The dynamics on Σ is the left shift $\sigma : \Sigma \to \Sigma$ given by $\sigma(x_0, x_1, \ldots,) = (x_1, x_2, \ldots)$, then π defines a semi-conjugacy $f \circ \pi = \pi \circ \sigma$. The corresponding symbolic measure $\tilde{\mu}$ of μ is given by $\mu = \pi_* \tilde{\mu}$, that is, for all Borel-measurable sets $B \in \mathcal{B}([0, 1])$, $\mu(B) = \tilde{\mu}(\pi^{-1} B)$.

Denote $\mathcal{P}^n := \bigvee_{j=0}^{n-1} f^{-j} \mathcal{P}$, each $P \in \mathcal{P}^n$ corresponds to an *n-cylinder* in Σ: let $\Sigma_n \subseteq \mathcal{A}^n$ denote all finite words of length n and for any $\mathbf{i} \in \Sigma$, the n-cylinder defined by $\mathbf{i}$ is

$$[\mathbf{i}] = [i_0, \ldots, i_{n-1}] := \{y \in \Sigma : y_j = i_j, \ j = 0, \ldots, n-1\}.$$

Then, $\pi[i_0, i_1, \ldots, i_{n-1}] = \bigcap_{j=0}^{n-1} f^{-j} P_{i_j} =: P_{\mathbf{i}}$. The *depth* of a cylinder $[\mathbf{i}]$ is the length of $\mathbf{i}$.

Furthermore, (f, μ) is required to have the following mixing property.

Definition 2.1. Say μ is *exponentially ψ-mixing* if there are $C_1, \rho > 0$ and a monotone decreasing function $\psi(k) \leq C_1 e^{-\rho k}$ for all $k \in \mathbb{N}$, such that the corresponding symbolic measure $\tilde{\mu}$ satisfies: for all $n, k \in \mathbb{N}$, $\mathbf{i} \in \Sigma_n$ and $\mathbf{j} \in \Sigma^* = \bigcup_{l \geq 1} \Sigma_l$,

$$\left| \frac{\tilde{\mu}([\mathbf{i}] \cap \sigma^{-(n+k)}[\mathbf{j}])}{\tilde{\mu}[\mathbf{i}] \tilde{\mu}[\mathbf{j}]} - 1 \right| \leq \psi(k).$$

3. *Examples*

Theorem 1.4 is applicable to the following systems.

Example 3.1. Finitely branched Gibbs–Markov maps: let f be a topologically transitive piecewise expanding Markov map with $\mathcal{A}$ finite. Here, f is said to be *Gibbs–Markov* if for some *potential* $\phi : \Sigma \to \mathbb{R}$ which is *locally Hölder* with respect to the symbolic metric d_s, there exists $G > 0$ and $P \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, all $x = (x_0, x_1, \ldots) \in \Sigma$,

$$\frac{1}{G} \leq \frac{\tilde{\mu}([x_0, \ldots, x_{n-1}])}{\exp(\sum_{j=0}^{n-1} \phi(\sigma^j x) - nP)} \leq G.$$

For maps of this kind, $|Df|$ is uniformly bounded; thus for each ball at scale r, it is possible to approximate any ball with finitely many cylinders of the same depth (see for example the proof of [JT, Lemma 3.2]), and by the Gibbs property of $\tilde{\mu}$, the asymptotic decay rate converges so $\dim_M(\mu)$ exists and is finite. Since Gibbs measures are exponentially ψ-mixing (see [Bow, Proposition 1.14]), by Theorem 1.1, we have

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_M(\mu)$$

for μ-a.e. x in the repeller of f.

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In the next example, when $r \to 0$ at a polynomial rate, $M_\mu(r)$ decays exponentially; hence, $\overline{\dim}_M(\mu)$ is infinite and the stretched Minkowski dimensions are needed.

Example 3.2. Similar to [**JT**, Example 7.4], consider the following class of infinitely full-branched maps: pick $\kappa > 1$ and set $c = \zeta(\kappa) = \sum_{n\in\mathbb{N}}(1/n^\kappa)$. Let $a_0 = 0$, $a_j = \sum_{j=1}^{n}(1/cj^\kappa)$ and define f by

$$\text{for all } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad f(x) = cn^\kappa(x - a_{n-1}) \text{ for } x \in [a_{n-1}, a_n) =: P_n.$$

Then, f is an infinitely full-branched affine map and we can associate this map with a full-shift system on $\mathbb{N}$: $x = \pi(i_0, i_1, \ldots)$ if for all $j \geq 1$, $f^j(x) \in P_{i_j}$.

Let $\omega > 1$ and construct $\tilde{\mu}$ the finite Bernoulli measure by

$$\tilde{\mu}([i_0, \ldots, i_{n-1}]) = \prod_{j=0}^{n-1} \omega^{-i_j},$$

so the push-forward measure $\mu = \pi_*\tilde{\mu}$ has $\mu(P_n) = \omega^{-n}$.

PROPOSITION 3.3. *For (f, μ) defined in the example above, $\overline{\dim}_M(\mu) = \infty$, but $\dim^s_M(\mu) = 1/(\kappa - 1)$.*

Proof. For each $r > 0$, the r-ball of minimum measure is found near 1. In particular, along the sequence $r_n = (1/2c)\sum_{j\geq n} j^{-\kappa} \approx 1/2c(\kappa - 1)n^{\kappa-1}$, the ball that realises $M_\mu(r_n)$ is contained in $\bigcup_{j=n}^{\infty} P_j$; hence,

$$\omega^{-n} \leq M_\mu(r_n) \leq \frac{\omega^{-n}}{1 - \omega^{-1}}.$$

Therefore,

$$\overline{\dim}_M(\mu) \geq \limsup_{n\to\infty} \frac{n \log \omega}{(\kappa - 1) \log n} = \infty,$$

whereas for all n,

$$\frac{\log n + \log(\log \omega + \log(1 - 1/\omega)/n)}{(\kappa - 1) \log n + \log(2c(\kappa - 1))} \leq \frac{\log |\log M_\mu(r_n)|}{-\log r_n} \leq \frac{\log n + \log \log \omega}{(\kappa - 1) \log n + \log(2c(\kappa - 1))}.$$

As for all $r > 0$, there is unique $n \in \mathbb{N}$ such that $r_{n+1} \leq r < r_n$, while $\lim_{n\to\infty}(\log r_{n+1}/\log r_n) = 1$. We can conclude with $\dim^s_M(\mu) = 1/(\kappa - 1)$. □

As in [**JT**, Example 7.4], it is very difficult for the system to cover small neighbourhoods of 1, so Theorem 1.1 says $\limsup_{r\to 0}(\log \tau_r(x)/-\log r) \geq \overline{\dim}_M(\mu) = \infty$, but since $\tilde{\mu}$ is Bernoulli and hence ψ-mixing, Theorem 1.4 asserts that

$$\lim_{r\to 0} \frac{\log \log \tau_r(x)}{-\log r} = \frac{1}{\kappa - 1} \quad \mu\text{-almost everywhere.}$$

4. *Proof of Theorem 1.4*

The proofs in this section are adapted from those of [**BJK**, Propositions 3.1 and 3.2]. We will only demonstrate the proofs for Theorem 1.4, that is, the asymptotics are determined

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by stretched Minkowski dimensions; the proofs for Theorem 1.1 are obtained by replacing all stretched exponential sequences in the proofs below by some exponential sequence, for example, for a given constant $s \in \mathbb{R}$, $e^{\pm n^s}$ will be replaced by $2^{\pm ns}$.

Assuming the inequalities in (1.1), we first prove (1.2), which requires the exponentially ψ-mixing condition.

Remark 4.1. Assuming the conditions of Theorem 1.4, we will prove that the statements hold along the subsequence $r_n = n^{-1}$ such that for each $r > 0$, there is a unique $n \in \mathbb{N}$ with $r_{n+1} < r \leq r_n$, while $\lim_{n\to\infty}(\log r_{n+1}/\log r_n) = 1$ (if $\overline{\dim}_M(\mu)$ or $\underline{\dim}_M(\mu)$ are finite, we choose $r_n = 2^{-n}$ instead). Since $\log \tau_r(x)$ is increasing as $r \to 0$,

$$\limsup_{n\to\infty} \frac{\log\log \tau_{r_n}(x)}{-\log r_n} = \limsup_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r},$$

and similarly for liminfs.

4.1. *Proof of (1.2).*

PROPOSITION 4.2. *Suppose (f, μ) is exponentially ψ-mixing and the upper stretched Minkowski dimension $\overline{\dim}_M^s(\mu)$ is finite, then for μ-a.e. $x \in \Lambda$,*

$$\limsup_{n\to\infty} \frac{\log\log \tau_r(x)}{-\log r} \leq \overline{\dim}_M^s(\mu).$$

Proof. Let $\varepsilon > 0$ and, for simplicity, denote $\overline{\alpha} := \overline{\dim}_M^s(\mu)$.

For any finite k-word $\mathbf{i} = x_0, \ldots, x_{k-1} \in \Sigma_k$, let $\mathbf{i}^- = x_0, \ldots, x_{k-2}$, that is, $\mathbf{i}$ dropping the last letter. Recall that for each $\mathbf{i} \in \Sigma^*$, $P_\mathbf{i} = \pi[\mathbf{i}]$, and we define

$$\mathcal{W}_r := \{\mathbf{i} \in \Sigma^* : \operatorname{diam}(P_\mathbf{i}) \leq r < \operatorname{diam}(P_{\mathbf{i}^-})\}.$$

By expansion, for each $n \in \mathbb{N}$, the lengths of the words in $\mathcal{W}_{n^{-1}}$ are bounded from above; hence, we can define

$$L(n) := \frac{\log n}{\log \gamma} + 1 \geq \max\{|\mathbf{i}| : \mathbf{i} \in \mathcal{W}_{n^{-1}}\}.$$

Given $y \in [0, 1]$ and $r > 0$ such that $B(y, r) \subset \operatorname{supp}(\mu)$, define the corresponding symbolic balls by

$$\tilde{B}(y, r) := \{[\mathbf{i}] : \mathbf{i} \in \mathcal{W}_r,\ P_\mathbf{i} \cap B(y, r) \neq \emptyset\}.$$

Note that if for some $x \in P_\mathbf{i}$ and $[\mathbf{i}] \in \tilde{B}(y, r)$, $d(x, y) \leq r + \operatorname{diam}(P_\mathbf{i}) \leq 2r$, then

$$B(y, r) \subset \pi\tilde{B}(y, r) \subset B(y, 2r).$$

Let $\mathcal{Q}_n$ be a cover of Λ with balls of radius $r_n = 1/2n$. Denote the collection of their centres by $\mathcal{Y}_n$ and $\#\mathcal{Q}_n = \#\mathcal{Y}_n \leq n$. Let $\tau(\mathcal{Q}_n, x)$ be the minimum time for the orbit of x to have visited each element of $\mathcal{Q}_n$ at least once,

$$\tau(\mathcal{Q}_n, x) := \min\{k \in \mathbb{N} : \text{for all } Q \in \mathcal{Q}_n, \text{ there exists } 0 \leq j \leq k : f^j(x) \in Q\}.$$

Then, $\tau_{1/n}(x) \leq \tau(\mathcal{Q}_n, x)$ for all n and all x since for all $y \in \Lambda$, there is $Q \in \mathcal{Q}_n$ and $j \leq \tau(\mathcal{Q}_n, x)$ such that $f^j(x) \in Q$ and $y \in Q$; hence, $d(f^j(x), y) \leq 1/n$. Let $\varepsilon > 0$ be

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an arbitrary number and for each $k \in \mathbb{N}$, set $L'(k) = \lceil L(k) + 1/\rho(k^{\overline{\alpha}+\varepsilon} + \log C_1)\rceil$, where C_1, ρ were given in Definition 2.1 and $\lceil t\rceil$ takes the least integer no smaller than t. We have

$$\begin{aligned}
&\mu(x : \tau_{1/n}(x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)) \leq \mu(x : \tau(\mathcal{Q}_n, x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)) \\
&\quad = \mu(x : \text{there exists } y \in \mathcal{Y}_n : f^j(x) \notin B(y, 1/2n) \text{ for all } j \leq e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)) \\
&\quad \leq \mu(x : \text{there exists } y \in \mathcal{Y}_n : f^{jL'(4n)}(x) \notin B(y, 1/2n) \text{ for all } j \leq e^{n^{\overline{\alpha}+\varepsilon}}) \\
&\quad = \mu\bigg(\bigcup_{y\in\mathcal{Y}_n} \bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} (f^{-jL'(4n)} B(y, 1/2n))^c\bigg) \leq \sum_{y\in\mathcal{Y}_n} \mu\bigg(\bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} (f^{-jL'(4n)} B(y, 1/2n))^c\bigg).
\end{aligned} \tag{4.1}$$

A cylinder $[\mathbf{i}]$ in $\tilde{B}(y, 1/4n)$ has depth at most $L(4n)$, then by our choice of $L'(4n)$ and the exponentially ψ-mixing property of $\tilde{\mu}$,

$$\mu(\tilde{B}(y, 1/4n) \cap f^{-L'(4n)} \tilde{B}(y, 1/4n)) \leq (1 + \exp(-((4n)^{\overline{\alpha}+\varepsilon}) + \log C_1)\mu(\tilde{B}(y, 1/4n)).$$

Similar calculations hold for $\mu(\bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} (f^{-jL'(4n)} B(y, 1/2n))^c)$ since the compliment of $\tilde{B}(y, 1/4n)$ can be written as a countable union of cylinders of depths no greater than $L(4n)$.

As $\pi(\tilde{B}(z, r)) \subseteq B(z, 2r)$ for all z and all $r > 0$,

$$\begin{aligned}
&\sum_{y\in\mathcal{Y}_n} \mu\bigg(\bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} (f^{-jL'(4n)} B(y, 1/2n))^c\bigg) \leq \sum_{y\in\mathcal{Y}_n} \tilde{\mu}\bigg(\bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} (\sigma^{-jL'(4n)} \tilde{B}(y, 1/4n))^c\bigg) \\
&\quad \leq \bigg(1 + \psi\bigg(\frac{1}{\rho}((4n)^{\overline{\alpha}+\varepsilon} + \log C_1)\bigg)\bigg)^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y\in\mathcal{Y}_n} \bigg(1 - \tilde{\mu}\bigg(\tilde{B}\bigg(y, \frac{1}{4n}\bigg)\bigg)\bigg)^{e^{n^{\overline{\alpha}+\varepsilon}}} \\
&\quad \leq (1 + e^{-n^{\overline{\alpha}+\varepsilon}})^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y\in\mathcal{Y}_n} \bigg(1 - \mu\bigg(B\bigg(y, \frac{1}{4n}\bigg)\bigg)\bigg)^{e^{n^{\overline{\alpha}+\varepsilon}}}.
\end{aligned} \tag{4.2}$$

By definition of $\overline{\alpha}$, for all n large such that $(\varepsilon/4)\log n \geq (\overline{\alpha} + \varepsilon/4)\log 4$, we have

$$\log\bigg(-\log M_\mu\bigg(\frac{1}{4n}\bigg)\bigg) \leq (\overline{\alpha} + \varepsilon/4)(\log 4n) \leq (\overline{\alpha} + \varepsilon/2)\log n.$$

So for all $y \in \operatorname{supp}(\mu)$ and all n large enough,

$$\mu\bigg(B\bigg(y, \frac{1}{4n}\bigg)\bigg) \geq e^{-n^{\overline{\alpha}+\varepsilon/2}} \geq \frac{e^{n^{\varepsilon/2}}}{e^{n^{\overline{\alpha}+\varepsilon}}}.$$

As for all $u \in \mathbb{R}$ and all large k, $(1 + u/k)^k \approx e^u$, combining (4.1) and (4.2), for some uniform constant $C_2 > 0$,

$$\mu(x : \tau_{1/n}(x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)) \leq (1 + e^{-n^{\overline{\alpha}+\varepsilon}})^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y \in \mathcal{Y}_{k+1}} (1 - e^{-n^{\overline{\alpha}+\varepsilon/2}})^{e^{n^{\overline{\alpha}+\varepsilon}}}$$

$$\leq (1 + e^{-n^{\overline{\alpha}+\varepsilon}})^{e^{n^{\overline{\alpha}+\varepsilon}}} n \left(1 - \frac{e^{n^{\varepsilon/2}}}{e^{n^{\overline{\alpha}+\varepsilon}}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \leq C_2 \exp(\log n - e^{n^{\varepsilon/2}}).$$

The last term is clearly summable over n, then by Borel–Cantelli, for all n large enough, $\tau_{1/n}(x) \leq e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)$. Since $\log L'(4n) \approx (\overline{\alpha} + \varepsilon) \log n \ll n^{\overline{\alpha}+\varepsilon}$, we have for μ-a.e. $x \in \Lambda$,

$$\limsup_{n\to\infty} \frac{\log \log \tau_{1/n}(x)}{\log n} \leq \limsup_{n\to\infty} \frac{\log \log(e^{n^{\overline{\alpha}+\varepsilon}} L'(4n))}{\log n} \leq \overline{\alpha} + \varepsilon.$$

By Remark 4.1, this upper bound for lim sup holds for all sequences decreasing to 0, and as $\varepsilon > 0$ was arbitrary, we can conclude that for μ-a.e. $x \in \Lambda$,

$$\limsup_{r\to 0} \frac{\log \log \tau_r(x)}{-\log r} = \limsup_{n\to\infty} \frac{\log \log \tau_{1/n}}{\log n} \leq \overline{\alpha}. \qquad \square$$

PROPOSITION 4.3. *Suppose (f, μ) is exponentially ψ-mixing and the lower stretched Minkowski dimension of μ, $\underline{\dim}^s_M(\mu)$, is finite, then for μ-a.e. $x \in \Lambda$,*

$$\liminf_{r\to 0} \frac{\log \log \tau_r(x)}{-\log r} \leq \underline{\dim}^s_M(\mu).$$

Proof. Again, for simplicity, denote $\underline{\alpha} := \underline{\dim}^s_M(\mu)$. Let $\varepsilon > 0$ and by definition of liminf, there is a subsequence $\{n_k\}_k \to \infty$ such that for all k,

$$\frac{\log(-\log M_\mu(1/n_k))}{\log n_k} \leq \underline{\alpha} + \varepsilon,$$

then repeating the proof of Proposition 4.2 by replacing n by n_k everywhere, one gets that for μ-a.e. x,

$$\liminf_{k\to\infty} \frac{\log \log \tau_{1/n_k}(x)}{\log n_k} \leq \underline{\alpha} + \varepsilon.$$

Again sending $\varepsilon \to 0$ and using the fact that liminf over the entire sequence is no greater than the liminf along any subsequence, the proposition is proved. $\square$

4.2. *Proof of the inequalities (1.1).*

PROPOSITION 4.4. *For μ-a.e. $x \in \Lambda$,*

$$\liminf_{n\to\infty} \frac{\log \log \tau_r(x)}{-\log r} \geq \underline{\dim}^s_M(\mu).$$

Proof. We continue to use the notation $\underline{\alpha} = \underline{\dim}^s_M(\mu)$. Let $\varepsilon > 0$ be arbitrary and by definition of $\underline{\alpha}$ for all large n, there exists $y_n \in \operatorname{supp}(\mu)$ such that $\mu(B(y_n, 1/n)) \leq e^{-n^{\underline{\alpha}-\varepsilon}}$. Let

$$T(x, y, r) := \inf\{j \geq 0 : f^j(x) \in B(y, r)\},$$

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so for all $n \in \mathbb{N}$ and all x, $\tau_{1/n}(x) \geq T(x, y_n, 1/n)$. Then, by invariance,

$$\begin{aligned}\mu(x : \tau_{1/n}(x) < e^{n^{\underline{\alpha}-\varepsilon}}/n^2) &\leq \mu(x : T(x, y_n, 1/n) < e^{n^{\underline{\alpha}-\varepsilon}}/n^2) \\ &= \mu(x : \text{there exists } 0 \leq j < e^{n^{\underline{\alpha}-\varepsilon}}/n^2 : \ f^j(x) \in B(y_n, 1/n)) \\ &\leq \sum_{j=0}^{e^{n^{\underline{\alpha}-\varepsilon}}/n^2-1} \mu(x : f^j(x) \in B(y_n, 1/n)) \\ &= \sum_{j=0}^{e^{n^{\underline{\alpha}-\varepsilon}}/n^2-1} \mu\bigg(f^{-j}B\bigg(y_n, \frac{1}{n}\bigg)\bigg) \leq \frac{e^{n^{\underline{\alpha}-\varepsilon}}}{n^2} e^{-n^{\overline{\alpha}-\varepsilon}} = \frac{1}{n^2},\end{aligned}$$

which is summable. By Borel–Cantelli, since $2 \log n \ll n^{\underline{\alpha}-\varepsilon}$, for μ-a.e. x,

$$\liminf_{n\to\infty} \frac{\log \log \tau_{1/n}(x)}{\log n} \geq \underline{\alpha} - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, the proposition is proved. □

Similar to Propositions 4.2 and 4.3, we get the following proposition.

PROPOSITION 4.5. *For μ-a.e. $x \in \Lambda$,*

$$\limsup_{r\to 0} \frac{\log \log \tau_r(x)}{-\log r} \geq \overline{\dim}^s_M(\mu).$$

Proof. Let $\varepsilon > 0$, then by definition of limsup, there exists a subsequence $\{n_k\}_k \to \infty$ such that for all k,

$$\frac{\log \log(-M_\mu(1/n_k))}{\log n_k} \geq \overline{\alpha} - \varepsilon.$$

Then, repeating the proof of Proposition 4.4 along $\{n_k\}_k$, one gets that for μ-a.e. x:

$$\limsup_{k\to\infty} \frac{\log \log \tau_{1/n_k}(x)}{\log n_k} \geq \overline{\alpha} - \varepsilon.$$

As ε can be arbitrarily small,

$$\limsup_{r\to 0} \frac{\log \log \tau_r(x)}{-\log r} \geq \limsup_{k\to\infty} \frac{\log \log \tau_{1/n_k}(x)}{\log n_k} \geq \overline{\alpha}.$$ □

5. *Irrational rotations*

The proof of (1.2) requires an exponentially ψ-mixing rate which is a strong mixing condition, and it is natural to ask if the same asymptotic growth in Theorem 1.4 remains the same under different mixing conditions, for example, exponentially ϕ-mixing and α-mixing, or even polynomial ψ-mixing. Although these questions are unresolved, in this section, we will show that the limsup and liminf of the asymptotic growth rate can differ if the system is not mixing at all.

Let $\theta \in (0, 1)$ be an irrational number and define $T(x) = T_\theta(x) = x + \theta \pmod 1$. Denote the one-dimensional Lebesgue measure on $[0, 1)$ by μ, then (T, μ) is an ergodic probability preserving system with $\dim_M(\mu) = 1$.

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Definition 5.1. For a given irrational number θ, the *type of* T_θ is given by the following number:

$$\eta = \eta(\theta) := \sup \left\{ \beta : \liminf_{n\to\infty} n^\beta \|n\theta\| = 0 \right\},$$

where for every $r \in \mathbb{R}$, $\|r\| = \min_{n\in\mathbb{Z}} |r - n|$.

Remark 5.2. (See [K]) For every $\theta \in (0, 1)$ irrational, $\eta(\theta) \geq 1$ and $\eta(\theta) = 1$ almost everywhere, but there exists irrational number with $\eta(\theta) \in (1, \infty]$, for example, the Liouville numbers.

For any irrational number $\theta \in (0, 1)$, there is a unique continued fraction expansion

$$\theta = [a_1, a_2, \ldots] := \frac{1}{a_1 + \frac{1}{a_2 + \cdots}},$$

where $a_i \geq 1$ for all $i \geq 1$. Set $p_0 = 0$ and $q_0 = 1$, and for $i \geq 1$, choose $p_i, q_i \in \mathbb{N}$ coprime such that

$$\frac{p_i}{q_i} = [a_1, \ldots, a_i] = \frac{1}{a_1 + \frac{1}{\cdots \frac{1}{a_i}}}.$$

Definition 5.3. The a_i terms are called the ith *partial quotient* and p_i/q_i the ith *convergent*. In particular (see [K]),

$$\eta(\theta) = \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}.$$

THEOREM 5.4. *For any irrational rotation* T_θ*,*

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_M(\mu) = 1 \leq \eta(\theta) = \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \mu\text{-a.e.}$$

By Remark 5.2, there exists irrational rotations such that the asymptotic cover time does not converge. The proof of this theorem relies on the algebraic properties of $\eta(\theta)$. For simplicity, we fix θ and write η from now on.

LEMMA 5.5. [KS, Fact 1, Lemma 7] *For each* $i \in \mathbb{N}$*, the following statements hold:*

(a) $q_{i+2} = a_{i+2}q_{i+1} + q_i$ *and* $p_{i+2} = a_{i+2}p_{i+1} + p_i$*;*

(b) $1/(2q_{i+1}) \leq 1/(q_{i+1} + q_i) < \|q_i\theta\| < 1/q_{i+1}$ *for* $i \geq 1$*;*

(c) *if* $0 < j < q_{i+1}$*, then* $\|j\theta\| \geq \|q_i\theta\|$*;*

(d) *for* $\varepsilon > 0$*, there exists uniform* $C_\varepsilon > 0$ *such that for all* $j \in \mathbb{N}$*,* $j^{\eta+\varepsilon}\|j\theta\| > C_\varepsilon$*.*

The following propositions use results given in [KS, Propositions 6 and 10].

PROPOSITION 5.6. *For* μ*-a.e.* x*,*

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \geq \eta. \tag{5.1}$$

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Proof. First, it is easy to see that for all $r > 0$ and all $x, y \in [0, 1)$, by the nature of rotation, $\tau_r(x) = \tau_r(y)$. In particular, $\tau_r(x) = \tau_r(Tx)$; hence, the function $x \mapsto \limsup_{r\to 0}(\log \tau_r(x)/-\log r)$ is T invariant; therefore, constant μ-almost everywhere by ergodicity of μ.

By [**KS**, Proposition 10], for almost every x, y,

$$\limsup_{r\to 0} \frac{\log W_{B(y,r)}(x)}{-\log r} \geq \eta,$$

where $W_E(x) := \inf\{n \geq 1 : T^n x \in E\}$ denotes the waiting time of x before visiting E. Hence, there exists a set of strictly positive measures consisting of points that satisfy

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \geq \limsup_{r\to 0} \frac{\log W_{B(y,r)}(x)}{-\log r} \geq \eta,$$

since for all $y \in [0, 1)$, $\tau_r(x) \geq W_{B(y,r)}(x)$. As $\limsup_{r\to 0}(\log \tau_r(x)/-\log r)$ is μ-almost everywhere constant, the inequality above holds for μ-a.e. x and hence the proposition is proved. □

PROPOSITION 5.7. *For μ-a.e. x,*

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \leq \eta.$$

Proof. Let $\mathcal{Q}_n := \{[2^{-n}j, 2^{-n}(j+1)) : j = 0, \ldots, 2^n - 1\}$ and set $\tau(\mathcal{Q}_n, x)$ as the minimum time for x to have visited each element of $\mathcal{Q}_n$. Again, we have $\tau_{2^{-n+1}}(x) \leq \tau(\mathcal{Q}_n, x)$ for all x. By Lemma 5.5(a) and (c), $\{\|q_i\theta\|\}_i$ is a decreasing sequence, and for each $n \in \mathbb{N}$, there exists a minimal j such that $\|q_j\theta\| < 2^{-n} \leq \|q_{j-1}\theta\|$, write $j = j_n$.

By [**KS**, Proposition 6] for all n, there is $\mu(W_{[0,2^{-n})} > q_{j_n} + q_{j_n-1}) = 0$. Notice that for all $a, b \in [0, 1)$,

$$\mu\{W_{[a,a+b)}(x) = k\} = \mu\{\{x : W_{[0,b)}(x) = k\} + a\} = \mu\{W_{[0,b)}(x) = k\}, \tag{5.2}$$

as $\mu = Leb$ is translation invariant. Then, by (5.2),

$$\begin{aligned}
\mu\{\tau(\mathcal{Q}_n, x) > q_{j_n} + q_{j_n-1}\} &= \mu\{x : \text{for all } Q \in \mathcal{Q}_n :\ W_Q(x) > q_{j_n} + q_{j_n-1}\} \\
&= \mu\bigg(x : \bigcup_{Q\in\mathcal{Q}_n} \{W_Q(x) > q_{j_n-1} + q_{j_n}\}\bigg) \leq \sum_{Q\in\mathcal{Q}_n} \mu(W_Q > q_{j_n-1} + q_{j_n}) \\
&= \sum_{j=0}^{2^n-1} \mu(W_{[2^{-n}j,2^{-n}(j+1))} > q_{j_n} + q_{j_n-1}) = \sum_{j=0}^{2^n-1} \mu(W_{[0,2^{-n})} > q_{j_n} + q_{j_n-1}) = 0.
\end{aligned}$$

Hence, by Borel–Cantelli, for all n large enough, $\tau_{2^{-n+1}}(x) \leq (q_{j_n} + q_{j_n-1})$ for μ-a.e. $x \in [0, 1)$.

Let $\varepsilon > 0$, and by Lemma 5.5(b) and (d), there exists C_ε such that

$$\log(q_{j_n} + q_{j_n-1}) \leq \log(2q_{j_n}) \leq \log \frac{2}{\|q_{j_n}\theta\|} \leq (\eta + \varepsilon) \log q_{j_n} + \log 2 - \log C_\varepsilon.$$

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Again by Lemma 5.5 and our choice of j_n, for μ-a.e. x and all n large enough,

$$\log \tau_{2^{-n+1}}(x) \le \log(q_{j_n} + q_{j_n-1}) \lesssim (\eta+\varepsilon) \log q_{j_n}$$
$$\le -(\eta+\varepsilon) \log \|q_{j_n-1}\theta\| \le (\eta+\varepsilon) n \log 2,$$

where $a \lesssim b$ means $a \le b$ up to a uniform constant. Hence, $\limsup_{n\to\infty}((\log \tau_{2^{-n}}(x))/ n \log 2) \le \eta+\varepsilon$ for μ-a.e. x. Again, since for each $r < 0$ there is a unique $n \in \mathbb{N}$ for which $2^{-n} < r \le 2^{-n+1}$, we can apply the subsequence trick again. As $\varepsilon > 0$ is arbitrarily small, the proposition is proved. □

PROPOSITION 5.8. *For μ-a.e. $x \in [0, 1)$,*

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = 1.$$

Proof. Let $\varepsilon > 0$, and using the same arguments as in the previous proof, that is, the cover time is greater than the hitting time of the ball of smallest measure at scale r, then along the sequence $r_n = 2^{-(n+1)}$, one gets for all $[a - r_n, a + r_n) \subset [0, 1)$,

$$\sum_{n\ge 1} \mu(\tau_{r_n}(x) < 2^{n(1-\varepsilon)}) \le \sum_{n\ge 1} \mu(W_{[a-2^{-n-1},a+2^{-n-1})}(x) < 2^{n(1-\varepsilon)})$$
$$\le \sum_{n\ge 1} \sum_{k=0}^{2^{n(1-\varepsilon)}} \mu(T^{-k}[a - 2^{-n-1}, a + 2^{-n-1})) = \sum_{n\ge 1} 2^{n(1-\varepsilon)} 2^{-n} = \sum_{n\ge 1} 2^{-\varepsilon n} < \infty.$$

Since for each r there is a unique n such that $r_n < r \le r_{n-1}$ while $\lim_n (\log r_n / \log r_{n-1}) = 1$, so by Borel–Cantelli,

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \liminf_{n\to\infty} \frac{\log \tau_{2^{-n}}(x)}{n \log 2} \ge 1 - \varepsilon,$$

and as ε is arbitrarily small, the lower bound is proved.

For the upper bound of liminf, recall that $\tau(\mathcal{Q}_n, x) \ge \tau_{2^{-n}}(x)$. We can repeat the proof of Proposition 5.7, apart from that this time, we choose $\{2^{-n_i}\}_i$ according to $\{q_i\}_{i\in\mathbb{N}}$: for each i, choose $n_i \in \mathbb{N}$ to be the smallest number such that

$$\|q_{i+1}\theta\| < 2^{-n_i} \le \|q_i\theta\|.$$

Hence, as in Proposition 5.7,

$$\mu(\tau(\mathcal{Q}_{n_i}, x) > q_{i+1} + q_i) \le \sum_{Q\in\mathcal{Q}_{n_i}} \mu(W_Q > q_{i+1} + q_i) = 0.$$

Again by Lemma 5.5(b), $q_{i+1} + q_i \le 2q_{i+1} \le (2/\|q_i\theta\|) < 2^{n_i+1}$ by our choice of n_i, so $\lim_{i\to\infty}(\log(q_i + q_{i+1})/n_i \log 2) \le 1$; therefore, for μ-a.e. x,

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \le \liminf_{i\to\infty} \frac{\log \tau_{2^{-n_i}}(x)}{n_i \log 2} \le \liminf_{i\to\infty} \frac{\log \tau(\mathcal{Q}_{n_i}, x)}{n_i \log 2} \le 1.$$ □

6. *Cover time for flows*

In this section, we prove results analogous to Theorem 1.1 regarding cover times for a class of flows similar to those discussed in [**RT**, §4].

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Let $\{f_t\}_{t\in\mathbb{R}}$ be a flow on a metric space $(\mathcal{X}, d_{\mathcal{X}})$ preserving an ergodic measure ν, that is, $\nu(f_t^{-1}A) = \nu(A)$ for every $t \geq 0$ and A measurable. Define the cover time of x at scale r by

$$\tau_r(x) := \inf\{T > 0 : \text{for all } y \in \Omega, \text{ there exists } t \leq T : \ d(f_t(x), y) < r\}.$$

We will assume the existence of a Poincaré section $Y \subset \mathcal{X}$, and let $R_Y(x)$ denote the first hitting time to Y, that is, $R_Y(x) := \inf\{t > 0 : f_t(x) \in Y\}$, with $\overline{R} := \int R_Y d\nu < \infty$. Define the Poincaré map by (Y, F, μ), where $F = f_{R_Y}$ and μ is the induced measure on Y given by $\mu = (1/\overline{R})\nu|_Y$. Additionally, assume the following conditions are satisfied:

(H1) $\dim_M(\mu)$ exists and is finite for (F, μ);

(H2) (Y, F, μ) is Gibbs–Markov, so Theorem 1.1 is applicable for μ-a.e. $y \in Y$;

(H3) $\{f_t\}_t$ has bounded speed: there exists $K > 0$ such that for all $t > 0$, $d(f_s(x), f_{s+t}(x)) < Kt$;

(H4) $\{f_t\}_t$ is topologically mixing and there exists $T_1 > 0$ such that

$$\bigcup_{0<t\leq T_1} f_t(Y) = \mathcal{X}; \tag{6.1}$$

(H5) there exists

$$C_f := \sup\{\operatorname{diam}(f_t(I))/\operatorname{diam}(I) : I \text{ an interval contained in } Y, 0 < t \leq T_1\} \in (0, \infty).$$

Remark 6.1. The last condition is satisfied when condition (H3) holds and the flow is, for example, Lipschitz, that is, there exists $L > 0$ such that for all $x, y \in \mathcal{X}$,

$$d_{\mathcal{X}}(f_t(x), f_t(y)) \leq L^t d_{\mathcal{X}}(x, y).$$

THEOREM 6.2. *Let (f_t, ν) be a measure-preserving flow satisfying conditions (H1)–(H5) and $\underline{\dim}_M(\nu) > 1$, then for ν-a.e. $x \in \Omega$,*

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \geq \underline{\dim}_M(\nu) - 1. \tag{6.2}$$

Furthermore, if $\overline{\dim}_M(\nu) = \dim_M(\mu) + 1$,

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \leq \overline{\dim}_M(\mu) \quad \nu\text{-a.e.} \tag{6.3}$$

Proof of (6.2). This proof is analogous to those of Proposition 4.3 and [**RT**, Theorem 4.1].

Fix some $y \in \Omega$ and $r > 0$, and consider the random variable

$$S_{T,r}(x) := \int_0^T \mathbf{1}_{B(y,r)}(f_t(x))\, dt.$$

Observe that by the bounded speed property, for all $T > r/K$,

$$\{x : \text{there exists } 0 \leq t \leq T \text{ such that } \ d(f_t(x), y)) < r\} \subset \{S_{2T,2r}(x) > r/K\},$$

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since if $d(f_s(x), y) < r$ for some s, then for all $t < r/K$, $d(f_{t+s}(x), y) < 2r$. Also set

$$T(x, y, r) := \inf\{t \geq 0 : f_t(x) \in B(y, r)\},$$

and similarly for all $r > 0$ and all x, z, $\tau_r(x) \geq T(x, y, r)$.

Let $\varepsilon > 0$ be arbitrary and by definition of $\underline{\alpha}$ for all large $n \in \mathbb{N}$, there exists $y_n \in \Omega$ such that $\nu(B(y_n, 2^{-n})) \leq 2^{-n(\underline{\alpha}-\varepsilon)}$. By Markov's inequality, for some $\mathcal{T}_n > 0$ to be decided later,

$$\begin{aligned}\nu(x : \tau_{2^{-n}}(x) < \mathcal{T}_n) &\leq \nu(x : T(x, y_n, 2^{-n}) < \mathcal{T}_n) \\ &= \nu(x : \text{there exists } 0 \leq t < \mathcal{T}_n : \ f_t(x) \in B(y_n, 2^{-n})) \\ &\leq \nu(x : S_{2\mathcal{T}_n, 2^{-n+1}}(x) > 2^{-n}/K) \leq K2^n \int_0^{2\mathcal{T}_n} \int \mathbf{1}_{B(y_{n-1}, 2^{-n+1})}(f_t(x))\, d\nu(x)\, dt \\ &\leq K2^{n+1}\mathcal{T}_n \nu(B(y_{n-1}, 2^{-n+1})) \leq 4K\mathcal{T}_n 2^{-(n-1)(\underline{\alpha}-\varepsilon-1)}.\end{aligned}$$

Choosing $\mathcal{T}_n = 2^{(n-1)(\underline{\alpha}-\varepsilon-1)}/n^2$, the last term above is summable along n; hence, by Borel–Cantelli, for ν-a.e. x

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \geq \liminf_{n\to\infty} \frac{\log \mathcal{T}_n}{n \log 2} = \underline{\alpha} - 1 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily small, the lower bound is $\underline{\alpha} - 1$, and by Remark 4.1, the proposition is proved. □

Note that the proof of lower bound is independent of the existence or mixing properties of the Poincaré map (Y, F, μ). For upper bound, we first prove that the cover time of the Poincaré F in Y is comparable to the cover time of the flow.

LEMMA 6.3. *Define*

$$\tau_r^F(x) := \min\{n \in \mathbb{N}_0 : \textit{for all } y \in Y, \textit{there exists } 0 \leq j \leq n : d(y, F^j x) < r\}.$$

There exists $\lambda = (1/C_f)$ *for* C_f *defined in condition (H5) such that* $\tau_r(x) \leq T_1 + \sum_{j=0}^{\tau_{\lambda r}^F(x)} R_Y(F^j x)$.

Proof. This is adapted from the proof of [JT, Lemma 6.4] and [RT, Theorem 2.1]. Here, F is by assumption Gibbs–Markov, so one can find $\mathcal{P}(r)$, a natural partition of Y using cylinder sets with respect to F, such that for each $P \in \mathcal{P}(r)$: (a) $\operatorname{diam}(P) \leq r/C_f$; and (b) for all $0 < t \leq T_1$, $f_t(P)$ is connected. Suppose $\tau_{r/C_f}^F(x) = k$, then the orbit $\{x, F(x), \ldots, F^k(x)\}$ must have visited every element of $\mathcal{P}$. By (6.1) for each $y \in \Omega$, there is $P \in \mathcal{P}(r)$ and $0 < s \leq T_1$ such that $y \in f_s(P)$ and, hence, there exists $j \leq k$ such that $d(f_s(F^j(x)), y) \leq C_f|P| < r$. Then, set $\lambda = 1/C_f$. The lemma is proved. □

Proof of (6.3). Now assume $\overline{\dim}_M(\nu) = \dim_M(\mu) + 1$. Let $\xi > 0$ be arbitrary and define the sets

$$U_{\xi,N} := \{x \in Y : |R_n(x) - n\overline{R}| \leq \xi n \text{ for all } n \geq N\},$$

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where $R_n(x) = \sum_{j=0}^{n-1} R_Y(F^j(x))$. By ergodicity, $\lim_N \mu(U_{\xi,N}) = 1$, so for N large, $\nu(U_{\xi,N}) > 0$; hence, by invariance,

$$\lim_{N\to\infty} \nu\bigg(\bigcup_{t=0}^{\xi N} f_{-t}(U_{\xi,N})\bigg) = 1. \tag{6.4}$$

Let $\varepsilon > 0$ be arbitrary. By (6.4), one can pick N^* such that for each ν typical $x \in \mathcal{X}$, there is some $t^* \le \xi N^*$ such that $f_{t^*}(x) \in Y$. By Theorem 1.1 applied to the Poincaré map and Lemma 6.3, for all sufficiently small $r > 0$, we have the following two inequalities:

$$\frac{\log \tau^F_{\lambda r}(f_{t^*}x)}{-\log \lambda r} \le \dim_M(\mu) + \varepsilon, \qquad \frac{\log(\tau_r(x) - T_1)}{-\log r} \le \frac{\log((\overline{R} + \xi)\tau^F_{\lambda r}(f_{t^*}x))}{-\log r}.$$

Then, as λ, $\overline{R}$ are constants and ε is arbitrary, for ν-a.e. x,

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \le \dim_M(\mu) = \overline{\dim}_M(\nu) - 1. \qquad \square$$

6.1. *Example: suspension semi-flows over topological Markov shifts.* In this section, we give an example of a flow for which $\dim_M(\nu) = \dim_M(\mu) + 1$ is satisfied, so Theorem 6.2 is applicable.

Let $\mathcal{A}$ be a finite alphabet and M an $\mathcal{A} \times \mathcal{A}$ matrix with $\{0, 1\}$ entries, we will consider two-sided topological Markov shift systems $(\Sigma, \sigma, \phi, \mu)$, where

$$\Sigma := \{x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathcal{A}^{\mathbb{Z}} : \text{for all } j, x_j \in \mathcal{A} \text{ and } M_{x_j,x_{j+1}} = 1\},$$

σ the usual left shift, ϕ a Hölder potential and μ is the unique Gibbs measure with respect to ϕ. We assume that $\dim_M(\mu) \in (0, \infty)$. The natural symbolic metric on Σ is $d(x, y) = 2^{-x\wedge y}$, where

$$x \wedge y = \sup\{k \ge 0 : x_j = y_j \text{ for all } |j| < k\}.$$

An n-cylinder in this setting is given by $[x_{-(n-1)}, \ldots, x_0, \ldots, x_{n-1}] := \{y \in \Sigma,\ y_j = x_j$ for all $|j| < n\}$, and it is a well-known fact that balls in Σ are precisely the cylinder sets. The left-shift map σ is bi-Lipschitz with Lipschitz constant $L = 2$. For a more detailed description of the shift space, see [**Bow**, §1].

Let $\varphi \in L^1(\mu)$ be a positive Lipschitz function, define the space

$$Y_\varphi := \{(x, s) \in \Sigma \times \mathbb{R}_{\ge 0} : 0 \le s \le \varphi(x)\}/\sim,$$

where $(x, \varphi(x)) \sim (\sigma(x), 0)$ for all $x \in I$. The suspension flow Ψ over σ is the function that acts on Y_φ by

$$\Psi_t(x, s) = (\sigma^k(x), v),$$

where $k, v \ge 0$ are determined by $s + t = v + \sum_{j=0}^{k-1} \varphi(\sigma^j(x))$. The invariant measure ν for the flow Ψ on Y_φ satisfies the following: for every $g : Y_\varphi \to \mathbb{R}$ continuous,

$$\int g\, d\nu = \frac{1}{\int_\Sigma \varphi\, d\mu} \int_\Sigma \int_0^{\varphi(x)} g(x, s)\, ds\, d\mu(x). \tag{6.5}$$

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The metric on Y_φ is the Bowen–Walters distance d_Y (see for example [BW]). Define another metric d_π on Y_φ: for all $(x_i, t_i)_{i=1,2} \in Y_\varphi$,

$$d_\pi((x_1, t_1), (x_2, t_2)) := \min \begin{Bmatrix} d(x, y) + |s - t|, \\ d(\sigma x, y) + \varphi(x) - s + t, \\ d(x, \sigma y) + \varphi(y) - t + s \end{Bmatrix},$$

and the following proposition says d_π is comparable to the Bowen–Walters distance.

PROPOSITION 6.4. [BS, Proposition 17] *There exists $c = c_\pi$ such that*

$$c^{-1} d_\pi((x_1, t_1), (x_2, t_2)) \le d_Y((x_1, t_1), (x_2, t_2)) \le c\, d_\pi((x_1, t_1), (x_2, t_2)).$$

Then, the Minkowski dimension of the flow-invariant measure ν is given by the following.

PROPOSITION 6.5. *For (μ) the Gibbs measure with respect to ϕ on the two-sided subshift and ν the flow invariant measure,* $\dim_M(\nu) = \dim_M(\mu) + 1$.

Proof. The proof is based on the proof of [RT, Theorem 4.3] for correlation dimensions.

By Proposition 6.4 for all $r > 0$,

$$(B(x, r/2c) \times (s - r/2c, s + r/2c)) \cap Y \subset B_Y((x, s), r),$$

where B_Y denotes the ball with respect to the metric d_Y, then for all $(x, s) \in Y_\varphi$, put $\overline{\varphi} = \int_\Sigma \varphi \, d\mu$, then

$$\nu(B_Y((x, s), r)) \ge \nu\bigg(B(x, r/2c) \times \bigg(s - \frac{r}{2c}, s + \frac{r}{2c}\bigg)\bigg),$$

$$\frac{\log \nu(B_Y((x, s), r))}{\log r} \le \frac{\log((r/c\overline{\varphi})\mu(B(x, r/2c)))}{\log r}.$$

Hence, $\overline{\dim}_M(\nu) = \limsup_{r\to 0} \log \min_{(x,s)\in \mathrm{supp}(\nu)} \nu(B_Y((x, s), r)/\log r) \le \dim_M(\mu) + 1$.

For lower bound, define

$$B_1 := B(x, cr) \times (s - cr, s + cr), \quad B_2 := B(\sigma x, cr) \times [0, cr),$$

$$B_3 := \{(y, t) : y \in B(\sigma^{-1} x, 2cr) \text{ and } \varphi(y) - cr \le t \le \varphi(y)\}.$$

Then, as in the proof of [RT, Theorem 4.3], $B_Y((x, s), r) \subset (B_1 \cup B_2 \cup B_3) \cap Y_\varphi$.

For all $r > 0$ and $(x, s) \in Y_\varphi$ by (6.5), and as μ is σ, σ^{-1} invariant,

$$\nu(B_1 \cap Y_\varphi) = 2cr\mu(B(x, cr))/\overline{\varphi}, \quad \nu(B_2, Y_\varphi) \le cr\mu(B(x, cr))/\overline{\varphi}$$
$$\nu(B_3 \cap Y_\varphi) \le cr\mu(\sigma^{-1} B(x, 2cr))/\overline{\varphi} = cr\mu(B(x, 2cr))/\overline{\varphi}.$$

Therefore,

$$\nu(B_Y((x, s), r)) \le \frac{1}{\overline{\varphi}}(3r\mu(B(x, cr)) + cr\mu(B(x, 2cr))),$$

which is enough to conclude that $\underline{\dim}_M(\nu) \ge \dim_M(\mu) + 1$. □

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REFERENCES

[BJK] B. Bárány, N. Jurga and I. Kolossváry. On the convergence rate of the chaos game. *Int. Math. Res. Not. IMRN* **2023**(5) (2023), 4456–4500.

[Bow] R. Bowen. *Equilibrium States and The Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470)*. Springer, Berlin, 1975.

[BS] L. Barreira and B. Saussol. Multifractal analysis of hyperbolic flows. *Comm. Math. Phys.* **214** (2000), 339–371.

[BW] R. Bowen and P. Walters. Expansive one-parameter flows. *J. Differential Equations* **12** (1972), 180–193.

[FFK] K. Falconer, J. Fraser and A. Käemäki. Minkowski dimension for measures. *Proc. Amer. Math. Soc.* **151** (2023), 779–794.

[JT] N. Jurga and M. Todd. Cover times in dynamical systems. *Israel J. Math.* (2024); in press.

[K] A. Khintchine. *Continued Fractions*. University of Chicago Press, Chicago, IL, 1964.

[KS] D. Kim and B. Seo. The waiting time for irrational rotations. *Nonlinearity* **16** (2003), 1861.

[M] P. Matthews. Covering problems for Brownian motion on spheres. *Ann. Probab.* **16**(1) (1988), 189–199.

[RT] J. Rousseau and M. Todd. Orbits closeness for slowly mixing dynamical systems. *Ergod. Th. & Dynam. Sys.* **44**(4) (2024), 1192–1208.

https://doi.org/10.1017/etds.2025.10216 Published online by Cambridge University Press